Dynamic Matching and Bargaining Games: A General Approach

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Abstract

Dynamic matching and bargaining games provide models of decentralized trading with frictions. A central objective of the literature is to investigate how equilibrium outcomes depend on the level of the frictions. In particular, will the outcome become efficient when frictions become small? Existing specifications of such games give different answers. To investigate what causes these difference we identify four simple conditions on trading outcomes and their relation to the level of frictions. We show that for every game which satisfies these conditions, the equilibrium outcome becomes efficient when frictions are small. We illustrate this approach with two games in which we show that our conditions hold, implying that the limiting outcome is efficient. We then proceed to specifications of dynamic matching and bargaining games in which limiting equilibrium outcomes do not need to become efficient. We show that our conditions do not hold in these games.

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1 Introduction

In a dynamic matching and bargaining game, a large population of traders interacts repeatedly in a decentralized market. Every trading period, traders are *matched* into small groups where they *bargain* over the terms of trade. If they fail to reach an agreement, they can wait at some costs until the next period to be rematched into a new group. These waiting costs are the *frictions* of trading in the decentralized market. A major question in the literature concerns the trading outcome with vanishing frictions: Will the outcome become efficient? Ideally, one would like not only to find answers for particular trading institutions but also to gain general understanding of the conditions under which trading with vanishing frictions has this property and when it does not. This is the task of this paper. The main contribution is the provision of a general, "detail free" framework for the analysis of decentralized markets. Recent contributions that fall into the framework of this paper include Moreno and Wooders (2001), Satterthwaite and Shneyerov (2006), and De Fraja and Sakovics (2001).

As a basic set-up we use the following steady state, dynamic matching and bargaining environment similar to the one used by Gale (1987):[1] There is a continuum of buyers who have unit demand and valuations $v \in [0,1]$ for an indivisible good and there is a continuum of sellers who have unit capacity and costs $c \in [0,1]$. These traders are matched into small groups. In these groups they bargain and, if they reach an agreement, they trade. The groups are connected to form a large market by allowing unsuccessful traders to be matched into new groups in the next period. Integration, however, is imperfect because there is a probability $\delta \in (0,1)$ that a trader dies while waiting. These are the *frictions* of trading. To keep the market in a steady state, there is an exogeneous inflow of new buyers and sellers at the end of each period.

This framework is general with respect to the matching technology and with respect to the bargaining protocol, i.e. we do not specify how traders are matched into groups. Also, we do not specify how bargaining within the groups takes place and what information is released before and during bargaining. We will see how existing models in the literature differ in how they fill in these details. But no matter how this is done, every specification of the model will give rise to an *outcome* which consists of (a) probabilities of trading for entering types and (b) expected equilibrium payoffs. Let $Q^S(c)$ denote the probability that a seller of type c sells his good and let $Q^B(v)$ denote the probability

[1] The main difference is our assumption that traders are finitely lived, see section 6.2 for the case of infinitely lived traders.

that a buyer of type v gets the good. Similarly, let $V^S(c)$ and $V^B(v)$ be the payoffs to these types. So an outcome is a vector $A = [Q^S, Q^B, V^S, V^B]$.

Outcomes are defined across models. We state four conditions on outcomes and we argue that these conditions are natural. To illustrate and to motivate these conditions, we introduce a *basic model* by specifying the matching technology and the bargaining protocol as in Lauermann (2006a): We assume that matching is pairwise and "groups" consist of one seller and one buyer. Within each match, the seller makes a take-it-or-leave-it price offer to the buyer. An important assumption is that information is private so that the seller cannot observe the valuation of the buyer.

Now, suppose there is some sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ which converges to zero, $\delta_k \to 0$ and for every $\delta_k \in (0,1)$ take some associated trading outcome A_k of some equilibrium of the specification described before. The first condition on outcomes A_k is that trading probabilities $Q_k^S(\cdot)$ and $Q_k^B(\cdot)$ are monotone in types. The second condition is that the difference of payoffs between types is bounded. Outcomes of the basic model have this property because of asymmetric information which implies that traders can mimic each others' strategies. The third condition is that whenever trading probabilities of some set of buyers do not converge to one as δ_k becomes small, these buyers become *available*, i.e. a seller can be certain to be matched with such a buyer at least once in his lifetime. The fourth condition states that if a seller of type c_x is almost certain to be matched with a buyer of type v_x or better and a buyer of type v_x is certain to be matched with a seller of type c_x or better, i.e. if types $c \leq c_x$ and $v \geq v_x$ are available, then their joint payoff becomes pairwise efficient and $V^S(c_x) + V^B(v_x)$ is at least $v_x - c_x$ in the limit. One might expect this condition to hold since otherwise there would be unrealized surplus left on the table and the respective traders c_x and v_x would be able to find trading partners with whom to realize that surplus.

Let $S(A)$ be the surplus of an outcome A and let S^* be its maximum. Our main result is this: Every sequence of outcomes $\{A_k\}_{k=1}^{\infty}$ which satisfies the four conditions stated before becomes efficient when δ becomes small:

$$\lim_{k \to \infty} S(A_k) = S^*.$$

Then we show that these conditions are met in many specifications of the framework. They hold not only with asymmetric information but also in a model similar to Gale's own specification with *symmetric* information. We give some intuition that the conditions hold even in the non-steady state model of Moreno and Wooders (2001). Also, we provide some intuition for the case of bargaining between one seller and many buyers, when the seller holds a second price auction.

Auctions are also used to specify the bargaining protocol in the model of Satterthwaite and Shneyerov (2006). But just as Gale (1987), they include an entry stage and assume that traders are infinitely lived. To show that our approach is also valid with these additional complications, we extend our general framework by including an entry decision and by considering the case where the exit rate δ is set equal to zero. In this new framework we need stronger conditions to ensure convergence to efficiency, discussed in detail in section 6.

To show that our conditions are "tight" we show that convergence to efficiency fails if our conditions are violated and relate this to the failure of convergence in the literature. In particular, we show that the failure in Lauermann (2006b) can be traced back to the failure of the second condition, the failure in Diamond (1971) can be traced to the failure of availability (the third condition), the failure in Serrano (2002) to the failure of weak efficiency (the fourth condition), and the failure in De Fraja and Sakovics (2001) to the observation that in their model pairwise efficiency for all types is not a sufficient condition for efficiency.

The rest of the paper is structured as follows. First, we introduce the basic model in section 2. Then we provide the general framework in section 3.1 and in section 3.2 we discuss necessary and sufficient conditions for outcomes to be efficient. We introduce the four conditions on outcomes in section 3.3. We prove our main result in the next section: Every sequence of outcomes that meets the four conditions becomes efficient. In section 5 we demonstrate that the conditions are met in some examples. We introduce some variations of the general framework by adding an entry stage (section 6.1) and assuming that traders are infinitely lived (section 6.2). Failures of convergence to efficiency are discussed in section 7.

2 The Basic Model[2]

There is a continuum of buyers and sellers who interact in a repeated market over infinitely many periods. Sellers have one unit of an indivisible good and their costs of

[2] This description follows closely the one in Lauermann (2006a)

trading are $c \in [0,1]$. Buyers want to acquire one unit of the good and their valuation for the good is $v \in [0,1]$. At the beginning of each period there is some pool of buyers and sellers. The traders from this pool are matched into pairs consisting of one seller and one buyer. Within each pair the seller announces a price offer $p \in [0,1]$ and the buyer announces whether he rejects or accepts the offer. If he accepts, the seller receives $p - c$ while the buyer receives $v - p$. Next, all buyers and sellers who have traded exit the pool. Likewise, a share δ of all those traders who failed to trade exits. Finally, new players enter the market and the period ends. The next period starts according to the same rules.

The inflow of buyers and the inflow of sellers has mass one each. The distribution of valuations among buyers in the inflow is exogeneously given by some c.d.f. $G^B(\cdot)$ and similarly, the distribution of costs is given by some distribution $G^S(\cdot)$. We assume that $G^B(\cdot)$ and $G^S(\cdot)$ have continuous and strictly positive densities. Let p^w be the price such that the mass of sellers in the inflow with costs below p^w is exactly equal to the mass of buyers with valuations above p^w:

$$G^S(p^w) = 1 - G^B(p^w). \tag{1}$$

Since the left hand side is strictly increasing while the right hand side is strictly decreasing, the solution to (1) is the unique. The function $G^S(\cdot)$ can be interpreted as *supply* and $1 - G^B(p^w)$ can be interpreted as *demand*. So p^w is the price at which supply equals demand and the price corresponds to the *Walrasian* market clearing price relative to the inflow.

The market constellation is characterized by a vector $\sigma = \left[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M\right]$ where $p(c) \in [0,1]$ is the price offered by a seller of type c, $r(v) \in [0,1]$ is the highest price accepted by a buyer of type v, $\Phi^S(\cdot)$ is the cumulative distribution function of costs in the pool of sellers, $\Phi^B(\cdot)$ is the corresponding distribution function for buyers, and M is the total mass of buyers in the pool which is equal to the total mass of sellers in a steady state. For the analysis, we assume that all functions under consideration are measurable. With Σ_M being the set of measurable functions $f : [0,1] \to [0,1]$, σ is an element of $\Sigma \equiv \Sigma_M^4 \times [0,1]$.

We say that a vector σ constitutes an *equilibrium* if strategies are mutually optimal given the distribution of types and if the distribution of types in the pool is consistent with the trading strategies and the exogeneous inflow. These conditions are now spelled out in detail.

First we turn to the sellers. If the seller offers a price p, let us denote by $D(p|\sigma)$ the probability that the buyer accepts the offer. Buyers accept a price p if $p \leq r(v)$ (see below) so $D(p|\sigma)$ is

$$D(p|\sigma) \equiv \int_{\{v|p \leq r(v)\}} d\Phi^B(v) \quad .^3 \tag{2}$$

Let $q^S(p|\sigma)$ be the probability that a seller can trade some time during his lifetime

$$q^S(p|\sigma, \delta) \equiv \frac{D(p|\sigma)}{1-(1-D(p|\sigma))(1-\delta)}, \tag{3}$$

which we also call *lifetime trading probability*, and which is the solution to the recursive formula

$$q^S(p|\sigma, \delta) = D(p|\sigma) + (1-D(p|\sigma))(1-\delta)\, q^S(p|\sigma, \delta).$$

The expected payoff to a seller when offering a price p is defined as

$$U^S(p, c|\sigma) \equiv q^S(p|\sigma, \delta)(p-c)$$

and we require that $p(c) \in \arg\max U^S(\cdot, c|\sigma)$ for all c in equilibrium.

For buyers, let $S(r|\sigma)$ denote the probability to receive an offer $p \leq r$ in any period. Again, we define the lifetime trading probability by

$$q^B(r|\sigma, \delta) \equiv \frac{S(r|\sigma)}{1-(1-S(r|\sigma))(1-\delta)}.$$

The expected price offer conditional on $p \leq r$ is denoted by $E[p|p \leq r, \sigma]$.[4] Payoffs when accepting all $p \leq r$ are given by

$$U^B(r, v|\sigma) \equiv q^B(r|\sigma, \delta)(v - E[p|p \leq r, \sigma]). \tag{4}$$

Let $V^B(v|\sigma) \equiv \sup_r U^B(r, v|\sigma)$. Suppose that the following condition holds

$$r(v) = v - (1-\delta) V^B(v|\sigma). \tag{5}$$

Then a buyer who receives an offer $p = r(v)$ is just indifferent between accepting and rejecting the offer: Then his payoff from accepting the offer, $v - r(v)$, is equal to his

[3] Later, $r(\cdot)$ is shown to be monotone, and $D(p|\sigma)$ simplifies to $\left(1-\Phi^B\left(r^{-1}(p)\right)\right)$; $r^{-1}(p) = \inf\{v, 1|r(v) \geq p\}$.

[4] If $Q^B(r) = 0$, then $E[p|p \leq r] \equiv r$.

payoff from rejection, which is the continuation payoff $(1-\delta) V^B(v|\sigma)$. If $p < r(v)$, the buyer is strictly better off when accepting the offer and when $p > r(v)$, the buyer is strictly better off accepting the offer. Hence, it is optimal for a buyer to accept a price if it is below $r(v)$ and to reject the price otherwise.[5]

We restrict attention to equilibria in which the pool does not change over time. If the distribution at the beginning of a period is given by $\Phi_t^S(\cdot)$ and the trading strategies are $r(\cdot)$ and $p(\cdot)$, then the distribution of sellers at the end of the period is sum of the entering sellers and the initial sellers who did neither trade nor die:

$$\Phi_{t+1}^S(c|\sigma) = G^S(c) + (1-\delta)\int_0^c (1 - D(p(\tau)))\, d\Phi_t^S(\tau).$$

The pool of traders is in a steady state if and only if the distribution of types does not change over time. For sellers we need that $\Phi_{t+1}^S(c|\sigma) = \Phi_t^S(c) = \Phi^S(c)$ for all c. This condition can be written as[6]

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M(D(p(\tau)|\sigma) + \delta(1 - D(p(\tau)|\sigma)))} \quad \text{for all } c. \tag{6}$$

A similar condition can be obtained for buyers:

$$\Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M(S(r(\tau)|\sigma) + \delta(1 - S(r(\tau)|\sigma)))} \quad \text{for all } v. \tag{7}$$

Summing up, we say σ is an equilibrium if it satisfies the above conditions:

Definition 1 *A steady state equilibrium vector $\sigma^* \in \Sigma$ consists of an optimal pair of strategies and a corresponding steady state pool, i.e. $\sigma^* = \left[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M\right]$ such that*

- $p(c) \in \arg\max U^S(p, c|\sigma^*)$ *for all c*
- $r(v) = v - (1-\delta) V^B(v|\sigma^*)$ *for all v*
- $\Phi^S(\cdot)$, $\Phi^B(\cdot)$, *and M satisfy the steady state conditions (6), (7).*

[5] In our model, we *assume* that buyers use reservation prices. In general, reservation price strategies are the unique optimal sequentially rational strategies when sampling without recall from a known stationary distribution of prices, see McMillan and Rothschild (1994).

