On The Essentiality of Money

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Abstract

We explore the extent to which money improves the efficiency of resource allocations in an economy with decentralized trade and no commitment with respect to future actions. We assume that all agents, with the exception of a subset with measure $\varepsilon > 0$, are completely anonymous and that all actions in meetings involving these agents are private. We obtain that, for any $\varepsilon > 0$, as long as agents are sufficiently patient, money is not required to achieve desirable allocations. We then modify our environment by assuming that a subset of agents with measure $\delta > 0$ has private information about their preferences. We obtain that, for all $\varepsilon \geq 0$, and all $\delta > 0$, money is essential to achieve desirable allocations. These results suggest that, from an efficiency point of view, private information about fundamentals plays a key role in the unveiling of the reasons for the essentiality of money.

1 Introduction

In this article we explore the extent to which money improves the efficiency of resource allocations in an economy where trade is difficult. Our benchmark is a search model of money with random and decentralized trade. In these models, exchange induces desirable allocations but agents cannot precommit to future actions: any current production must be compensated by some future consumption. We assume that almost all agents are completely anonymous and that all actions in meetings involving these agents are observed only by the agents that directly participate in such meetings. A small part of the population, instead, is formed by public agents, that is, agents whose meetings can be observed by everyone.[1]

[1] The assumption that some but not all agents have public histories was first introduced in search models of money by Cavalcanti and Wallace (1999a,b). Cavalcanti and Wallace (1999a) show that inside money issued by agents with public histories (banks) improves the allocation of resources when outside money is

We provide conditions under which, for any measure of public agents, money is not required to achieve desirable allocations if agents are sufficiently patient. We show that there exists a social norm of gift-exchange that achieves the first best.[2] The intuition for this result runs as follows. Consider an arrangement where an agent always produces if called upon to do so, as long as he has always observed the same behavior by other agents in the past. If, instead, he observes a deviation in a meeting, he stops producing from that period on. Under this arrangement, if an agent does not produce in a private meeting, he triggers a non-cooperative behavior that eventually leads to a collective punishment. This punishment occurs when some agent that was directly or indirectly "infected" by the initial deviation meets with a public agent, at which point everyone in the economy observe that a deviation has occurred and stop producing.

Kocherlakota (1998) and Kocherlakota and Wallace (1998) argue that money constitutes a form of imperfect memory in environments where trade decisions cannot be publicly observed. Our result suggests that, if agents are patient, a small amount of public information achieves desirable allocations without the need for money.[3] This does not imply that money cannot play a relevant role. However, it does suggest that this role may go beyond being a memory of past transactions. We explore this argument further by focusing on a distinct feature of our environment, namely the assumption that there exists common knowledge of preferences. The idea that money plays a relevant role under private information is not new. Alchian (1977) studied the role of money in an economy where agents do not have complete knowledge of one another's characteristics. Banerjee and Maskin (1996) study the emergence of media of exchange when agents cannot distinguish between the qualities of goods. In search models of money, the role of private information was explored by Williamson and Wright (1994), Trejos (1999), and Berentsen and Rocheteau (2004). Consistent with these works, we show that money is essential to achieve desirable allocations in the presence of private information.

An explicit modelling of the frictions that make money essential is a prevailing requisite within the so-called fundamental models of money, and of search models of money in particular (Wallace (2001)). In short, if money is not essential and there is an alternative

scarce. Cavalcanti and Wallace (1999b) show that the set of implementable outcomes using outside money is a strict subset of the set using inside money.

[2] The reference paper on social norms is Kandori (1992). See also Ellison (1994) and Okuno-Fujiwara and Postlewaite (1995).

[3] Kocherlakota and Wallace (1998) show that, for any given discount factor, money is essential when past trade decisions are publicly recorded with a lag, as long as this lag is sufficiently large. In their analysis, the size of the lag is a function of the discount factor.

mechanism that can sustain better allocations, there is no reason for money to circulate. We think that a natural complement to this requirement is that essentiality should be preserved under small departures from assumptions that are deemed to be of relevance, i.e., assumptions that are used to explain the essentiality of money. For instance, if one builds a model with a continuum of agents and argues that the size of the population is relevant for money to be essential, it is desirable that money preserves its essentiality under small departures from the continuum. Based on this reasoning, our results suggest that private information about fundamentals plays a key role in the unveiling of the reasons for the essentiality of money.

The paper proceeds as follows. In the next section, we describe the environment. In section 3, we discuss strategies and the equilibrium concept. In section 4, we provide the conditions under which there exists a social norm that achieves the first best. In section 5 we show that money is essential if there is private information about preferences. Section 6 concludes. An Appendix collects some proofs and results that are omitted from the main text.