[6] We get this by algebraic manipulation of $\Phi^S(c) = \Phi_{t+1}^S(c)$ by observing that $\int_0^c d\Phi^S(\tau) = \int_0^c dG^S(\tau) + \int_0^c (1-\delta)(1 - D(p(\tau)))\, d\Phi^S(c)$ and then $\int_0^c d\Phi^S(\tau) - \frac{dG^S(\tau)}{1-(1-\delta)(1-D(p(\tau)))} = 0$.

3 The General Approach

Now we start with the analysis of the outcomes of the general framework. In the next subsection we introduce some basic notation and make some preliminary observations. Then we show that outcomes are efficient if they are "Walrasian" and we derive a sufficient condition for efficiency to prepare our proof of convergence. Finally, we introduce the four conditions that we use to characterize outcomes.

3.1 Outcomes

An outcome is a vector $A = \left[V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)\right]$, where $V^S(c)$ is the expected payoff of an entering seller with type c and $Q^S(c)$ is his (lifetime) trading probability. Similarly, $V^B(v)$ is the expected payoff of an entering buyer of type v and $Q^B(v)$ is his (lifetime) trading probability. We define $T^S(\cdot)$ and $T^B(\cdot)$ such that

$$V^S(c) = T^S(c) - c\,Q^S(c) \quad \text{and} \quad V^B(v) = v\,Q^B(v) - T^B(v). \tag{8}$$

i.e. $T^S(c) \equiv V^S(c) + cQ^S(c)$ and $T^B(v) \equiv vQ^B(v) - V^B(v)$. We assume that there is no discounting,[7] so $T^S(\cdot)$ and $T^B(\cdot)$ can be interpreted as expected transfers. Given an outcome A, the surplus of entering traders is defined as

$$S(A) \equiv \int_0^1 V^B(v)\,dG^B(v) + \int_0^1 V^S(c)\,dG^S(c).$$

We assume that all functions are measurable, i.e. $A \in \Sigma_M^4$ and $S(\cdot) : \Sigma_M^4 \to \mathbb{R}$. Since these functions are also bounded (by definition) this allows us to use the Lebesgue integral to define a distance between two functions, $d(\cdot,\cdot) : \Sigma_M^2 \to [0,1]$ with $d(f_1, f_2) = \int_0^1 |f_1(x) - f_2(x)|\,dx$.[8] We use $d(\cdot,\cdot)$ to define convergence in Σ_M. Furthermore, in many cases we can restrict further the set of functions to the set of monotone functions. This will turn out to be helpful for our proofs because every sequence of monotone functions has a convergent subsequence, i.e. sets of monotone functions are sequentially compact.[9] For future references, let $\Sigma_+ \subset \Sigma_M$ be the subset of weakly increasing

[7] When we consider infinitely lived agents in section 6.2 we also discuss the implications of a model with discounting. We show that the basic insights still apply.

[8] Note that $d(\cdot,\cdot)$ is only a semimetric: $d(f_1, f_2)$ does not imply $f_1 = f_2$. Still $d(f_1, f_2)$ is non-negative and symmetric, and it satifies $d(f_1, f_1) = 0$ and the triangle-inequality. We endow Σ_M with the semimetric topology (see Aliprantis, Border (1994, p. 23), defined in the natural way by using open $\varepsilon-$balls $B_\varepsilon(f_1) = \{f \in \Sigma_M | d(f_1, f) < \varepsilon\}$ to define open sets just as in a metric space.

[9] By Helly's selection theorem (see Kolmogorov, Fomin 1970, p. 372) every sequence $\{f_N\}_{N=1}^{\infty}$ of

functions and let $\Sigma_- \subset \Sigma_M$ be the subset of weakly decreasing functions.

A natural consistency requirement on an outcome is that total transfers collectively made by all buyers are equal to total transfers received by all sellers, $\int_0^1 T^B(v)\, dG^B(v) = \int_0^1 T^S(c)\, dG^S(c)$. From (8), it follows that this is equivalent to the following condition on A:

$$\int_0^1 \left(v\, Q^B(v) - V^B(v)\right) dG^B(v) = \int_0^1 \left(V^S(c) + c\, Q^S(c)\right) dG^S(c)\,. \tag{9}$$

Define

$$S_Q(A) \equiv \int_0^1 v\, Q^B(v)\, dG^B(v) - \int_0^1 c\, Q^S(c)\, dG^S(c)$$

Condition (9) is equivalent to

$$S(A) = S_Q(A)\,. \tag{10}$$

This equality reflects the idea that for the purpose of welfare analysis only the allocation of the good matters while transfers "cancel".

Similar to the balance of transfers, the total mass of buyers who trade is required to be equal to the total mass of sellers who trade:

$$\int_0^1 Q^S(c)\, dG^S(c) = \int_0^1 Q^B(v)\, dG^B(v)\,. \tag{11}$$

Economically, this condition corresponds to the scarcity of the good: For every buyer who enjoys consumption there must be some seller who incurs costs. We define the set $\hat{Q}$ of all trading outcomes satisfying balance of total trades:

$$\hat{Q} \equiv \left\{Q^S(\cdot), Q^B(\cdot) \in \Sigma_M^2 \mid \text{condition (11) holds}\right\}.$$

An outcome A satisfies *mass balance* if it satisfies the two consistency conditions:

Definition 2 ***Mass balance.*** *An outcome* $A = \left[V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)\right]$ *satisfies mass balance of trades and transfers if*

$$A \in \hat{A} \equiv \left\{A \mid Q \in \hat{Q} \text{ and } S(A) = S_Q(A)\right\}.$$

We say a sequence of outcomes $\{A_k\}_{k=1}^{\infty}$satisfies mass balance if each of its members A_k is in $\hat{A}$.

monotone functions has a pointwise convergent subsequence $\{f_{N'}\}_{N=1}^{\infty}$. Lebesgue's bounded convergence theorem implies $d\left(f_{N'}, \bar{f}\right) \to 0$ for some $\bar{f}$. The limit $\bar{f}$ is clearly monotone itself.

3.2 Efficiency

Now, we want to know which outcomes are efficient and we want to find conditions that ensure that a sequence of outcomes converges to efficiency to prepare for the proof. Our object of interest is the maximal surplus that can be reached subject to the resource constraint $Q \in \hat{Q}$:

$$S^* \equiv \sup_{A \in \hat{A}} S_Q(\cdot).$$

Basic economic intuition suggests that the optimal allocation is the following: All buyers with valuations above the market clearing price p^w get the good while all sellers with costs below p^w sell theirs; buyers with lower valuations and sellers with higher costs do not trade. Let Q^W be the set of *Walrasian* allocations of the good that are equivalent[10] to this rule:

$$Q^W \equiv \left\{ Q \in \hat{Q} \mid \int_0^1 \left| Q^S(c) - 1_{c \leq p^w}(c) \right| dc = 0, \int_0^1 \left| Q^B(v) - 1_{v \geq p^w}(v) \right| dv = 0 \right\}. \tag{12}$$

And indeed, the economic intuition proves to be correct (see the appendix for details):

Lemma 1 *For all outcomes that satisfy mass balance, i.e. for all $A \in \hat{A}$: $S(A) = S^*$ if and only if $Q \in Q^W$.*

Accordingly, the maximal surplus S^* is given by:

$$S^* = \int_{p^w}^1 v dG^B(v) - \int_0^{p^w} c dG^S(c). \tag{13}$$

Let $\hat{Q}_+$ be the set of trading probabilities which are monotone and which satisfy mass balance of trades, i.e. $\hat{Q}_+ \equiv \left\{ Q \in \hat{Q} \mid Q^S \in \Sigma_-, Q^B \in \Sigma_+ \right\}$. Because $\hat{Q}_+$ is sequentially compact, we can show that the former lemma also holds in the limit: a sequence of outcomes $\{Q_k\}_{k=1}^{\infty}$ becomes efficient if and only if it converges to the set Q^W. We say that a sequence $\{Q_k\}_{k=1}^{\infty}$ converges to Q^W if its distance to any element of Q^W becomes zero in every component. The proof of the following lemma is relegated to the appendix:

Lemma 2 *For every sequence $\{A_k\}_{k=1}^{\infty}$ with $A_k \in \hat{A}$ and with $Q_k \in \hat{Q}_+$:*

$$\lim_{k \to \infty} S_Q(A_k) = S^* \quad \textit{if and only if} \quad \lim_{k \to \infty} Q_k = Q^W.$$

[10] Two functions $Q_1 \in Q^W$ and $Q_2 \in Q^W$ are equivalent if $d(Q_1^s, Q_2^s) = d(Q_1^B, Q_2^B) = 0$.

Next, we derive a simple sufficient condition for the efficiency of an outcome: Suppose the outcome is such that for any cost c_x and for any valuation v_x, joint payoffs $V^S(c_x) + V^B(v_x)$ are weakly larger than the private surplus $v_x - c_x$. Then this implies that A is efficient, i.e. $S(A) = S^*$:

Lemma 3 ***Sufficiency***. *If some outcome satisfies mass balance, i.e. if $A \in \hat{A}$ and if for all v and for all c: $V^S(c) + V^B(v) \geq v - c$, then $S(A) = S^*$.*

Proof: Let $\bar{p} \equiv \inf_{c \leq p^w} \left(V^S(c) + c\right)$. Then $V^S(c) + V^B(v) \geq v - c$ for all v, c implies $V^B(v) \geq v - \inf_{c \leq p^w} \left(V^S(c) + c\right)$ for all v. Together with the definition of $\bar{p}$ we use this to bound $S(A)$:

$$\begin{aligned} S(A) &\geq \int_{p^w}^{1} V^B(v)\, dG^B(v) + \int_0^{p^w} V^S(c)\, dG^S(c) \\ &\geq \int_{p^w}^{1} (v - \bar{p})\, dG^B(v) + \int_0^{p^w} (\bar{p} - c)\, dG^S(c) \\ &= S^* + \bar{p}\left(G^S(p^w) - \left(1 - G^B(p^w)\right)\right) = S^*. \end{aligned}$$

where the last line follows from the definition of p^w. By the restriction $A \in \hat{A}$ and by the definition of S^*, $S(A) \leq S^*$. Therefore $S(A) \geq S^*$ implies $S(A) = S^*$. ∎

By continuity of $S(\cdot)$, the last lemma carries over to sequences (see the appendix for details). For technical reasons, we restrict the elements of A_k to the set of outcomes $\hat{A}_+ \subset \hat{A}$ which satisfy mass balance and which are monotone in each component, $\hat{A}_+ \equiv \hat{A} \cap [\Sigma_- \times \Sigma_+ \times \Sigma_- \times \Sigma_+]$:

Lemma 4 *For every sequence $\{A_k\}_{k=1}^{\infty}$ with $A_k \in \hat{A}_+$*

$$\lim_{k \to \infty} S(A_k) = S^* \qquad \text{if} \qquad \lim_{k \to \infty} \inf \left[V_k^S(c) + V_k^B(v)\right] \geq v - c \quad \text{for all } v,\ c.$$

3.3 General Conditions

Now we state the four conditions. For this, we take some sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ and for each exit rate δ_k we take some outcome A_k. This gives us a sequence $\{A_k\}_{k=1}^{\infty}$. In order to characterize limiting functions (whenever they exist) we denote their pointwise limits by upper bars. For sequences of trading probabilities

$$\bar{Q}^S(c) \equiv \lim_{k \to \infty} Q_k^S(c), \text{ and } \bar{Q}^B(v) \equiv \lim_{k \to \infty} Q_k^B(v),$$

and for sequences of payoffs

$$\bar{V}^S(c) \equiv \lim_{k\to\infty} V_k^S(c) \text{ and } \bar{V}^B(v) \equiv \lim_{k\to\infty} V_k^B(v).$$

The first two conditions are conditions on each element of the sequence while the other two are conditions on its limit. We motivate the former two by the trading situation with asymmetric information in the basic model.[11] (In section 5.1 we prove in detail that the conditions hold.) The main observation is that with asymmetric information, a type c can mimic the strategy of another type c_x. If he does so, he receives a transfer $T_k^S(c_x)$ and trades with probability $Q_k^S(c_x)$. Thus, for $V_k^S(c)$ to be the equilibrium payoff for type c, $V_k^S(c)$ must be at least as large as $T_k^S(c_x) - c\,Q_k^S(c_x)$. The same observations apply to buyers. So payoffs and trading probabilities have to satisfy *incentive compatibility constraints*. Because payoffs can be shown to satisfy the single crossing condition, this requires in particular that trading probabilities are monotone (for details, see section 5.1):

Condition 1 ***Monotonicity.*** *For every member A_k of the sequence $\{A_k\}_{k=1}^{\infty}$:*

$$Q_k^S(\cdot) \in \Sigma_- \quad \text{and} \quad Q_k^B(\cdot) \in \Sigma_+.$$

Furthermore, incentive compatibility requires that the difference of payoffs between any two types is bounded: Suppose the seller c_x trades with probability one and suppose in addition that type c has a cost advantage $(c_x - c)$, then the profit to c from this strategy is $(c_x - c)$ higher than the payoff of c_x. Hence, equilibrium payoffs to c must be at least $(c_x - c)$ higher than the equilibrium payoff of c_x, with $(c_x - c)$ being the *rent* of this seller. In general, the trading probability of c_x might be below one and so the payoff difference must be somewhere between zero and $(c_x - c)$. Note, that the difference cannot become larger than one, since otherwise c_x would like to mimic the type c. Again, the same applies to buyers:

Condition 2 ***No Rent Extraction.*** *For every member A_k of the sequence $\{A_k\}_{k=1}^{\infty}$ and for every c, $c_x \in [0,1]$ there is some $a \in [0,1]$ such that*[12]

[11] See the remark on page 22 on why the condition also holds in the model by Gale (1987) with *symmetric* information.