2 Environment

Our environment is a modified version of Trejos and Wright (1995) and Cavalcanti and Wallace (1999a,b). Consider an economy with a unit continuum of agents and a continuum of goods. Utility in every period is given by $u(x) - y$ where x is the amount consumed and y is the amount produced. The function u is defined on $[0, \infty)$ and $u(.)$ is increasing, twice differentiable, $u(0) = 0$, $u'' < 0$ and $u'(0) = \infty$. The efficient amount of production is given by y^*, where $u'(y^*) = 1$. Agents maximize expected utility with a discount factor $\beta \in (0, 1)$. In the economy there is also a storable, indivisible and intrinsically useless object, which we denote as money. We assume that an agent can hold at most one unit of money at a time.

A key feature of the environment is that trade is decentralized and agents face frictions in the exchange process. We formalize this idea by assuming that agents meet anonymously and randomly in pairs. In each pairing, there is a probability σ that one agent will be assigned as a producer and the other agent will be assigned as a consumer, and a probability σ that these roles will be reversed. Finally, there is a probability $1 - 2\sigma$ of a no coincidence of wants.

The economy comprises two different types of agents: public and private agents. The measure of public agents is equal to $\varepsilon > 0$, where ε is small. There is a record keeping device that allows everyone in the economy to observe the actions in all meetings in which public agents participate (public meetings). This device updates information with a lag of size $k \in \mathbb{N}$. Precisely, at the end of every period t, it records the information on all public meetings up to period $k\tau$, where τ is the highest integer such that $k\tau \leq t$.

3 Strategies and Equilibrium

Our environment is an example of an extensive game with imperfect information. In every period t, the history of an agent (say, agent i) includes the past actions of meetings in which agent i participated, and the information released up to that point on the past actions in public meetings. In what follows, we simplify the description of actions by assuming that, upon entering a meeting, the only action available to agent i is an announcement of his intentions for that meeting. This announcement occurs before the agent knows whether he will be a producer or not, and it is binding within the meeting. If money is not used as a medium of exchange, the announcement is simply the quantity of goods y_i that agent i is willing to produce. As a result, the set of available actions for agent i is given by $Y = [0, \infty)$. We denote a generic action profile in a meeting in period t by $y_t^n = (y_{it}, y_{m(i)t}) \in Y \times Y$, where y_{it} is the action of agent i and $y_{m(i)t}$ is the action of $i's$ match. If we denote the history up to period t by h^t, and the set of all possible histories up to period t by H^t, a strategy for agent i is a sequence $\sigma_i = \{\sigma_{it}\}$, with $\sigma_{it} : H^t \to Y$. Notice that agent i can always announce zero production, so the assumption that announcements are binding within the meeting does not violate agent's i participation constraint. Finally, if money is used as a medium of exchange in a transaction, the announcement by agent i is a vector $\widetilde{y}_i$ that depends on the trading rules prevailing in the economy. In the next section we consider an exchange process that does not require the use of money. For this reason, we postpone a more detailed discussion of the trading rules with money to section 5.

In every period t, the decision of agent i depends not only on his history but also on his beliefs. The belief of agent i in period t is an assessment of the action profile in all meetings up to period t. Because the action profile in a meeting in which the agent participated is observable by the agents in the meeting, and the record of action profiles in public meetings becomes observable by all agents in the economy, beliefs are only required with respect to private meetings in which the agent did not participate and public meetings that occur after the last period in which the record keeping device was updated. A precise assessment of beliefs is necessary to make sure that an agent's strategy is sequentially rational after every possible history. For a given strategy profile, beliefs are trivially computed by using Bayes rule along the path of play induced by the strategy profile. However, Bayes rule cannot be applied to histories that are reached with zero probability. The notion of sequential equilibrium provides criteria for the assessment of beliefs at these histories. In a sequential equilibrium, a strategy profile σ is said to be supported by a consistent belief μ if μ is the limit of a sequence of beliefs μ_s, which are computed by Bayes rule from a sequence of strictly mixed strategy profiles σ_s that converges to σ.

The concept of sequential equilibrium was originally designed for extensive games with a finite number of actions and players. However, it presents a natural interpretation in our setting. Assume, for instance, that a player observes a deviation from the path of play. He explains this observation by resorting to the trembling-hand theory. Now, since trembles by a finite number of players are infinitely more likely than trembles by a continuum of players, he must infer that the deviation was caused by and is restricted to a finite number of players.[4] Following this interpretation, we assume that, upon observing a deviation from the path of play, an agent must conclude that this deviation was caused by a finite number of mistakes. We further assume that an agent updates his belief conditional on the information that is released by the record keeping device. Precisely, after observing a deviation from the path of play for the first time, the agent infers that this deviation cannot have occurred before the last period in which the actions in public meetings were publicly observed and no deviation was detected. One can argue that an agent's belief should also attach a positive though small probability that a deviation has occured before the last period in which the record keeping device was updated and no deviation was detected. As it will become clear in the next section, such beliefs would actually increase the incentives to follow the equilibrium strategy after the observation of a deviation.

Formally, a strategy profile $\sigma = \{\sigma_i\}_{i\in[0,1]}$ is an equilibrium if the following conditions are satisfied: (i) for every history consistent with the path of play, an agent must assign zero probability to the event that a deviation has occurred; (ii) σ is a Nash equilibrium i.e., for each agent $i \in [0,1]$, there are no profitable deviations along the path of play; (iii) for every history that occurs with zero probability along the path of play, an agent assumes that the mistake that caused the initial deviation cannot have occurred before the last period in which the actions in public meetings were publicly observed and no deviation was detected; (iv) for each agent $i \in [0,1]$, σ_i is sequentially rational after every history that occurs with

[4] A similar reasoning is put forth by Takahashi (2007). Takahashi investigates whether a community with a continuum of agents supports cooperation in the repeated prisoner's dilemma when agents are anonymously and randomly matched in pairs. In his set up, because all agents are private, he assumes that the player's belief with respect to the distribution of records of play in the continuum population should not be affected by the observation of a deviation, as this deviation was most probably caused by a finite number of mistakes.

zero probability along the path of play.

4 Social Norms and the Inessentiality of Money

In what follows, we describe a strategy that induces a social norm of gift-exchange and implements the first best. Throughout this section, we say that an agent cooperates (deviates) if he announces (does not announce) a quantity equal to y^*. Consider the following social norm, which prescribes an agent's action after any possible history.

"An agent starts cooperating in period 1. If cooperation was the outcome in all meetings that the agent observed up to period t, he cooperates in period $t+1$. If a deviation occurred in some meeting that the agent participated but there was no public observation of a deviation, the agent deviates in period $t+1$. Consider now a history in which a *new* deviation is observed in the most recent period in which the record keeping device updated its information. If this deviation was not the result of a past punishment phase, the agent enters a punishment phase that starts right after the announcement of the deviation. In the punishment phase, the agent does not cooperate for T periods, and after that he resumes behavior as in period 1. If instead the *new* deviation was the result of all agents implementing a past punishment phase, the agent chooses his action based only on the observation of the meetings in which he participated since the end of the last punishment phase."

Along the path of play, if an agent follows the social norm, he always cooperates if called upon to do so. Consider a private agent in period $k(t-1)+n$. He follows the social norm upon meeting with a private agent as long as (where $n \in \{1, ..., k\}$)

$$\sigma\left[u(y^*) - y^*\right] \geq \begin{array}{c} (1-\beta^{k-(n-1)})\sigma u(y^*) + \beta^{k-(n-1)}(1-\varepsilon)^{2^{k-(n-1)}-2} V_{2^{k-(n-1)}} + \\ \beta^{k-(n-1)}\left[1-(1-\varepsilon)^{2^{k-(n-1)}-2}\right] V_{pub} \end{array}, \qquad (1)$$

where $V_{pub} = \beta^T \sigma\left[u(y^*) - y^*\right]$ is the expected payoff right after the public observation of a deviation, and $V_{2^{k-(n-1)}}$ is the expected payoff when there are $2^{k-(n-1)}$ defectors among the private agents and the public record did not detect any *new* deviation. Alternatively, if the agent meets with a public agent in period $k(t-1)+n$, he cooperates if and only if (where $n \in \{1, ..., k\}$)

$$\sigma\left[u(y^*) - y^*\right] \geq (1-\beta^{k-(n-1)})\sigma u(y^*) + \beta^{k-(n-1)} V_{pub}. \qquad (2)$$