[12] Here and in the following conditions, by referring to limits we implicitly condition on their existence. So the statement includes the implicit qualifier "whenever the pointwise limits $\bar{V}^S(c)$, $\bar{V}^S(c_x)$, and $\bar{Q}^S(c_x)$ exist, then....".

$$\begin{aligned} V_k^S(c) &\geq V_k^S(c_x) + a(c_x - c), \\ \textit{and} \quad \bar{V}^S(c) &\geq \bar{V}^S(c_x) + (c_x - c) \qquad \textit{if } \bar{Q}^S(c_x) = 1. \end{aligned}$$

For every v_x, v there is some $a \in [0,1]$ such that

$$\begin{aligned} V_k^B(v) &\geq V_k^B(v_x) + a(v - v_x), \\ \textit{and} \quad \bar{V}^B(v) &\geq \bar{V}^B(v_x) + (v - v_x) \qquad \textit{if } \bar{Q}^B(v_x) = 1. \end{aligned}$$

The no rent extraction property immediately implies monotonicity and continuity of the payoff functions, something we will utilize in the proof. In particular, monotonicity and continuity carry over to the limiting functions $\bar{V}^S$ and $\bar{V}^B$.[13]

Remark: With asymmetric information we can state the bound more tightly as $V_k^S(c) \geq V_k^S(c_x) + Q_k^S(c_x)(c_x - c)$ and similarly for buyers (see (18)). However, here we want to find conditions which are just strong enough to imply the convergence result but still weak enough so that they hold in a broad range of models. In particular, we want to include the possibility of symmetric information which precludes us to state the conditions in the way described before.

For the two other conditions we introduce the concept of *availability*. The availability of the set of buyers with valuations above some v'' is the probability that a seller is matched with some $v \in [v'', 1]$ at least once during his lifetime. Intuitively, the share of a set of buyers in the pool is proportional to the probability that they do not trade but stay in the market until they die, i.e. the higher $\left(1 - Q_k^B(v)\right)$ is, the larger is their share. If $Q_k^B(\cdot)$ does not converge to one with $\delta_k \to 0$ on some interval $[v'', v']$, $\left(1 - Q_k^B(\cdot)\right)$ does not vanish. In the basic model, this implies that a seller becomes certain to be matched with a buyer of type $v \in [v'', v']$ at least once in his lifetime. This idea is formalized by the introduction of an operator $L^B(\cdot) : [0,1]^2 \times \Sigma_M^4 \to [0,1]$, where $L^B(v'', \delta_k, A_k)$ is interpreted as the probability that a seller is matched with a buyers of type $v \geq v''$ at least once, given the exit rate δ_k and the outcome A_k.[14] Let $L^B(v) \equiv L^B(v, \delta_k, A_k)$ and let $\bar{L}^B(v) = \liminf_{k \to \infty} L_k^B(v)$. The observation given before becomes the condition

[13] By the condition, all payoff functions must be Lipschitz continuous with Lipschitz constant 1, since $\left|V^S(c) - V^S(c_x)\right| \leq |c - c_x|$. Therefore, every sequence of such functions is equicontinuous and hence its limit must be equicontinuous whenever it exists, see Kolmogorov, Fomin p. 102.

[14] For an example, see the definition of $M^B(\cdot)$ in the basic model in equation (16).

that if $\limsup Q_k^B(v') < 1$, then $L_k^B(v'') \to 1$. Introducing a similar function $L^S(\cdot)$ for sellers, we state

Condition 3 ***Availability***. *If $\bar{Q}^B(v') < 1$ for some v', then for all $v'' < v'$: $\bar{L}^B(v'') = 1$. If $\bar{Q}^S(c') < 1$ for some c', then for all $c'' > c'$: $\bar{L}^S(c'') = 1$.*

Suppose it is commonly known that types c_x and v_x are available, i.e. buyers and sellers are mutually sure to meet some $c \leq c_x$ and some $v \geq v_x$, respectively. Then one might expect their joint payoffs to be ex ante pairwise efficient. Otherwise, it becomes certain that (a) there is still "money on the table" between these types and (b) they are certain to meet each other so that they can realize this additional surplus:

Condition 4 ***Weak pairwise efficiency***. *For every pair of types c_x and v_x for which $\bar{L}^S(c_x) = 1$ and $\bar{L}^B(v_x) = 1$:*

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

Remark: The latter two conditions are separated because this will make it easier to check them in the models. Alternatively, we could have stated that if for some v_x, c_x, $\bar{Q}^B(v_x) < 1$ and $\bar{Q}^S(c_x) < 1$, then for all $v < v_x$ and $c > c_x$, $\bar{V}^S(c) + \bar{V}^B(v) \geq v - c$.

4 Main Result

In this section, we state and prove our main result: Suppose there is a pair of functions L^S, L^B and a sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ such that a given sequence of outcomes $\{A_k\}_{k=1}^{\infty}$ satisfies the conditions stated before. Then surplus along this sequence becomes efficient:

Proposition 1 *Suppose some sequence $\{A_k\}_{k=1}^{\infty}$ satisfies mass balance, monotonicity, no rent extraction, availability, and weak pairwise efficiency for some sequence $\{\delta_k\}_{k=1}^{\infty}$ and for some pair of functions L^B, L^S. Then the outcome becomes efficient, i.e.*

$$\lim_{k \to \infty} S(A_k) = S^*.$$

From the monotonicity condition and the no rent extraction condition, all components of each element A_k are monotone, i.e. $A_k \in \hat{A}_+$. By Helly's selection principle (see Kolmogorov, Fomin 1970) we can find a pointwise convergent subsequence $\{A_{k'}\}_{k'=1}^{\infty}$.

Let its limit be denoted by $\left(\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B\right)$. We first show that $\left(\bar{Q}^S, \bar{Q}^B\right)$ is in the set of Walrasian allocations Q^W for every such subsequence. Then we show that this is sufficient for $\lim_{k\to\infty} S_Q(A_k) = S^*$.

Given the subsequence $\{A_{k'}\}_{k'=1}^{\infty}$, define threshold types c_x and v_x as the lowest cost and highest valuation such that traders with these types do not trade with certainty in the limit, i.e.

$$c_x \equiv \inf\left\{c, 1|\bar{Q}^S(c) < 1\right\} \quad \text{and} \quad v_x \equiv \sup\left\{v, 0|\bar{Q}^B(v) < 1\right\}.$$

First, we show that the no rent extraction conditions implies

$$\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \qquad \text{for all } c,$$

and

$$\bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x) \qquad \text{for all } v.$$

So the payoffs to all types can be bounded from below once we know the payoffs of the cut-off types. The first inequality follows directly for all types $c \in [c_x, 1]$ by the no rent extraction condition, observing that $(c_x - c)$ is negative. For types $[0, c_x]$, the inequality is trivially true if $c_x = 0$; if $c_x > 0$, choose some $\varepsilon \in (0, c_x)$ and note that $\bar{Q}^S(c_x - \varepsilon) = 1$ by definition of c_x and by monotonicity of $\bar{Q}(\cdot)$ (which is implied by monotonicity of each element $Q_{k'}$) Hence, for all $c \leq c_x - \varepsilon$, the no rent extraction condition implies that $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) - \varepsilon$. By continuity of $\bar{V}(\cdot)$, (see the statements following the no rent extraction condition), and by ε being arbitrary, we get $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$. So the first inequality holds for all $c \in [0, 1]$. The second inequality follows for all buyers by symmetric reasoning.

Adding the two inequalities yields a lower bound on the joint surplus of all c and v:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + V^B(v_x) - (v_x - c_x). \qquad (14)$$

We use the availability and the weak efficiency conditions to show that the right hand side is at least $(v - c)$:

We consider two cases for the ordering of c_x and v_x. First, suppose $c_x < v_x$. Take some $\varepsilon \in (0, v_x - c_x)$. By definition of c_x and v_x, and by monotonicity of $\bar{Q}^S(\cdot)$ and

$\bar{Q}^B(\cdot)$, we have $\bar{Q}^S(c_x + 0.5\varepsilon) < 1$ and $\bar{Q}^B(v_x - 0.5\varepsilon) < 1$. The availability condition implies that $\bar{L}^S(c_x + \varepsilon) = \bar{L}^B(v_x - \varepsilon) = 1$. By the weak efficiency condition:

$$\bar{V}^S(c_x + \varepsilon) + \bar{V}^B(v_x - \varepsilon) \geq v_x - c_x - 2\varepsilon.$$

By continuity of $\bar{V}^S(\cdot) + \bar{V}^B(\cdot)$ and by ε being arbitrary:

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

Now consider the case $v_x \leq c_x$. Since $(v_x - c_x)$ is non-positive and payoffs are non-negative, we immediately get that

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

So for both possible orderings of c_x and v_x the sum of the last four terms in (14) is positive. Hence, payoffs are pairwise efficient, i.e. for all v and for all c:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c.$$

According to lemma 3, pairwise efficiency is a sufficient condition for the subsequence to become efficient since payoffs V_k^S and V_k^B are monotone. Therefore $\lim_{k\to\infty} S(A_{k'}) = S^*$ along the subsequence. Hence, by the necessity part of lemma 2, limiting trading probabilities must be Walrasian, i.e. $\left(\bar{Q}^S, \bar{Q}^B\right)$ must be in Q^W.

Because the choice of the subsequence was arbitrary, this implies that the limit of every convergent subsequence is in Q^W. Because $\hat{A}$ is sequentially compact, this implies $\lim_{k\to\infty}\left(Q_k^S, Q_k^B\right) = Q^W$ for the original sequence.[15] By the sufficiency part of lemma 2, if the limit of the sequence is in Q^W, then the sequence converges to efficiency and $\lim_{k\to\infty} S(A_k) = S^*$, as claimed.

[15] In a sequentially compact space, if all convergent subsequences of some sequence have a common limit, then the sequence itself converges to that limit, see lemma 6 in the appendix for the general argument.

5 Application of the Main Result

5.1 Basic Model

Take a decreasing sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \to 0$. By proposition (4) in Lauermann (2006a), for every k there exists an equilibrium σ_k of the basic model. Fixing one equilibrium for each k yields a sequence $\{\sigma_k^*\}_{k=0}^{\infty}$. With every equilibrium σ_k^* associate an outcome A_k, using the mapping $A(\cdot,\cdot) : \Sigma \times [0,1] \to \Sigma_M^4$ defined in the natural way: Q_k^S is given by $Q^S(c|\sigma_k, \delta_k) \equiv q^S(p_k(c)|\delta_k, \sigma_k)$ and the other components of $A(\cdot,\cdot)$ are defined similarly. By propositions (2) and (3) in Lauermann (2006a), this outcome satisfies *mass balance*. By substituting Q_k^B into the steady state conditions we can write them entirely in terms of A_k:

$$\Phi^B(v|\delta_k, A_k) = \int_0^v \frac{1 - Q_k^B(\tau) + \delta Q_k^B(\tau)}{M_k \delta_k} dG^B(\tau), \tag{15}$$

where M_k can be derived from $\Phi^B(1|\delta_k, A_k) = 1$. With $\Phi^S(c|\delta_k, A_k)$ defined similarly, let us define L^S and L^B

$$L^B(v|\delta_k, A_k) = \frac{1 - \Phi^B(v|\delta_k, A_k)}{1 - \Phi^B(v|\delta_k, A_k)(1-\delta_k)} \tag{16}$$

$$\text{and } L^S(c|\delta_k, A_k) = \frac{\Phi^S(c|\delta_k, A_k)}{1 - (1 - \Phi^S(c|\delta_k, A_k))(1-\delta_k)}. \tag{17}$$

Now we check the conditions.

Monotonicity. For $Q_k^S(\cdot)$: Suppose the function is not monotone decreasing for some k and for some $c_H > c_L$, $Q_k^S(c_H) \equiv Q_H > Q_L \equiv Q_k^S(c_L)$. Then with $p_L \equiv p_k(c_L)$ and $p_H \equiv p_k(c_H)$ optimality requires $U^S(p_H, c_H|\sigma_k, \delta_k) \geq U^S(p_L, c_H|\sigma_k, \delta_k)$, which is equivalent to

$$Q_H(p_H - c_H) \geq Q_L(p_L - c_H),$$

and this implies that for costs $c_L < c_H$

$$Q_H(p_H - c_L) > Q_L(p_L - c_L),$$

and thus, $U^S(p_H, c_L|\sigma_k, \delta_k) > U^S(p_L, c_L|\sigma_k, \delta_k)$, contradicting optimality of $p_L \equiv p_k(c_L)$ for c_L. Similar reasoning holds for $Q_k^S(\cdot)$.

No Rent Extraction. For $V_k^S(\cdot)$: Again, we use a direct implication of optimality:

$$V_k^S(c) - V_k^S(c_x) \geq U^S(p_k(c_x), c|\sigma_k, \delta_k) - U^S(p_k(c_x), c_x|\sigma_k, \delta_k),$$

which implies that for all c the condition holds, since by definition of $U^S(\cdot, \cdot|\sigma_k)$ the above inequality is equivalent to

$$V_k^S(c) \geq V_k^S(c_x) + q^S(p_k(c_x)|\delta_k, \tilde{\sigma}_k^*)(c_x - c), \tag{18}$$

and similarly for V_k^B.

Availability. For $\left\{L_k^B\right\}_{k=1}^{\infty} \equiv \left\{L^B(\cdot|\delta_k, A_k)\right\}_{k=1}^{\infty}$. Evaluating the steady state condition (15) at $\Phi_k^B(1|\delta_k, A_k)$ shows $M_k\delta_k \leq 1$ Choosing any $v' < v$ we get as lower bound on $1 - \Phi^B(v'|\delta_k, A_k)$:

$$1 - \Phi^B\left(v'|\delta_k, A_k\right) \geq \int_{v'}^{v} \left[1 - Q_k^B(\tau)\right] dG^B(\tau).$$

From monotonicity of Q_k^B, $Q_k^B(v') \leq Q_k^B(v)$ for all $v' < v$. By assumption, $dG^B(\tau)$ is continuous and strictly positive, so there is some $g_L > 0$ such that $dG^B(v) \geq g_l \;\; \forall v$. Together:

$$1 - \Phi^B\left(v'|\delta_k, A_k\right) \geq \left(1 - Q_k^B(v)\right)\left(v - v'\right) g_l,$$

and so for all sequences $Q_k^B(\cdot)$ with $\bar{Q}_k^B(v) < 1$:

$$\lim_{k\to\infty} \inf L_k^B\left(v'\right) \geq \frac{\left(1 - \bar{Q}_k^B(v)\right)(v - v')\, g_l}{1 - \left(1 - \left(1 - Q_k^B(v)\right)(v - v')\, g_l\right)} = 1.$$

and similarly for $\left\{L_k^S\right\}_{k=1}^{\infty}$.

Weak Efficiency: Suppose for some c_x and v_x, $\bar{L}^S(c_x) = \bar{L}^B(v_x) = 1$. By the no rent extraction condition, $V_k^B(\cdot)$ is increasing with a slope in $[0,1]$. Thus, $r_k(v)$ is increasing by $r_k(v) = v - (1 - \delta_k) V_k(v)$. Therefore, the set of types accepting a price $p = r_k(v_x)$ is at least the set $[v_x, 1]$. Therefore, the trading probability $D(r_k(v_x)|\sigma_k)$ is at least $1 - \Phi^B(v_x)$. By definition of L_k^B and q^S:

$$q_k^S(r_k(v_x)|\delta_k, \tilde{\sigma}_k^*) \geq L_k^B(v_x)$$

and therefore

$$\begin{aligned} U_k^S\left(r_k\left(v_x\right), c_x|\delta_k, \tilde{\sigma}_k^*\right) &\geq L_k^B\left(v_x\right)\left(r_k\left(v_x\right)-c_x\right) \\ &= L_k^B\left(v_x\right)\left(v_x-\left(1-\delta_k\right) V_k^B\left(v_x\right)-c_x\right), \end{aligned}$$

where the last line follows from the equilibrium condition for $r_k(v_x)$. By $V_k^S(c_x) \geq U_k^S(r_k(v_x), c_x|\delta_k, \tilde{\sigma}_k^*)$ for all k:

$$\begin{aligned} \lim_{k\to\infty} \inf V_k^S\left(c_x\right) &\geq \lim_{k\to\infty} \inf L_k^B\left(v_x\right)\left(v_x-\left(1-\delta_k\right) V_k^B\left(v_x\right)-c_x\right) \\ &= \left(v_x-\bar{V}_k^B\left(v_x\right)-c_x\right), \end{aligned}$$

which implies $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$ (whenever they exist) as claimed.

So $\{A_k\}_{k=1}^{\infty}$ satisfies our four condition and:

Corollary 1 *For every exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \to 0$ and for every sequence of associated equilibrium outcomes $\{A_k\}_{k=1}^{\infty}$ of the basic model:*

$$\lim_{k\to\infty} S\left(A_k\right) = S^*.$$

5.2 Symmetric Information and Intermediate Bargaining Power

We change the basic model by assuming that traders in each match observe each others' types and that both of them, the buyer and the seller, have a chance to propose a price offer. Let $\alpha \in (0,1)$ be the probability that the seller is chosen to propose a price and let $(1-\alpha)$ be the probability that the buyer is chosen. This is similar to the analysis in Gale (1987) but he considers a discrete set of types instead of a continuum (with discrete types efficiency is harder to get, see 7).

Strategies now account for the role of the trader and for the type of the opponent. Let Σ_{M^2} be the set of measurable functions $f : [0,1]^2 \to [0,1]$. Strategies are $\left[p^S, p^B, r^S, r^B\right] \in \Sigma_{M^2}^2 \times \Sigma_M^2$ where $p^B(v,c)$ is the price proposed by a buyer of type v to a seller of type c and $r^S(c)$ is the reservation price of a seller c. $p^S(c,v)$ and $r^B(v)$ are the corresponding proposals and reservation prices of sellers and buyers. A market constellation is a vector $\sigma_F \in \Sigma_F$ with $\Sigma_F \equiv \Sigma_{M^2}^2 \times \Sigma_M^4 \times \mathbb{R}_+$ and with a typical element $\sigma_F = \left[p^S, p^B, r^S, r^B, \Phi^S, \Phi^B, M\right]$.

Let $P^S\left(p^S, c|\sigma_F\right)$ be the probability that a seller who is chosen as a proposer trades in a given period when using $p^S = p^S(\cdot,\cdot)$

$$P^S\left(p^S, c|\sigma_F\right) \equiv \int_{v|r^B(v)\geq p^S(c,v)} d\Phi^B(v),$$

and let $R^S\left(r^S, c|\sigma_F\right)$ be the probability that the seller trades when chosen to respond:

$$R^S\left(r^S, c|\sigma_F\right) \equiv \int_{v|p^B(v,c)\geq r^S(c)} d\Phi^B(v),$$

then the per period probability of trading is $D^S\left(p^S, r^S, c|\sigma_F\right) = \alpha P^S\left(p^S, c|\sigma_F\right) + (1-\alpha) R^S\left(r^S, c|\sigma_F\right)$. Let $E^{RS}\left[p|p \geq r^S(c), c, \sigma_F\right]$ be the expected price conditional on trade when responding and $E^{PS}\left[p|p \leq r^B(v), \sigma_F\right]$ be the expected price conditional on trade when proposing. Expected payoffs are implicitly defined via

$$\begin{aligned} U^S\left(p^S, r^S, c|\sigma_F\right) &= \alpha P^S\left(p^S, c|\sigma_F\right)\left(E^{PS}[p] - c\right) + (1-\alpha) R^S\left(r^S, c|\sigma_F\right)\left(E^{RS}[p] - c\right) \\ &\quad + (1-\delta)\left(1 - D^S\left(p^S, r^S, c|\sigma_F\right)\right) U^S\left(p^S, r^S, c\right), \qquad (19) \end{aligned}$$

with $E^{PS}[p] = E^{PS}\left[p|p \leq r^B(v), \sigma_F\right]$ and $E^{RS}[p] = E^{RS}\left[p|p \geq r^S(c), c, \sigma_F\right]$. Let $U^{PS}\left(p, v|p^S, r^S, c, \sigma_F\right)$ be the payoff when matched with a type v, proposing p and continuing according to $\left(p^S, r^S\right)$:

$$U^{PS}\left(p, v, c|p^S, r^S, \sigma_F\right) = \begin{cases} p - c & \text{if } p \leq r^B(v) \\ (1-\delta) U^S\left(p^S, r^S, c|\sigma_F\right) & \text{otherwise} \end{cases}.$$

We define the corresponding functions for buyers analogously.

The steady state conditions do not change. They are

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M\left(D^S\left(p^S, r^S, \tau|\sigma_F\right) + \delta\left(1 - D^S\left(p^S, r^S, \tau|\sigma_F\right)\right)\right)} \quad \forall c \qquad (20)$$

$$\text{and } \Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M\left(D^B\left(p^B, r^B, \tau|\sigma_F\right) + \delta\left(1 - D^B\left(p^B, r^B, \tau|\sigma_F\right)\right)\right)} \quad \forall v \qquad (21)$$

We define an equilibrium, with x and y denoting types of traders. We require that the price offer by the proposer must be optimal for every possible type of the responder and we require that the reservation price has the same properties as derived in the basic model. These requirements incorporate the idea of "sequential rationality":

Definition 3 *A steady state equilibrium vector with full information,* $\sigma_F^* \in \Sigma$ *consist of an optimal pair of strategies and a corresponding steady state pool such that*

1. $\left(p^j, r^j\right) \in \arg\max U^j\left(\cdot,\cdot,x|\sigma_F\right) \quad \forall x$ *and* $j \in \{B,S\}$

2. $p^j\left(x\right) \in \arg\max U^{Pj}\left(\cdot,x,y|p^j,r^j,\sigma_F\right)$ $\forall x,y$ *and* $j \in \{B,S\}$

3. $r^B\left(v\right) = v - \left(1-\delta\right)U^B\left(p^B,r^B,v|\sigma_F\right)$ *and* $r^S\left(c\right) = \left(1-\delta\right)U^S\left(p^S,r^S,c|\sigma_F\right) + c$ $\forall v,\, c$

4. $\Phi^S\left(\cdot\right)$, $\Phi^B\left(\cdot\right)$, M *satisfy the steady state conditions (20), (21).*

We show that payoffs can be rewritten in a very compact way. Firstly, the optimal price offer of a buyer v to a seller of type c is clearly never strictly above $r^S\left(c\right)$ but either equal to the reservation price or equal to some unacceptable price below, $p < r^S\left(c\right)$. Hence, the expected price offer to the seller conditional upon acceptance is $E^{RS}\left[p|p \geq r^S\left(c\right),c,\sigma_F\right] = r^S\left(c\right)$ and similarly for buyers. This implies in particular that a responder is indifferent between accepting and rejecting an offer. Therefore, expected payoffs do not change if a trader plans to simply reject all offers. Thus, payoffs depend only on the price offers a trader makes when being a proposer himself. To derive this payoff let q^{PS} be the lifetime trading probability when trading only as a proposer with, using the offer strategy p^S, where $q^{PS}\left(\cdot,\cdot|\cdot\right)$ is the solution to

$$q^{PS}\left(p^S,c|\sigma_F\right) = \alpha P^S\left(p^S\right) + \left(1-\delta\right)\left(1-\alpha P^S\left(p^S\right)\right)q^{PS}\left(p^S,r^S,c|\sigma_F\right).$$

Rewriting the payoff definition (19) using q^{PS} and $E^{RS}\left[p|p \geq r^S\left(c\right),c,\sigma_F\right] = r^S\left(c\right)$ yields

$$U^S\left(p^S,r^S,c|\sigma_F\right) = q^{PS}\left(p^S,c|\sigma_F\right)\left(E^{PS}\left[p|p \leq r^B\left(v\right),\sigma_F\right] - c\right) \tag{22}$$

and similarly for buyers

$$U^B\left(p^B,r^B,v|\sigma_F\right) = q^{PB}\left(p^B,c|\sigma_F\right)\left(v - E^{PB}\left[p|p \geq r^S\left(c\right),\sigma_F\right]\right). \tag{23}$$

Now take a sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \to 0$ as before and assume that for every δ_k there is some equilibrium. Let this be σ_{Fk} which gives us a sequence $\{\sigma_{Fk}\}_{k=1}^{\infty}$. Let A_k be the outcome of equilibrium σ_{Fk} with $A_k = A\left(\sigma_k^F,\delta_k\right)$ defined in the natural way. We check only the no rent Extraction condition, because the other conditions are immediate. For this, let $U^S\left(p_k^S\left(\cdot,\cdot\right),1,c|\sigma_{Fk},\delta_k\right)$ be the payoff to a seller of type

c when offering a price $p_k^S(\cdot, c)$ when chosen to propose while rejecting any price offer when chosen to respond. From (22):

$$V_k^S(c) = U^S\left(p_k^S(\cdot,\cdot), 1, c|\sigma_{Fk}, \delta_k\right)$$

and from optimality

$$\begin{aligned} V_k^S(c_x) - V_k^S(c) &\geq U^S\left(p_k^S(\cdot, c), 1, c_x|\sigma_{Fk}, \delta_k\right) - U^S\left(p_k^S(\cdot, c), 1, c|\sigma_{Fk}, \delta_k\right) \\ &\geq q^{PS}\left(p_k^S(\cdot,\cdot), c|\sigma_F\right)(c - c_x). \end{aligned} \tag{24}$$

and together with symmetric reasoning for buyers the first parts of the condition holds. For the limiting part, we show that if the lifetime trading probability Q_k^S converges to one, then $q^{PS}\left(p_k^S(\cdot,\cdot), c|\sigma_F\right)$ converges to one as well. Therefore, (24) implies that whenever $Q_k^S \to 1$, we get $V_k^S(c_x) - V_k^S(c) \geq (c - c_x)$. For details, see the appendix.