Notice that there is no need to write down the conditions for public agents, since a deviation by a public agent induces the same incentives as a deviation by a private agent in a meeting with a public agent. In Appendix A (Lemma 3) we show that the expected payoff when only private deviations are observed is higher than the expected payoff after the public observation of a deviation. Intuitively, it takes longer for the punishment phase to occur after a deviation against a private agent. We also show that, if agents are sufficiently patient, the expected payoff of a deviation against a private agent converges to the expected payoff of a deviation against a public agent. The reason is that an initial deviation against private agents will eventually reach a public agent, at which point the expected payoff will be the same as the payoff of a direct deviation against a public agent. When agents are very patient, this is the payoff that really matters. As a result, there exists β' such that, for all $\beta > \beta'$, a sufficient condition that prevents deviations along the path of play so that (1) and (2) are satisfied is

$$\sigma\left[u(y^*) - y^*\right] > (1 - \beta^{k-(n-1)})\sigma u(y^*) + \beta^{k-(n-1)} V_{pub}. \tag{3}$$

Moreover, since $\sigma u(y^*) > V_{pub}$, we can restrict attention to

$$\sigma\left[u(y^*) - y^*\right] > (1 - \beta^{k})\sigma u(y^*) + \beta^{k} V_{pub}. \tag{4}$$

For all β, there always exists $T = T(\beta)$, where $T + 1$ is the smallest integer such that

$$\beta V_{pub} = \beta^{T+1}\sigma\left[u(y^*) - y^*\right] \leq \sigma\left[u(y^*) - y^*\right] - \eta y^*, \tag{5}$$

where $\eta \in (0, 1 - \sigma)$. As a result, a sufficient condition for (4) is

$$\sigma\left[u(y^*) - y^*\right] > (1 - \beta^{k})\sigma u(y^*) + \beta^{k-1}\left\{\sigma\left[u(y^*) - y^*\right] - \eta y^*\right\}. \tag{6}$$

We can rewrite (6) as

$$\beta^{k-1}\eta y^* - \beta^{k-1}\sigma u(y^*)\,(1-\beta) - \sigma y^*(1 - \beta^{k-1}) > 0. \tag{7}$$

Clearly, for every k, there exists $\beta(k) \geq \beta'$ such that (7) holds for all $\beta > \beta(k)$. Consider now the incentives to follow the social norm out of the equilibrium path. There are three main contingencies that we need to take into account. First, right after a public observation of a deviation, agents enter a punishment phase that lasts T periods. Clearly, as long as (1) and (2) are satisfied and an agent wants to cooperate when all other agents are also cooperating, he follows the rules of the social norm. This reasoning is valid for any belief the agent may have regarding the actions in past meetings that he did not observe. This happens because, if all agents are following the social norm, the agent knows that no other agent is going to cooperate for T periods and all agents will resume cooperation after that. Second, there are histories of deviations in public meetings that are not yet publicly known. After these histories, the optimal behavior is to deviate until the period in which the information on the deviation is made public. After that, the agent behaves as in the first contingency. The reason is that the agent's actions up to the period in which the information of the deviation is made public do not affect the future play of the game.

The more interesting case arises after histories of deviations in private meetings. The reason is that an agent may be tempted to cooperate in order to slow down the spread of non-cooperative behavior throughout the economy. This temptation varies with the agent's belief. For instance, if the agent observes a deviation in some period t, a possible belief explaining his observation is that the mistake that originated the deviation occurred in period 1, and that this deviation spread throughout private meetings from period 1 to period t. Based on this belief, the agent can count the expected number of private agents that are directly or indirectly "infected" by the initial deviation. This allows him to compute the expect payoff of following the rules of the social norm and the expected payoff of deviating from its rules. However, it may be argued that this belief is not reasonable because it assumes that the initial deviation never reached a public agent, which is unlikely to happen if t is large. This suggests that a more robust explanation is that the mistake originating the deviation cannot have occurred before the last period in which the actions in all public meetings were observed and no deviation was detected. As stated in condition (iii) of the equilibrium concept, we restrict attention to beliefs that conform with this explanation.

We now characterize sufficient conditions for an equilibrium when agents form beliefs consistent with the equilibrium definition. Precisely, we assume that beliefs are formed as follows. The first time that an agent observes a private deviation in period $k(t-1)+n$ ($n \in \{0, ..., k-1\}$) and there is no public observation of a deviation on the information released at the end of period $k(t-1)$, the agent explains his observation in the following manner: (i) if $n=0$, the mistake originating this deviation occurred in period $k(t-1)$; (ii) if $n \in \{1, ..., k-1\}$, the mistake originating this deviation occurred in period $k(t-1)+1$.

Consider a private agent i that has observed a deviation by another private agent in period $k(t-1)+n$ and there is no public observation of a deviation. Agent i expects that at the end of period $k(t-1)+k$, the number of agents reached by this deviation must be at least equal to 2^{k-2}, if agent i excludes himself. This is true irrespective of the actions chosen by agent i between period $k(t-1)+1$ and $k(t-1)+k$. Out of this set of 2^{k-2} agents, agent i knows that the agent that deviated against him is a private agent. Therefore, he assesses a probability of at most $(1-\varepsilon)^{2^{k-2}-1}$ that all other agents reached by the deviation are also private agents.