Now the other conditions follow and we sketch the idea: Given the no rent extraction condition, payoffs $V^S(\cdot)$ and $V^B(\cdot)$ are monotone. From the equilibrium conditions it follows that two traders v_x and c_x who are matched trade if and only their joint trading surplus $v_x - c_x$ is larger than their joint continuation payoff $(1-\delta)\left[V^S(c_x) + V^B(v_x)\right]$. Together with $V^B(\cdot)$ being increasing at a rate smaller than one - from the no rent extraction condition - a buyer with a higher valuation trades with a larger set of sellers and hence, the trading probability $Q_k^B(\cdot)$ is monotone increasing in v. Analogous reasoning implies the same for sellers. Weak pairwise efficiency is a direct implication of the above observation. Finally, availability follows by the same reasoning as in the basic model, because we are using exactly the same matching technology. Hence:

Corollary 2 *For every sequence of exit rates* $\{\delta_k\}_{k=1}^{\infty}$ *and equilibrium outcomes* $\{A_k\}_{k=1}^{\infty}$ *of the full information model with intermediate bargaining power* $\alpha \in (0,1)$

$$\lim_{k\to\infty} S(A_k) = S^*.$$

Remark: The crucial step for proving convergence with symmetric information is the following observation: Although it is true that a trader of type c does not need to *receive* the same offers as a type c_x, he can *make* the same offers when chosen to be the proposer. Even more so: as we have seen in (22) and (22), payoffs depend *only* on the offers made as a proposer. Therefore, a seller of type c can mimic the strategy of another type c_x in much the same way as a seller in our basic model can mimic the pricing strategy of another seller. In Lauermann (2006b) I look at the case with symmetric information where buyers are never chosen to be the proposer. There, convergence to efficiency does not hold.

5.3 Further Applications

Variants of the basic model that can be analyzed as before include one time entry and second price auctions with reservation prices. In a model with one time entry, time is running from $t = 0$ to infinity. In period zero, a unit mass of buyers and a unit mass of sellers arrive with types distributed according to distribution functions G^S and G^B with the same properties as in the basic model. There is no further inflow in the subsequent periods; Thus, the pool in $t \geq 1$ consists only of those who did not trade before and who did not die before. So the pool depletes over time.[16] Otherwise, assume that information is asymmetric and sellers make price offers. We can characterize outcomes just as we did with steady state models by considering the ex ante trading probabilities and payoffs to the entering traders in the first period. Their joint expected surplus is the natural welfare criterion. Clearly, mass balance should hold with respect to the ex ante outcome. With Q_0^S, Q_0^B denoting the first period expected lifetime trading probabilities, $Q_0^S, Q_0^B \in Q^W$ is a necessary and sufficient condition for efficiency, with Q^W as defined in (12). For this model one can show that our conditions hold: By asymmetric information, traders can mimic each other. Just as in the basic model, this implies that trading probabilities are monotone and ex ante payoffs have a bounded slope. For the availability condition, note that if the ex ante trading probability of some buyers is not one, then these buyers will stay for many periods in the market. One can show that this implies that a seller will become certain to be matched with them some time, i.e. availability holds. Finally, weak efficiency holds by similar reasoning as in the basic model. Thus, our main result applies even to non-steady state markets and the outcome will become efficient with δ converging to 0.

We can include auctions in the basic model as follows: Suppose matches consist of one seller and a random number of buyers and suppose the seller conducts a second price auction: Upon observing the number of buyers in his group, he announces a reservation price p. Then the buyers submit their bids. Restricting attention to equilibria in dominant strategies, these bids are equal to the reservation prices derived before. This allows a simple characterization of the equilibrium. Suppose we keep the basic model otherwise, that is, we keep the assumption that there is an exogeneous inflow and that

[16] To avoid the problem of having no traders left in the market after some period, Moreno and Wooders assume that only a share $\beta \in (0, 1)$ of all traders in the market is actually matched in every period.

there is some death rate δ. Our conditions hold in this model as well: Monotonicity and no rent extraction follow from asymmetric information, and availability and weak efficiency follow by similar reasoning as in the basic model. Therefore, if sellers can use auctions to sell their goods, with vanishing δ the outcome becomes efficient.

6 Extensions

6.1 Including an Entry Stage

Suppose we include an entry stage into the basic model, i.e. suppose that new traders must decide whether they want to enter the pool or not. If they enter the pool, they must pay some entry costs $e \in (0,1)$. Let $Z^S(\cdot) : [0,1] \to \{0,1\}$, $Z^S(\cdot) \in \Sigma_M$ denote the entry decision, with $Z^S(c) = 1$ indicating the decision of type c, v to become active. Let $V^S(\cdot)$ denote the expected payoffs to a seller if he enters, gross of e. ($V^S(\cdot)$ is also calculated for those who do not actually become active). Let $Z^B(\cdot)$ and $V^B(\cdot)$ be the corresponding functions for buyers.

We assume that sellers enter whenever this is profitable, i.e. $Z^S(c) = 1$ whenever $V^S(c) \geq e$ and we assume that $Z^S(c) = 0$ otherwise. For buyers we assume the same , $Z^B(v) = 1$ whenever $V^B(v) \geq e$. Let c_0^e be the highest type of a seller such that entry is profitable, $c_0^e \equiv \sup\left\{c, 0 | V^S(c) \geq e\right\}$ and let v_0^e be the lowest type of a buyer such that entry is profitable, $v_0^e \equiv \inf\left\{v, 1 | V^B(v) \geq e\right\}$. By this definition, types $c > c_0^e$ or $v < v_0^e$ do not enter.[17]

Given the entry stage, the matching technology of the basic model has to be changed to account for the possibility that the two sides of the market might not be identical. But no matter how this is done, types who do not enter are not available. Therefore, the probability to match some set of buyers might be zero even though the lifetime trading probability of these types is strictly below one. One can show that this failure of availability leads to a failure of convergence to efficiency in the basic model, see section 7. Therefore, stronger forces towards efficiency are needed. In the models by Gale and by Satterthwaite and Shneyerov these forces come from curtailing the bargaining power of the seller. Formally, these models satisfy a stronger condition than condition 4 (weak efficiency). Sequences of trading outcomes that satisfy this stronger condition converge to efficiency even if they satisfy only a weaker availability condition, due to the entry stage.

[17] If payoffs are monotone, all $c < c_0$ and all $v > v_0$ enter.

An outcome A^E of a model with an entry stage is a vector $\left[V^S, V^B, Q^S, Q^B, Z^S, Z^B\right] \in \Sigma_M^6$. Surplus conditional on $\left(Q^S, Q^B, Z^S, Z^B\right)$ and gross of entry costs is

$$S_Q^E\left(A^E\right) = \int_0^1 vZQ^B\left(v\right)dG^B\left(v\right) - \int_0^1 cZQ^S\left(c\right)dG^S\left(c\right).$$

with $ZB^B\left(v\right) \equiv Z^B\left(v\right)Q^B\left(v\right)$ and $ZQ^S\left(c\right) \equiv Z^S\left(c\right)Q^S\left(c\right)$. These latter functions are the *effective* trading probabilities and we work with them throughout this section. Mass balance with entry is satisfied if the expected transfers collectively received by all entering sellers are equal to the transfers collectively made by all buyers. Equivalently, the mass of entering seller who trade must be equal to the mass of entering buyers who trade:

Definition 4 ***Mass Balance with Entry***. *An outcome A^E satisfies balance of transfers and trades if*

$$S^E\left(A^E\right) = \int_0^1 Z^S\left(\tau\right)V^S\left(\tau\right)dG^S\left(\tau\right) + \int_0^1 Z^B\left(\tau\right)V^B\left(\tau\right)dG^B\left(\tau\right) = S_Q^E\left(A^E\right) \quad (25)$$

and if

$$\int_0^1 ZQ^S\left(c\right)dG^S\left(c\right) = \int_0^1 ZQ^B\left(v\right)dG^B\left(v\right). \quad (26)$$

We say that an outcome A^E is *Walrasian* if the effective trading probabilities are in Q^W, i.e. if $\left(ZQ^S, ZQ^B\right) \in Q^W$. By reasoning analogously to the lemmas without entry, an outcome is efficient if and only if it is Walrasian:

Lemma 5 *For all outcomes that satisfy mass balance with entry, $S^E\left(A^E\right) = S^*$ if and only if $ZQ \in Q^W$. For every sequence $\left\{A_k^E\right\}_{k=1}^\infty$ such that A_k^E satisfies mass balance with entry and such that $ZQ \in \Sigma_- \times \Sigma_+$:*

$$\lim_{k\to\infty} S_Q^E\left(A_k^E\right) = S^* \quad \textit{if and only if} \quad \lim_{k\to\infty}\left(ZQ_k\right) = Q^W.$$

Take any sequence of outcomes $\left\{A_k^E\right\}_{k=1}^\infty$ and a connected sequence of frictions $\{\delta_k, e_k\}_{k=1}^\infty$ with $(\delta_k, e_k) \to (0,0)$. If the limit of the effective trading probabilities exists we denote them by $\overline{ZQ}^S$ and $\overline{ZQ}^B$ and if the limit of the cut-off types $c_0^{e_k}$ and $v_0^{e_k}$ exists, we denote them by c_0 and v_0. Now we restate the conditions: The monotonicity condition becomes a condition on effective trading probabilities:

Condition 5 ***Monotonicity with Entry***. *For every member* A_k^E*,*

$$\overline{ZQ}_k^S \in \Sigma_- \quad \textit{and} \quad \overline{ZQ}_k^B \in \Sigma_+.$$

The no rent extraction condition remains unchanged. But as said in the introduction, we weaken availability and we assume that it holds only for types $c \leq c_0$ and $v \geq v_0$. With $L_E^S : [0,1]^2 \times \Sigma_M^6 \to [0,1]$ defined analogously to L^S without entry and similarly for L_E^B:

Condition 6 ***Weak Availability***. *If* $\overline{ZQ}^S(c')$ *and* c_0 *exist and if* $\overline{ZQ}^S(c') < 1$ *for some* $c' < c_0$*, then*

$$\bar{L}_E^S(c) = 1 \quad \textit{for all } c \in (c', c_0).$$

If $\overline{ZQ}^B(v')$ *and* v_0 *exist and if* $\overline{ZQ}^B(v') < 1$ *for some* $v' > v_0$*, then*

$$\bar{L}_E^S(v) = 1 \quad \textit{for all } v \in (v_0, v').$$

We strengthen weak pairwise efficiency by requiring availability only on one side of the market. But it has to hold only for pairs involving either v_0 or c_0. As we will see, the limiting payoff of these cut-off types are zero and then $\bar{V}^S(c') + \bar{V}^B(v_0) \geq v_0 - c'$ implies $\bar{V}^S(c') \geq v_0 - c'$. The following condition is formulated such that it is met by the models of Satterthwaite and Shneyerov and by the model of Gale:

Condition 7 ***Strong Pairwise Efficiency***. *If* $\bar{L}^S(c') = 1$ *for some* c' *and if* v_0 *exists then*

$$\bar{V}^S(c') \geq v_0 - c'.$$

If $\bar{L}^B(v') = 1$ *for some* v' *and if* c_0 *exists then*

$$\bar{V}^B(v') \geq v - c_0.$$

Remark: In the basic model, the first part of this condition does not hold. As discussed in detail in section **??** if all sellers offer a common price p^N strictly above p^w, they are rationed and do not trade with certainty, i.e. $\overline{ZQ}^S(c') < 1$ and $\bar{L}^S(c') = 1$ for $c' \leq v_0 = p^N$. Nevertheless, they have no incentive to decrease their offers if $Z^B(v) = 0$ for all $v < p^N$. Hence, payoffs become $\bar{V}^S(c') = \overline{ZQ}^S(c')(p^N - c') < p^N - c'$.

Note, that for all models with entry there is an equilibrium in which no trader enters. If a sequence of outcomes includes such outcomes as subsequence, its limit cannot become efficient. Hence, we restrict attention to *non-trivial sequences*, where entry does not vanish along any subsequence, i.e.

$$\lim_{k\to\infty}\sup v_0^{e_k} < 1 \quad \text{and} \quad \lim_{k\to\infty}\inf c_o^{e_k} > 0.$$

Under the stronger efficiency condition we can:

Proposition 2 *Suppose some non-trivial sequence* $\left\{A_k^E\right\}_{k=1}^{\infty}$ *satisfies mass balance and monotonicity with entry, no rent extraction, weak availability and strong pairwise efficiency for some pair of functions* L^B *and* L^S *and for some sequence* $\{\delta_k\}_{k=1}^{\infty}$ *and* $\{e_k\}_{k=1}^{\infty}$ *with* $e_k \to 0$. *Then the outcome becomes efficient, i.e.*

$$\lim_{k\to\infty} S^E\left(A_k^E\right) = S^*.$$

As before, we take some convergent subsequence of outcomes and denote the limit by $\left(\bar{V}^S, \bar{V}^S, \overline{ZQ}^S, \overline{ZQ}^B, \bar{Q}^S, \bar{Q}^B\right)$. Let v_x be the lowest valuation and c_x the highest cost that does not trade for sure in the limit:

$$v_x = \sup\left\{v, 0|\overline{ZQ}^B(v) < 1\right\} \qquad \text{and} \quad c_x = \inf\left\{c, 1|\overline{ZQ}^S(c) < 1\right\},$$

and we take a further subsubsequence indexed by k' such that the cutoffs $v_0^{e_{k'}}$ and $c_0^{e_{k'}}$ converge to some v_0 and c_0. Now we want to show that $(ZQ_{k''}) \to Q^W$. We will argue at the end of the proof that this implies $S\left(A_k^E\right) \to S^*$ for the sequence itself.

The no rent extraction condition remained unchanged. Noting that $\lim Q_{k'}^S(c) = 1$ whenever $\overline{ZQ}^S(c) = 1$ and symmetrically, $\lim Q_{k'}^B(v) = 1$ whenever $\overline{ZQ}^S(v) = 1$, the no rent extraction condition implies just as before:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + V^B(v_x) - (v_x - c_x). \tag{27}$$

and we want to derive again a lower bound on the joint payoff of c_x and v_x by showing that $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$:

If $v_x \leq c_x$, we are done by payoffs being non-negative. So suppose $v_x > c_x$. We consider three subcases for the relation between c_x, c_0, v_0, v_x. Subcase 1 is $c_x < c_0 < v_0 < v_x$. Then, by definition of c_x, $\overline{ZQ}^S(c_x + 0.5\varepsilon) < 1$ for $\varepsilon \in (0, \min\{c_0 - c_x; v_x - v_0\})$.

Thus, $\bar{L}^S(c_x+\varepsilon)=1$ by weak availability and, by symmetric reasoning, $\bar{L}^B(v_x-\varepsilon)=1$. Therefore, strong efficiency implies

$$\bar{V}^S(c_x+\varepsilon)\geq v_0-c_x-\varepsilon \qquad \text{and} \qquad \bar{V}^B(v_x-\varepsilon)\geq v_x-c_0-\varepsilon.$$

Together with continuity of payoffs, ε being arbitrary, and $v_0\geq c_0$ we get

$$\bar{V}^S(c_x)+\ \bar{V}^B(v_x)\geq v_x-c_x+v_0-c_0\geq v_x-c_x$$

as claimed.

Subcase 2a: $c_x=c_0$ and $v_0<v_x$. By definition of v_x $\overline{ZQ}^B(v_x-0.5\varepsilon)<1$, for $\varepsilon\in(0,v_x-v_0)$. So by availability we have $\bar{L}^B(v_x-\varepsilon)=1$. Hence, strong efficiency implies

$$\bar{V}^B(v_x-\varepsilon)\geq v_x-c_0-\varepsilon$$

and by continuity of payoffs and $c_x=c_0$ this implies $\bar{V}^B(v_x)\geq v_x-c_x$. Hence, by $\bar{V}^S(c_x)\geq 0$ we get the desired inequality $\bar{V}^B(v_x)+\bar{V}^S(c_x)\geq v_x-c_x$. Subcase 2b: $c_x<c_0$ and $v_0=v_x$. By analogous reasoning: $\bar{V}^B(v_x)+\bar{V}^S(c_x)\geq v_x-c_x$.

Subcase 3: $c_x=c_0$ and $v_0=v_x$. Note first, that marginal types must make zero profits in the limit: If $\lim_{k\to\infty}\sup V_k^S(c_o^{e_k})>0$ then for some ε small enough, $\lim_{k\to\infty}\sup V_k^S(c_o^{e_k}+\varepsilon)>0$ by (Lipschitz-) continuity of payoffs. This contradicts the definition of the marginal type. Hence $\lim_{k\to\infty}V_k^S(c_o^{e_k})=0$ and $\lim_{k\to\infty}V_k^B(v_o^{e_k})=0$,by symmetry. With this observation, we show that this subcase leads to a contradiction: If $c_x=c_0$ and $v_0=v_x$, with $c_x<v_x$, then the mass of sellers who trade becomes

$$\lim_{k'\to\infty}\int_0^1 ZQ_{k'}^S(c)\,dG^S(c)=\int_0^{c_x}dG^S(c)=G^S(c_x)$$

and similarly the mass of buyers who trade becomes

$$\lim_{k'\to\infty}\int_0^1 ZQ_{k'}^B(v)\,dG^B(v)=\int_{v_x}^1 dG^B(v)=1-G^B(c_x)$$

By mass balance of total trades (26), it must be that the mass of entering sellers becomes equal to the mass of entering buyers and $G^S(c_x)=1-\left(G^B(v_x)\right)$. Furthermore, by $\bar{Q}^S(c)=1$ for all $c<c_x=c_0$, no-rent extraction implies

$$\bar{V}^S(c_0)\geq\bar{V}^S(c)+(c-c_0)$$

and thus $\bar{V}^S(c) \leq \bar{V}^S(c_0) + (c_0 - c)$ for $c < c_0$. From before, $\bar{V}^S(c_0) = 0$ and so we get $\bar{V}^S(c) \leq c_0 - c$. By symmetric reasoning, $\bar{V}^B(v) \leq v - v_0$. We use this to get an upper bound on the limit of $S\left(A^E_{k'}\right)$:

$$\begin{aligned} \lim_{k' \to \infty} \inf S^E\left(A^E_{k'}\right) &\leq \int_0^{c_x} (c_0 - c)\, dG^S(c) + \int_{v_x}^1 (v - v_0)\, dG^B(v) \\ &\leq \int_{v_x}^1 v dG^B(v) - \int_0^{c_x} c dG^S(c) - G^S(c_x)(v_x - c_x) \\ &< \int_{v_x}^1 v dG^B(v) - \int_0^{c_x} c dG^S(c) = \lim_{k' \to \infty} S^E_Q\left(A^E_{k'}\right), \end{aligned}$$

where we used that $G^S(c_x)$ is equal to $1 - \left(G^B(v_x)\right)$ for the second line and the hypothesis of the subcase, $(v_x - c_x) > 0$, for the third line. By $S^E_Q\left(A^E_{k'}\right)$ having a limit different from $S^E\left(A^E_{k'}\right)$, the mass balance identity (25), $S^E\left(A^E_{k'}\right) = S^E_Q\left(Q^S(\cdot), Q^B(\cdot)\right)$ is violated for k' large enough. Hence, this subcase is impossible since by choice of the (sub-)sequence $\left\{A^E_{k'}\right\}$ each of its elements does satisfy mass balance.

Hence, $\bar{V}^B(v_x) + \bar{V}^S(c_x) \geq v_x - c_x$ in all possible cases. Thus, inequality (27) implies that limiting payoffs are pairwise efficient for all types c and v:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c\,. \tag{28}$$

By reasoning analogously to the second part of the main result, this implies

$$\lim_{k \to \infty} S^E\left(A^E_k\right) = S^*.$$

6.2 Infinitely Lived Traders

In the literature, it is a common assumption that traders are infinitely lived and that they discount future payoffs, see e.g. Gale (1987). We will show in this section how to apply our approach. In particular, we will have to extend the description of outcomes to account for the difference between discounted and undiscounted transfers and similarly we have to account for the difference between discounted and undiscounted trading probabilities. These differences have an important implication for the resource constraint which we discuss below.

With infinitely lived agents every trader who enters the market must ultimately trade for otherwise a steady state with a finite pool is impossible. This makes the inclusion of an entry stage necessary. As before, let $Z^j(\cdot) \in \{0, 1\}$ denote the entry decision, with

$Z^S(c) = 1$ and $Z^B(v) = 1$ indicating the decision of types c and v to become active. Let $T^S_\infty(\cdot)$ be a measurable function mapping $[0,1]$ into $\mathbb{R}_+$, where $T^S_\infty(c)$ denotes the undiscounted expected payments received by a seller of type c. Similarly, let $T^B_\infty(v)$ denote the undiscounted payment made by a buyer of type v. The undiscounted trading probabilities are $Q^S_\infty(\cdot) \in \Sigma_M$ and $Q^B_\infty(\cdot) \in \Sigma_M$. For individual *discounted* payoffs, we denote by $Q^S_\beta(\cdot) \in \Sigma_M$ and $Q^B_\beta(\cdot) \in \Sigma_M$ the discounted probabilities of trading and by $T^S_\beta(\cdot) \in \Sigma_M$ and $T^B_\beta(\cdot) \in \Sigma_M$ the discounted transfers received and made, where β indicates the use of a discount factor. Payoffs are given by

$$V^S(c) = T^S_\beta(c) - c\,Q^S_\beta(c) \quad \text{and} \quad V^B(v) = vQ^B_\beta(v) - T^B_\beta(v). \tag{29}$$

An outcome is given by $A^\infty = \left[V^S, V^B, Q^S_\infty, Q^B_\infty, T^S_\infty, T^B_\infty, Z^S, Z^B, Q^S_\beta, Q^B_\beta\right]$ and surplus is

$$S(A^\infty) = \int_0^1 V^S(c)\,dG^S(c) + \int_0^1 V^B(v)\,dG^B(v).$$

The mass balance condition becomes

Condition 8 ***Mass Balance with Infinitely Lived Players****. An outcome A^∞ satisfies mass balance if*

$$\int_0^1 Z^S(c)\,Q^S_\infty(c)\,dG^S(c) = \int_0^1 Z^B(v)\,Q^B_\infty(v)\,dG^B(v) \tag{30}$$

$$\text{and} \quad \int_0^1 Z^S(c)\,T^S_\infty(c)\,dG^S(c) = \int_0^1 Z^B(v)\,T^B_\infty(v)\,dG^B(v). \tag{31}$$

It is simple to derive an expected surplus maximizing allocation subject to mass balance: All sellers and all buyers enter while transfers are zero. Discounted trading probabilities are one for buyers and zero for sellers and then

$$S^{\max} = \int_0^1 vG^B(v) > S^*.$$

Here, we use that discounted trading probabilities and undiscounted trading probabilities can be different, i.e. we can have $0 \le Q^S_\beta \le Q^S_\infty$. By mass balance, we need Q^S_∞ to be large so that as many buyers as possible can trade while we need Q^S_β to be small so that costs become small. Because there is no bound on the difference between these two, we can set $Q^S_\infty(c) = 1$ while at the same time $Q^S_\beta(c) = 0$ for all c. Intuitively, with time running from minus to plus infinity, sellers trade infinitely many periods after their entry.

By $S^{\max} > S^*$, we cannot use S^* as defined in (13) as an upper bound on the surplus as we do in the proof of our convergence result. Therefore we impose some additional constraints on the outcome which are present in the existing models as well. Firstly, we assume that transfers are made only through prices:

Condition 9 ***Prices only***. *There are functions $p^S(\cdot) \in \Sigma_M$ and $p^B(\cdot) \in \Sigma_M$ such that for all c and for all v:*

$$\begin{aligned} T^S_\beta(c) &= Q^S_\beta(c)\, p^S(c) \qquad \textit{and} \qquad T^B_\beta(v) = Q^B_\beta(v)\, p^B(v) \\ T^S_\infty(c) &= p^S(c) \qquad\qquad\quad \textit{and} \qquad T^B_\infty(v) = p^B(v) \end{aligned} \tag{32}$$

Secondly, we require that whenever the trading probability of an entering seller of type c is one, his expected transfer must cover at least his costs, i.e. we require that $T^S_\infty \geq c$ whenever $Z^S(c)\, Q^S_\infty(c) = 1$. By $Q^S_\infty(c) = 1$ if $Z^S(c) = 1$ and stating the same for buyers, this implies that we want outcomes to satisfy

Condition 10 ***Individual Rationality***. *An outcome is individually rational if*

$$\begin{aligned} \forall c \text{ st. } Z^S(c) &= 1\text{:} \quad T^S(c) \geq c, \\ \forall v \text{ st. } Z^B(v) &= 1\text{:} \quad T^B(v) \leq v. \end{aligned}$$

Let $\hat{A}^{IR}$ be the set of outcomes which satisfy mass balance, prices only, and individual rationality. Together with the definition of payoffs, (29), surplus for any $A \in \hat{A}^{IR}$ is given by

$$S^\infty(A) = \int_0^1 Z^S(c)\, Q^S_\beta(c) \left(p^S(c) - c\right) dG^S(c) + \int_0^1 Z^B(v)\, Q^B_\beta(v) \left(v - p^B(v)\right) dG^B(v)\,.$$

Now we want to show that S^* as defined in (13) is the constrained maximum. First, note that the terms in the integral are positive, i.e. for all c such that $Z^S(c) = 1$, we have $\left(p^S(c) - c\right) \geq 0$ and likewise for buyers. Hence, in order to maximize $Z^\infty(\cdot)$, all entering traders must trade immediately, i.e. it has to hold that for almost all c: $Q^S_\beta(c) = 1$ if $Z^S(c) = 1$ similarly for buyers. In addition, mass balance (31) now requires $\int_0^1 Z^S(c)\, p^S(c) = \int_0^1 Z^B(v)\, p^B(v)$. Together with the observation on trading probabilities, a necessary condition for A' to be in $\arg\max_{A \in \hat{A}^{IR}} S^\infty(\cdot)$ is that

$$S^\infty\left(A'\right) = \int_0^1 v\, Z^B(v)\, dG^B(v) - \int_0^1 c\, Z^S(c)\, dG^S(c)\,,$$

and now the problem of maximizing $S^{\infty}(\cdot)$ is similar to our earlier problem. Indeed, let Q^{EW} be the set of "Walrasian" outcomes,

$$A^{EW} \equiv \left\{ A \mid \int_0^1 \left| Q^S_\beta(c) Z^S(c) - 1_{c \leq p^w}(c) \right| dc, \int_0^1 \left| Q^B_\beta(v) Z^B(v) - 1_{v \geq p^w}(v) \right| dv = 0 \right\},$$

then by reasoning analogously to lemma 1, Q^{EW} is the set of the maximizers of the surplus

$$A^{EW} \equiv \arg \max_{A \in \hat{A}^{IR}} S^{\infty}(\cdot).$$

In the models by Gale (1987) and Satterthwaite and Shneyerov (2006) traders are restricted to use prices. In addition, trade is voluntary so that no seller would accept to trade at a price below costs and no buyer would accept to trade at a price above his valuation. Thus, the set of equilibrium outcomes is a subset of $\hat{A}^{IR}$ and our approach is valid.