Now, if agent i meets with an agent in period $k(t-1)+n+1$ ($n \in \{0,...,k-1\}$) and deviates from the social norm in an attempt to slow down the spread of non-cooperative behavior, his expected payoff is at most

$$-(1-\beta)\sigma y^* + (1-\beta^{k-n})\sigma u(y^*) + \beta^{k-n}(1-\varepsilon)^{2^{k-2}-1} V_{2^{k-2}} + \beta^{k-n}\left[1-(1-\varepsilon)^{2^{k-2}-1}\right] V_{pub}. \quad (8)$$

Note that the expected payoff of not following the social norm does not depend on whether agent's i match in period $k(t-1)+n$ is public or private.

If agent i meets a public agent and follows the social norm, his expected payoff is

$$(1-\beta^{k-n})\sigma u(y^*) + \beta^{k-n} V_{pub}. \quad (9)$$

If agent i meets a private agent in period $k(t-1)+1$ and follows the social norm, his expected payoff is

$$(1-\beta^k)\sigma u(y^*) + \beta^k (1-\varepsilon)^{2^{k+1}-3} V_{2^{k+1}} + \beta^k \left[1-(1-\varepsilon)^{2^{k+1}-3}\right] V_{pub}. \quad (10)$$

Finally, if agent i meets a private agent in period $k(t-1)+n+1$ ($n \in \{1,...,k-1\}$) and follows the social norm, his expected payoff is

$$(1-\beta^{k-n})\sigma u(y^*) + \beta^{k-n}(1-\varepsilon)^{2^k-3} V_{2^k} + \beta^{k-n}\left[1-(1-\varepsilon)^{2^k-3}\right] V_{pub}. \quad (11)$$

Since the expected payoff when only private deviations are observed is higher than the expected payoff after the public observation of a deviation (Lemma 3 in Appendix A), a sufficient condition for an agent to follow the social norm after observing a private deviation is (where $n \in \{0,...,k-1\}$)

$$\sigma y^* \geq \frac{\beta^{k-n}}{1-\beta}(1-\varepsilon)^{2^{k-2}-1}\left(V_{2^{k-2}} - V_{pub}\right). \quad (12)$$

The left hand-side of (12) is the current cost of a deviation from the social norm, while the right-hand side is an upper bound on the expected future benefit of the slowdown in the spread of non-cooperative behavior. From the expression for $V_{2^{k-2}} - V_{pub}$ in the Appendix A, we can rewrite (12) as

$$\sigma y^* \geq \beta^{k-n}\frac{1-\beta^k}{1-\beta}(1-\varepsilon)^{2^{k-2}-1}\left[\sigma u(y^*) - V_{pub}\right]\sum_{s=0}^{\infty}\beta^{sk}(1-\varepsilon)^{\frac{2^{2k-2}(2^{sk}-1)}{2^k-1}}. \quad (13)$$

Given the definition of $T(\beta)$ in (5), T is the highest integer that satisfies

$$V_{pub} = \beta^T \sigma\left[u(y^*) - y^*\right] \geq \sigma\left[u(y^*) - y^*\right] - \eta y^*. \quad (14)$$

As a result, a sufficient condition for (13) is

$$\frac{\sigma}{\sigma+\eta} \geq \beta^{k-n} \frac{1-\beta^k}{1-\beta} (1-\varepsilon)^{2^{k-2}-1} \sum_{s=0}^{\infty} \beta^{sk} (1-\varepsilon)^{\frac{2^{2k-2}(2^{sk}-1)}{2^k-1}}. \tag{15}$$

The left-hand side of (15) is increasing in β, and we can further restrict our attention to

$$\frac{\sigma}{\sigma+\eta} \geq k(1-\varepsilon)^{2^{k-2}-1} \sum_{s=0}^{\infty} (1-\varepsilon)^{\frac{2^{2k-2}(2^{sk}-1)}{2^k-1}}. \tag{16}$$

Clearly, for any $\varepsilon > 0$, there exists $k(\varepsilon)$ such that (16) is satisfied for all $k \geq k(\varepsilon)$. Notice that the key requirement for an agent to follow the social norm after observing a private deviation is that the length k of the intervals in which the record keeping device updates the information on past public meetings is sufficiently large. The reason is that in order for an agent not be tempted to slow down the spread on non-cooperative behavior, he needs to believe that there is a reasonably high probability that a public agent is reached by a deviation, even if he cooperates.

Proposition 1 describes the conditions under which the social norm is an equilibrium.

Proposition 1 *Let $\varepsilon > 0$. There exists $k(\varepsilon)$ such that, for every $k \geq k(\varepsilon)$, the social norm is an equilibrium for all $\beta > \beta(k)$.*

Proof. In any equilibrium, agents must assign a zero probability that a deviation has occurred after any history consistent with the path of play. This requirement, together with Lemma 3 in Appendix A, and the reasoning in (1)-(7) implies that, irrespective of the value of $\varepsilon > 0$, for all $k \in \mathbb{N}$, there exists $\beta(k)$ such that the social norm is a Nash equilibrium whenever $\beta > \beta(k)$. Consider now the incentives to follow the social norm out of the equilibrium path. We start with simpler histories, where the social norm is sequentially rational irrespective of the beliefs the agent may have. First, an agent has no incentive to cooperate during the punishment phase since all other agents are also not cooperating. Moreover, as long as the social norm is a Nash equilibrium, an agent wants to resume cooperation after the punishment phase is over. Second, an agent must ignore the information on a new public deviation that is related to a past punishment phase when all other agents will do the same. In this case, the relevant history for the agent starts in the period right after the end of the past punishment phase. Notice that beliefs do not matter for all these histories because agents coordinate their actions using the information released by the record keeping device. Another class of histories that is simple to analyze are histories of deviations in meetings with public agents that are not yet publicly known and that are not part of the punishment phase. After such histories, the social norm dictates that the agent should deviate. Clearly, this behavior is sequentially rational irrespective of beliefs because the agent's actions up to the period in which the information of the deviation is made public do not affect the future behavior of the other agents. Finally, there are histories in which beliefs matter. These are the histories in which a private agent observes a deviation in a meeting with another private agent and there is no public observation of a deviation that agents need to take into account, i.e., there is no public observation of a deviation that is going to trigger a punishment phase. After these histories, an agent may have an incentive to slow down the spread of non-cooperative behavior. In turn, this incentive depends on the agent's belief regarding the total number of agents that are not cooperating. After these histories, beliefs are formed as follows. If an agent observes a deviation in a private meeting in some period t, the agent explains his observation by inferring that the initial mistake originating this deviation occurred either in the same period in which the record keeping device was most recently updated (if the deviation was observed during that period) or in the period right after this update occurs (if the deviation was observed after that period). Notice that this requirement is consistent with item (iii) of the equilibrium definition. Under these beliefs, Lemma 3 in Appendix A and the reasoning in (8)-(16) implies that, for all $\varepsilon > 0$, there exists $k(\varepsilon)$ such that it is sequentially rational to follow the social norm right after observing a deviation in a private meeting. Moreover, as long as the record keeping device does not release any information of a public deviation, it is also sequentially rational to follow the social norm and deviate in all periods after the observation of a deviation in a private meeting. The reason is that the expected number of agents affected by the spread of the non-cooperative behavior does not decrease after an initial observation of a deviation, and the incentive to slow down the contagion decreases with the expected number of defectors. ■

Proposition 1 states that the first best can be implemented by a social norm equilibrium as long as agents are sufficiently patient, for any measure of public agents and for an arbitrarily large lag in the updating of past transactions. Patience is the key requirement for an agent to follow the social norm along the equilibrium path. Intuitively, patience is a substitute of memory in a society that keeps a poor track of its past history. In this sense, a society that needs money to achieve desirable allocations is not a society that lacks memory but a society that lacks patience.

What does an arrangement in which all exchanges are mediated by the use of money accomplish in our economy? First, in any equilibrium where money is the only medium of exchange, an agent must have it in positive amounts in order to be able to consume. A direct implication of this requirement is that the first best cannot be achieved in the absence of a mechanism that guarantees that an agent will always have enough money to pay for his transactions. For instance, such a mechanism does not exist in a search model of money unless money is perfectly divisible. If money is not perfectly divisible, a positive measure of individuals will run out of money after a finite number of periods. As a result, if some of these agents meet other agents and are assigned as consumers, they will have nothing to offer in exchange and thus will not be able to consume. If instead, money is perfectly divisible, Kocherlakota (2002) shows that there exists a monetary equilibrium that implements the first best. The mechanism that supports the first best constructs a one to one mapping between individual histories and the decimal expansions of money holdings. This mapping allows money to become a recording device that keeps a perfect track of an agent's past decisions. This provides a way to efficiently punish agents that not produce the efficient amount of goods when called upon to do so. In summary, money is able to induce the same allocation as the social norm if and only if it is perfectly divisible. Any departure from perfect divisibility precludes a monetary arrangement to reach the first best.