7 Failures

In this section we demonstrate how to use our approach in order to understand why convergence to efficiency fails in some specifications of dynamic matching and bargaining games. In the first subsection, we discuss who the failure of convergence with symmetric information can be atributed to the failure of the no rent extraction condition. In the following section, we discuss how the simulteneity of decisions in double auctions can lead to the failure of weak efficiency. Finally, we discuss a model with cloning and show that the mass balance conditions does not hold in this case.

We do not provide a specification in which monotonicity is the only condition to fail since there is not such model known in the literature. The failure of availability with an entry stage is discussed at the end of the second subsection and interpreted as a coordination failure when traders have to decide simultaneously on whether to enter the market.

7.1 No Rent Extraction fails with Full Information and Asymmetric Bargaining Power

Suppose sellers in the basic model can observe the valuation of the buyer prior to making an offer. Clearly, this makes trading *within* each pair efficient: They trade whenever their trading surplus $(v - c)$ is larger than their joint continuation payoff

$(1-\delta)\left(V^S(c)+V^B(v)\right)$. But overall efficiency of trading in the market as a whole decreases: with $\delta \to 0$, the trading outcome does no longer become efficient. Here, we want to show which of our conditions is violated to understand why convergence to efficiency fails. A full discussion of the model can be found in the note by Lauermann (2006b).[18]

Note, that this is essentially the set-up of section 5.2: There, traders in each pair can mutually observe their valuations and costs. With probability α the seller is chosen to be the proposer of a price offer, while with probability $(1-\alpha)$ the buyer is chosen. While in section 5.2 we assume that α must be interior, $\alpha \in (0,1)$, here we assume that the seller has all the bargaining power, i.e. $\alpha = 1$. Let $\left\{A_k^F\right\}_{k=1}^{\infty}$ be a sequence of equilbrium outcomes of the model of section 5.2 with α set equal to 1. We can characterize the outcomes by two observations: Firstly, sellers appropriate all the trading surplus: no buyer receives strictly positive payoffs and $V^B(v) \equiv 0$. The price offer to a buyer is either equal to his type or too high to be acceptable, i.e. $p^S(c,v) \geq v$ for all c, v.[19] Secondly, the limiting outcome is as follows: there is some cut-off $\bar{v} \in (0,1)$ such that a seller of type $c = 0$ trades only with types $v > \bar{v}$. The limiting lifetime trading probabilities of buyers is zero if $v < \bar{v}$ and one if $v > \bar{v}$, i.e. $\bar{Q}^B = 1_{v>\bar{v}}$.[20]

While the sequence $\left\{A_k^F\right\}_{k=1}^{\infty}$ can be shown to satisfy monotonicity, availability, and weak efficiency,[21] the no rent extraction condition fails: Since $\bar{Q}^B(v_x) = 1$ for any $v_x > \bar{v}$, the condition would require that payoffs increase with a slope of one, i.e. for types $v' > v_x$, it must be that $\bar{V}^B(v') \geq \bar{V}^B(v_x) + (v' - v_x) > 0$. However, the payoff to any type v' is still zero and his rent $(v' - v_x)$ is *extracted*: part of this rent will go to the sellers but part of it is "wasted".[22] Because of this, the equilibrium outcome is not efficient in the limit.

[18] In Lauermann (2006b) sellers have homogeneous costs $c \equiv 0$ to ease exposition. Here, sellers are heterogeneous to keep the consistency of the underlying economy across specifications.

[19] Prices offers are always larger than or equal to reservation prices as argued in section 5.2, i.e. $p^S(c,v) \geq r^B(v)$. By definition, $v - r^B(v) = (1-\delta)V^B(v)$ and by $V^B(v) \leq v - r^B(v)$, $v - r^B(v) = 0$.

[20] Suppose not. Because trading probabities $\bar{Q}^B(\cdot)$ can be shown to be monotone, this would imply that for some interval (a,b), $\bar{Q}^B(v) \in (0,1)$ for all $v \in (0,1)$ ($\bar{Q}^B(v) \in \{0,1\}\, \forall v$ is never an equilibrium outcome). Then, for any $v' \in (a,b)$, types $v \geq v'$ are *available* and a seller $c = 0$ who trades only with $v \geq v'$ at prices $p^S(0,v) = \max\{v', v\}$ would trade with certainty and receive a payoff $\lim_{k\to\infty} U^S\left(0, p^S\right) \geq v' > a$. Contradiction!

[21] Weak efficiency is immediate with symmetric information, availability holds because the matching technology is unchanged to the case $\alpha \in (0,1)$ and monotonicity holds essentially because sellers profits satisfy the strict single crossing condition, i.e. sellers with lower costs prefer to trade with a higher probability at a lower price.

[22] See Lauermann (2006b) for details: Rents are wasted by rationing among sellers (i.e. some sellers with $c < p^w$ do not trade with certainty although they should) and by inefficient sellers trading (i.e. some sellers with $c > p^w$ who should not trade, can do so).

At the end of section 5.2 we discuss why with intermediate bargaining power, i.e. with $\alpha \in (0,1)$, the no rent extraction condition holds. Essentially, buyers can mimick each other's offer strategies. Another possibility to restore the no rent extraction property in the case of $\alpha = 1$ is to assume that the buyer's valuation is not perfectly observable (i.e. buyers have some "privacy"): the appendix of lauermann (2006b) contains an extension in which sellers receive only a signal about the valuation of the buyer and this signal contains noise. Although this signal can be arbitraily precise, with $\delta \to 0$ buyers can patiently wait until their type is misconceived as being very low so that they receive a low price offer. In particular, if some buyer v_x becomes certain to trade at some expected price $p \leq v_x$ in the limit, then any type $v' > v_x$ can wait until he receives the same offers and trade at an expected price $p \leq v_x$. The payoff to v' is therefore at least $(v' - v_x)$ larger than the payoff to v_x. Thus, the no rent extraction condition holds and the outcome becomes efficient in the limit.

Remark 1 *Prices with symmetric information are "monopolistic", i.e.* $p \geq v$*, by the same reasoning as in Diamond (1971): Sellers can use the waiting costs* $\delta \in (0,1)$ *to "hold-up" buyers. However, in the models that are used to derive the familiar Diamond paradox, this outcome is still efficient because buyers and sellers are assumed to be homogeneous.*[23] *Here, inefficiencies firstly stem from the fact that sellers rather incure rationing than trading at low prices with low valuation buyers and secondly they stem from the possibility of trading for sellers who have costs above* p^w *and who should not.*

7.2 Weak Efficiency fails without Sequential Rationality

Serrano (2002) is the first to specify the bargaining protocol as simultaneous double auction.[24] He shows that equilibrium outcomes do not need to becomes efficient. Without going into the details, we can replicate his result in our framework: Suppose we assume in the basic model that the buyer and the seller *simultaneously* announce a reservation price r and price offer p, respectively. Trade happens at the price p whenever the reservation price is below the price offer. If we leave the rest of the model unchanged, the following is an equilibrium for every δ_k: $p(c) \equiv 1$ and $r(v) \equiv 0$. In the equilibrium outcome, trading probabilities are zero for all types and $S(A_k) = 0$ for all k.

[23] In the original model, individual buyers have elastic demand for multiple units while sellers are restricted to linear prices. This causes inefficiencies which vanish once sellers can make a price-quantity offer.

[24] His interest, however, stems from the prior use of simultaneous auctions in dynamic matching and bargaining games with *common values*, see e.g. Wolinsky (1990).

While the sequence of outcomes satisfy monotonicity, no-rent extraction, and availability, weak efficiency fails: For any $v > c$, the trading surplus $(v - c)$ is strictly larger than the limiting payoffs, $\liminf \left(V^S(c) + V^B(v)\right)$ which are 0. Bargaining is inefficient because of miscoordination between the traders. As observed by Serrano, this failure happens because we cannot use sequential rationality to rule out such equlibria.[25]

Note the similarity to the failure of convergence with an entry stage: Setting a price above the highest valuation (and setting a reservation price below the lowest cost) is similar to the decision not to become an active trader. And just as it is a best response not to take an interior bargaining position if no other trader does so, it is a best response not to become active if not other trader does. But note also that just as we can restore sequential rationality by introducing trembles to the price setting decisions we can restore equilibria with trading when traders tremble at the entry decision stage.[26]

7.3 Mass Balance fails with Cloning

Cloning refers to the assumption that every trader who leaves the market is replaced by an exact copy of his type, a *clone*. With this assumption, the inflow depends on the trading outcome and is *endogeneous*. The pool of traders, however, does not change over time and is exogeneous. Models with cloning have been used by Rubinstein and Wolinsky (1985), by Gale (1987)[27], and recently by De Fraja and Sakovics (2001). With cloning, equilibrium outcomes might fail to converge to efficiency.[28] We want to understand why.

To understand the negative results we use the symmetric information model of section 5.2.[29] To recall the model: All traders from the pool are matched into pairs. In each pair they observe each others' valuation v and cost c. Then, with probability $\alpha \in (0, 1)$ the sellers is chosen to be the *proposer* of a price while with probability $(1 - \alpha)$ the buyer is chose to be the proposer. The other trader, the *responder*, can either accept or reject the offer. Afterwards, all those pairs in which the responder accepts the offer, trade and

[25] In our basic model, sequential rationality enter via the assumption that buyers use a reservation price which is equal to the continuation payoff.

[26] See Gale (1987, p30) who argues that equilibria without entry are not stable.

[27] Gale considers several specifcations of the inflow process, including one-time entry (in section 5 of his paper), cloning (in the first part of section 6) and exogenous inflows (in the second part of section 6). For his critique of the cloning model, see footnote 7.3.

[28] With efficiency defined as in section 3.2. It is not clear, whether this is the appropriate concept of efficiency, see also footnote 7.3.

[29] The main differences are the following: Rubinstein and Wolinsky assume that $c \equiv 0$ and $v \equiv 1$. Gale (1987) and DeFraja and Sakovics (2001) include an entry stage and have discounting instead of an exit rate. DeFraja and Sakovics in addition use a noisy search technology and assume that a buyer is matched with a random number of sellers. None of these differences affects the main conclusions.

leave the pool. An additional share δ of those who did not trade leaves (dies). Now the new traders enter. But different from the model in section 5.2, the inflow consists of exact clones of the leaving traders. Therefore, independent of who actually traded, the distribution of traders in the pool at the end of the period is always equal to the distribution in the beginning. Let these distributions be $G^S(\cdot)$ and $G^B(\cdot)$.

For every δ_k, we fix an equilibrium outcome $A_k^C = \left[V^S, V^B, Q^S, Q^B\right]$ of the cloning variant. Since we are using exactly the same matching and bargaining technology as in section 5.2, our four conditions still hold. Therefore, from the first part of the proof of the main proposition 1, we know that the outcome must become pairwise efficient for every convergent subsequence, i.e.

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c.$$

Actually, the limiting outcome can be fully characterized quite easily: It is standard to verify[30] that there is a price $p^N(\alpha)$ such that for $\delta_k \to 0$ limiting payoffs become $\bar{V}^S(c) = \max\left\{p^N(\alpha) - c, 0\right\}$ and $\bar{V}^B(v) = \left\{v - p^N(\alpha), 0\right\}$. Limiting trading probabilities become $\bar{Q}^S(c) = 1_{c<p^w}$ and $\bar{Q}^B(v) = 1_{v>p^w}$. The price $p^N(\alpha)$ itself is given as the unique solution to the following condition:

$$(1-\alpha)\int_0^{p^N(\alpha)} \left(p^N(\alpha) - c\right) dG^S(c) = \alpha \int_{p^N(\alpha)}^1 \left(v - p^N(\alpha)\right) dG^B(v).$$

Note, that for $\alpha = \frac{1}{2}$, the price $p^N\left(\frac{1}{2}\right)$ equates the expected surplus of buyers and sellers.

The price $p^N(\alpha)$ depends on the distribution of bargaining power and it is strictly increasing in α, i.e. the larger the bargaining power of sellers the higher the price. Thus, generically, the limiting outcome fails to be Walrasian: Only for a single point α^* it happens that $p^N(\alpha^*) = p^w$, while for all $\alpha \in (0,1)\setminus\alpha^*$, $p^N(\alpha) \neq p^w$ and hence $\left(\bar{Q}^S, \bar{Q}^B\right) \notin Q^W$.[31] To illustrate the failure we look at the extreme case with $\alpha \to 0$ when buyers enjoy all the bargaining power. In this case, the condition requires that the price must become zero, $\lim_{\alpha\to 0} p^N(\alpha) = 0$. (Note that we take the limit of outcomes with respect to $\delta \to 0$ first and with $\alpha \to 0$ afterwards). Thus, expected payoffs among buyers become approximately $\int_0^1 v dG^B(v)$ while expected payoffs to sellers become zero.

[30] Using the techniques by Gale (1987), see for example the teaching notes by Wright, http://www.ssc.upenn.edu/~rwright/courses/rw.pdf .