4.1 Robustness

We have assumed that meetings with public agents can be observed by everyone in the economy. This assumption is not crucial. As long as agents fear that an initial deviation may eventually lead to a generalized punishment in case it reaches public agents and is then eventually disseminated throughout the rest of the economy, similar results would arise. The important features are that agents are sufficiently patient and that the incentives to slow down the spread of non-cooperative behavior are not too strong.

We describe the incentives to slow down non-cooperative behavior with the assumption that information on the outcome of public meetings is updated with a lag, and with the belief that the initial mistake underlying an observed deviation occurred at some point in the recent past. Another mechanism that captures the same feature obtains with the assumption that agents meet in groups of size $k \geq 2$. We explore this mechanism in detail in Appendix B. We show that, for any measure of public agents $\varepsilon > 0$, there exists a minimum group size $k(\varepsilon)$ such that the social norm is an equilibrium if agents are sufficiently patient and $k \geq k(\varepsilon)$. We also show that, even though $k(\varepsilon)$ goes to infinity when ε converges to 0, the probability that a public agent participates in a randomly chosen meeting converges to zero when ε converges to zero. The advantage of this formulation is that it sustains a social norm equilibrium when each agent, after observing a deviation for the first time, believes that it is the first deviation to occur in the economy. Its disadvantage is that it requires relatively large groups when the measure of public agents is small, and one can argue that the need for media of exchange reduces when agents meet in large groups.

We also assumed that the only action available to agent i is an announcement of his intentions, and that this announcement is binding within the meeting. This assumption simplifies the computation of the number of agents that deviate when the non-cooperative behavior spreads throughout the economy. A more natural assumption is that a deviation occurs only when a producer meets a consumer and the former does not produce for the latter. The drawback of this assumption is that it requires to keep track of all meetings in which non-cooperative producers meet cooperative consumers as these are the only meetings which increase the number of defectors. This makes the analysis more cumbersome. The difference between this approach and the approach that we followed is that ours induce a faster dissemination of a non-cooperative behavior. It is true that a faster spread of deviations reduces the incentive of an agent to deviate along the equilibrium path and may increase the incentive of an agent to follow the social norm out of the equilibrium path. However, in terms of our results, the only effect of a slower spread of defection is that it may require a larger updating lag k and a higher patience β for the social norm to be an equilibrium.

5 Social Norms and the Essentiality of Money

In this section we modify our set up by assuming that a group of agents with a measure $\delta > 0$ have private information about their preferences. Precisely, each "private information agent" receives an i.i.d. preference shock at the beginning of every period. This shock has two possible realizations. Conditional on being assigned as a consumer, there is a probability π that he will obtain an utility $u(y)$, and a probability $1 - \pi$ that he will obtain an utility $\phi u(y)$, where $\phi \in (0, 1)$. The efficient production when the utility function is $\phi u(y)$ is given by $\phi u'(y^{\phi}) = 1$, while the efficient production with respect to the expected utility $\pi u(y) + (1 - \pi)\,\phi u(y) - y$ is given by $[\pi + (1 - \pi)\,\phi]\,u'(y^{\pi,\phi}) = 1$. Finally, an agent can recognize whether another agent is subject to preference shocks, but the actual realization of the shock is private information.

In this modified setting, the actions in meetings that involve agents with private information are as follows. If money is not used as a medium of exchange in a transaction, an agent with private information about his preference announces whether he is of a high type (h) and his utility function is $u(y)$, or whether he is of a low type (l) and his utility function is $\phi u(y)$. Moreover, upon meeting an agent whose tastes are private information, every agent announces a pair of quantities $\widetilde{y} = (y_h, y_l)$, where y_h (y_l) is the amount of goods that he is willing to produce for an agent that announces h (l). Finally, upon meeting an agent without private information, all agents announce y as the production level they are committed to produce in case they are assigned as producers.

A social norm that implements the first best needs to redefine the concept of cooperation in order to take into account the presence of private information. We say that an agent cooperates if he announces the pair $y^{**} = (y^*, y^\phi)$ upon meeting a private information agent. Moreover, a private information agent, besides announcing y^{**} upon meeting another agent with private information, also announces h (l) if he is of the high (low) type. Finally, upon meeting an agent without private information, an agents announce y^* as the production level he is committed to produce in case he is assigned as producer. Given this definition, the social norm that induces the first best along the path of play, is the same as the one in section 4. Lemma 1, however, shows that this social norm is not an equilibrium.