[31] DeFraja and Sakovics interpret this and similar results at length as indicating the overwhelming importance of "local market conditions", reflected here in the distribution of bargaining power.

Hence, $\lim_{\alpha\to 0}\lim_{\delta_k\to 0} S\left(A_k^C\right) = \int_0^1 v dG^B(v) - 0$ - which is strictly larger than S^*. How can this happen?

Note that in this limiting outcome almost all buyers trade with certainty while almost no seller trades: In the limit we have

$$\begin{aligned}\int_0^1 \bar{Q}^S(c)\, dG^S(c) &= 0 \\ &< 1 = \int_0^1 \bar{Q}^B(v)\, dG^B(v).\end{aligned}$$

Therefore, for some δ_k and α small enough, the mass balance condition (11) is violated.

Why is it possible with cloning that all buyers can trade? Note that for any $\alpha \in (0,1)$ in a given period, the probability for a buyer to find a sellers who accepts to trade at $p^N(\alpha)$ is $G^S\left(p^N(\alpha)\right) > 0$. So with $\delta \to 0$, a buyer becomes certain to be matched with a seller who accepts to trade. In the original model, without cloning, this is not true: If all trade happens at a price $p^N(\alpha)$ close to zero, sellers with costs $c \leq p^N(\alpha)$ would become scarce and the share of such sellers in the pool becomes zero.[32]

Thus, it is only the mass balance condition that is violated in this model. In particular, the generic failure of convergence to the Walrasian outcome is due *only* to the assumption of cloning. It does not relate to the specification of the matching and bargaining protocol.

DeFraja and Sakovics (2001) introduce cloning explicitly as a *technical* assumption.[33] Since this assumption has such a strong implication for the results and since they do not claim that cloning is meant to reflect economic conditions, one might try to find means other than cloning to solve possible technical problems. Nonetheless, it would clearly be interesting to find economic environments that are well modeled by the cloning assumption. In such an environment, the analysis by DeFraja and Sakovics (2001) suggests that the Walrasian prediction might be questionable.[34]

[32] As argued in Lauermann (2006a), the *lifetime* trading probabiltiy $q^S\left(p^N(\alpha)\right)$ would become $G^S\left(p^N(\alpha)\right)$ in the limit.

[33] DeFraja and Sakovics have infinitely lived traders and assume an entry stage (see section 6.2 of our paper). They write that they assume cloning to ensure stationarity of the mass of traders who decide not enter the pool (see p. 846). They do not explicitly state any problem that would arise otherwise. Actually, in Gale (1987, section 6) and Satterthwaite and Shneyerov (2006), the mass of non-entering types is not stationary without causing problems.

[34] But even in such environments, Gale (1987) argues more generally that it is not appropriate to define the Walrasian outcome with respect to the distributions in the pool, i.e. with respect to $G^S(\cdot)$ and $G^B(\cdot)$. Instead, one should define the outcome with respect to the inflow, see Gale (1987, p23).

Note that in the model by Rubinstein and Wolinsky (1985) no inefficiencies arise: Sellers have homogeneous costs $c = 0$ and buyers have homogeneous valuations $v = 1$. Independent of the trading price, all surplus is realized and the outcome is efficient. However, the trading price might not be the Walrasian price.

8 Conclusion

We have introduced a new approach to the analysis of decentralized market with vanishing frictions. By directly characterizing sequences of trading outcomes independent of the fine details of the trading institution, we have shown which conditions imply convergence to efficiency across different models. We then validated this approach by showing that sequences of equilibrium outcomes for models in the literature satisfy these conditions.

Our framework of characterizing outcomes independent of fine details is also suggestive to the analysis away from the limit, showing how to make robust predictions from the very structure of decentralized trading. In addition to these theoretical contributions, our framework can provide guidance to empirical research in evaluating the prediction of market clearance and its robustness by clarifying the underlying assumptions. In particular, when bargaining power is very unevenly distributed and the stronger side has very good information, the outcome might be far away from the efficient one even for small frictions.

Several open questions remain. First, one would like to be able to have a uniform bound on the efficiency loss. Are there general conditions which guarantee across models that the realized surplus is within ε of the efficient surplus for δ small enough? Secondly, we assumed that p^w is known ex ante. In many markets, however, traders are uncertain with respect to demand and supply and p^w is a random variable. Can we expect decentralized market to converge to efficiency even if traders have to learn the state of the market? Finally, can decentralized trading solve the problem of coordinating economic activity across markets for different goods? Dynamic matching and bargaining games seem to be a promising tool for the analysis of these specific problems of decentralized decision taking.

A Appendix

A.1 Proof of Convergence of Sequences

The sets of monotone functions Σ_+ and Σ_- are sequentially compact and satisfy the conditions of the following lemma (see footnote 3.1). The proof of the first part is standard:

Lemma 6 *Let (X, τ) be a sequentially compact topological space. Suppose there is some sequence $\{x_n\} = x_1, x_2, ...$ in X and some $y \in X$ such that every convergent subsequence converges to y. Then*

$$x_n \to y$$

Similarly, suppose there is some subset $Y \subset X$ such that every convergent subsequence of $\{x_n\}$ converges to some $y \in Y$ (possibly y is different for each subsequence). Then for every neighborhood $G \supset Y$ there is some $N(G)$, st. $x_n \in G$ for all $n \geq N(G)$, and we say $x_n \to Y$.

Proof: Suppose not. Then by the definition of convergence there is some neighborhood G of y that does not contain all elements of the sequence from some index onwards, i.e. for every N there is some $n'(N) \geq N$ such that $x_{n'(N)} \notin G$. This allows the construction of a subsequence $\{x_{n'}\}$ such that $x_{n'} \notin G$ for all n'. By X being sequentially compact, there is some convergent subsubsequence of $\{x_{n'}\}$. By the hypothesis of the lemma, this subsubsequence converges to y. This contradicts $x_{n'} \notin G$ for all n'. The second statement follows similarly: Suppose not, then there would be some neighborhood $G \supset Y$ and some subsequence $\{x_{n'}\}$ such that $x_{n'} \notin G$ for all n'. Again, we can find a convergent subsubsequence by X being sequentially compact which implies a contradiction to the definition of $\{x_{n'}\}$ ■

A.2 Proof of Lemma 2

The "if" part follows directly from continuity of $S_Q(\cdot)$ and from lemma 1. For the only if part, recall that we say $\lim_{k\to\infty} Q_k = Q^W$ if $d(Q_k, Q') \to 0$ for all $Q' \in Q^W$. By Helly's selection theorem, every sequence of monotone functions has a convergent subsequence. Take such a subsequence and let $\bar{Q}$ denote its limit. Lebesgue's bounded convergence theorem implies that $S_Q(\bar{Q}) = S^*$. Therefore $\bar{Q} \in Q^W$ from lemma 1. Hence, every convergent subsequence has its limit in Q^W and thus the sequence itself converges to Q^W, see lemma 6 ■

A.3 Proof of Lemma 3

Suppose the limiting statement does not hold. By the Bolzano-Weierstrass theorem, this implies that there is some $\varepsilon > 0$ and some subsequence indexed by k', such that $S(A_{k'})$ converges and $\lim_{k'\to\infty} S(A_{k'}) \leq S^* - \varepsilon$. Take some subsubsequence indexed by k'' such that $V^S_{k''}$, $V^B_{k''}$ converge pointwise. Such a subsubsequence exists by $\left(V^S_{k''}, V^B_{k''}\right) \in \Sigma_+ \times \Sigma_-$ and Helly's selection theorem. Let $\bar{p}$ be defined as before, $\bar{p} \equiv \inf_{c \leq p^w} \left(\bar{V}^S(c) + c\right)$. Along the subsubsequence $\lim_{k''\to\infty} V^S_{k''}(c) \geq \bar{V}^S(c) \geq \bar{p} - c$ for all $c \leq p^w$ and similarly, $\lim_{k''\to\infty} \inf V^B_{k''}(v) \geq v - \bar{p}$ for all $v \geq p^w$ by the condition of the lemma. Hence

$$\lim_{k''\to\infty} \inf S(A_{k''}) \geq \int_{p^w}^{1} (v - \bar{p})\, dG^B(v) + \int_0^{p^w} (\bar{p} - c)\, dG^S(c) = S^*,$$

where the last equality follows the observation in the first part of the proof. This is a contradiction to the starting hypothesis $\lim_{k'\to\infty} S(A_{k'}) \leq S^* - \varepsilon$ ■

A.4 Proof of Lemma 1

Let

$$\begin{aligned} S_M\left(M^T\right) &\equiv \max_{Q\in\hat{Q}} S_Q(\cdot) \\ & \quad st.\ M^T = \int_0^1 Q^S(c)\, dG^S(c) = \int_0^1 Q^B(v)\, dG^B(v) \qquad (33) \end{aligned}$$

and note that

$$\max_{Q\in\hat{Q}} S_Q(\cdot) = \max_{M^T\in[0,1]} S_M(\cdot).$$

Let $p^S\left(M^T\right)$ be such that $G^S\left(p^S\right) = M^T$ and $p^B\left(M^T\right)$ be such that $1 - G^B\left(p^B\right) = M^T$. Then clearly

$$\begin{aligned} Q(M) &\equiv \arg\max_{Q\in\hat{Q}} S_Q(\cdot) \quad \text{st. } (33) \\ &= \left\{ Q \in \hat{Q} | \int_0^1 \left| Q^S(c) - 1_{c\leq p^S(M)}(c) \right| dc = 0,\ \int_0^1 \left| Q^B(v) - 1_{v\geq p^B(M)}(v) \right| dv = 0 \right\} \end{aligned}$$

and thus

$$S_M\left(M^T\right) = \int_{p^B(M^T)}^1 v dG^B(v)\, dv - \int_0^{p^S(M^T)} c dG^S(v)\, dv.$$

Note that $S_M(\cdot)$ is continuously differentiable in M^T and the second derivative of $S_M\left(M^T\right)$ is $-\left(\frac{1}{dG^B(p^B(M^T))}+\frac{1}{dG^S(p^S(M^T))}\right)$ so that surplus is strictly concave in M^T. Therefore, a necessary and sufficient condition for $M^* \in \arg\max S_M(\cdot)$ is that the first derivative is zero:

$$p^B(M^*) - p^S(M^*) = 0, \tag{34}$$

which implies that the cutoffs must be the market clearing price p^w: By definition of M^T, $M^T = G^S\left(p^S\left(M^T\right)\right) = 1 - G^B\left(p^B\left(M^T\right)\right)$. This is true at $p^B(M^*) = p^S(M^*)$ only for $p^B(M^*) = p^S(M^*) = p^w$. Thus:

$$Q^W = Q\left(G^s\left(p^w\right)\right) = \arg\max_{Q\in\hat{Q}} S_Q(\cdot) \qquad \blacksquare$$

A.5 Proof of $q^{PS} \to 1$

We want to show that $\lim_{k\to\infty} Q_k^S(c) = 1$ implies $\lim_{k\to\infty} q^{PS}\left(p_k^S(\cdot,c), c|\sigma_F, \delta_k\right) = 1$. Note that $V_k^S(\cdot)$ is decreasing and $V_k^B(\cdot)$ is increasing. This implies that reservation prices $r_k^S(\cdot)$ and $r_k^B(\cdot)$ are monotone. Furthermore, if $r_k^B(v) = p_k^S(c,v)$ then $r_k^S(c) = p_k^B(v,c)$ because $p_k^S(c,v) = r_k^B(v)$ if and only if the continuation payoff is below the reservation price, i.e. if and only if

$$\begin{aligned}(1-\delta_k)\, U^S\left(p_k^S, r_k^S, c|\sigma_{kF}\right) &\leq r_k^B(v) - c \\ &= v - (1-\delta)\, U^B\left(p^B, r^B, v|\sigma_F\right) - c\end{aligned}$$

and hence $v - r_k^S(c) \geq (1-\delta)\, U^B\left(p^B, r^B, v|\sigma_F\right)$. Therefore, the probability to trade is independent of whether being a proposer or being a responder. So $D^S\left(p_k^S, r_k^S, c|\sigma_{Fk}, \delta_k\right) = P^S\left(p_k^S, r_k^S, c|\sigma_{Fk}, \delta_k\right)$. This implies that if $\lim_{k\to\infty} Q_k^S(c) = 1$ then $\lim_{k\to\infty} q^{PS}\left(p_k^S(\cdot,c), c|\sigma_F, \delta_k\right) = 1$: With $P_k^S\left(p_k^S\right) \equiv P^S\left(p_k^S, r_k^S, c|\sigma_{Fk}, \delta_k\right)$

$$\lim_{k\to\infty} Q_k^S(c) = \lim_{k\to\infty} \frac{P_k^S\left(p_k^S\right)}{1-(1-\delta_k)\left(1-P_k^S\left(p_k^S\right)\right)} = 1,$$

implies $\lim_{k\to\infty} \delta_k \left[P_k^S\left(p_k^S\right)\right]^{-1} = 0$ and therefore $\lim_{k\to\infty} \delta_k \left[\alpha P_k^S\left(p_k^S\right)\right]^{-1} = 0$. Thus $q^{PS}\left(p_k^S(\cdot,c), c|\sigma_F, \delta_k\right) \to 1$ by

$$q^{PS}\left(p_k^S(\cdot,c), c|\sigma_F, \delta_k\right) = \frac{\alpha P_k^S\left(p_k^S\right)}{1-(1-\delta_k)\left(1-\alpha P_k^S\left(p_k^S\right)\right)} \qquad \blacksquare$$

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