Lemma 1 *The social norm that implements the first best in the presence of private information about preferences is not an equilibrium.*

Proof. Assume that the social norm is an equilibrium, and consider the decision of an agent whose preferences are private information and knows that he is of the low type. If he follows the social norm he announces l and, conditional on being assigned as a consumer, obtains an utility $\phi u(y^\phi)$. If, instead, he deviates and says h, conditional on being assigned as a consumer, his current utility becomes $\phi u(y^*)$, which is higher than $\phi u(y^\phi)$. Because there is a positive probability that this agent is of the high type, all agents that observe the outcome of this meeting attach a zero probability to the event that a deviation has occurred. Therefore, this is a profitable deviation as it increases the current expected payoff and does not trigger any future punishment. ■

A direct implication of Lemma 1 is that, in any social norm equilibrium, $\widetilde{y} = (y, y)$. Given this restriction, what is the social norm equilibrium that maximizes the expected surplus in all meetings? Proposition 2 provides an answer to this question. The proof is in Appendix C.

Proposition 2 *Let* $\pi u(y^{\pi,\phi}) + (1-\pi)\,\phi u\left(y^{\pi,\phi}\right) > y^*$. *For all* $\delta > 0$*, there exists a social norm equilibrium that maximizes the expected surplus. In this equilibrium, the efficient level* y^* *is produced in all meetings where the agent assigned as a consumer has known preferences* *and a production of $y^{\pi,\phi}$ occurs in all meetings where the agent assigned as a consumer is a private information agent.*

We now describe a trading scheme that complements the social norm by using money in transactions that involve agents with private information. We inject money in the economy by assuming that, at the beginning of period 1 a measure m of agents is randomly chosen and receives one unit of money. Money plays no role in meetings between agents whose preferences are known. Consider now meetings that involve at least one private information agent. There are two possibilities. First, there are meetings between a private information agent (say agent a) and an agent with known preferences (say agent b). In these meetings, we require that agent a always announces y^* as the production level he is committed to produce in case he is assigned as producer. This announcement is the same as in section 4, i.e., it does not depend on money holdings and/or private information. However, the announcement of agent a that binds his decision as a consumer depends on his type and money holdings, while the announcement of agent b that binds his decision as a producer depends on his money holdings. Throughout we assume that money holdings are not ex-ante observable. We claim and prove below that the following announcements are self-enforcing. If the high type is realized and agent a has one unit of money, he announces $(y^*, y^*, 1)$, where the first (second) entry is the amount of goods that he wants to consume in exchange for 1 (0) units of money, and the third entry is his money holdings. If agent a does not have money and/or the low type is realized, he announces $(\infty, y^{\pi,\phi}, i)$, where $i = 0, 1$. As for agent b, if he has 1 [0] units of money he announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$ $[(y^*, y^{\pi,\phi}, 0)]$, where the first (second) entry in each announcement is the amount of goods that he is willing to produce in exchange for 1 [0] units of money, and the third entry is his money holdings. We define $y_{ab} = \left\{(y^*, y^*, 1); (\infty, y^{\pi,\phi}, 0)\right\}$ as the set of cooperative announcements that bind agent's a decisions as a consumer, and $y_{ba} = \left\{(y^{\pi,\phi}, y^{\pi,\phi}, 1); (y^*, y^{\pi,\phi}, 0)\right\}$ as the set of cooperative announcements that bind agent's b decisions as a producer. The next step is to describe how each profile of announcements within the set of cooperative announcements is implemented. We assume the following implementation scheme. If agent b announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$ $[(y^*, y^{\pi,\phi}, 0)]$ and is assigned as a consumer, he transfers 1 [0] units of money to agent a. Assume now that agent a is assigned as a consumer. If agent a has announced $(\infty, y^{\pi,\phi}, i)$ and agent b has announced $(y^*, y^{\pi,\phi}, 0)$ or $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$, agent a consumes $y^{\pi,\phi}$ and does not transfer any money. Moreover, If agent a announces $(y^*, y^*, 1)$ and agent b announces $(y^*, y^{\pi,\phi}, 0)$, agent a consumes y^* and gives one unit of money to agent b. Finally, if agent a announces $(y^*, y^*, 1)$ and agent b announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$, agent a consumes $y^{\pi,\phi}$

and agent b transfers one unit of money to agent a. We also need to consider meetings where both agents have private information about their preferences. In this case, we assume that they announce $y^{\pi,\phi}$. Hence, in all such meetings, there is no transfer of money and $y^{\pi,\phi}$ is produced. Notice that an agent is always free to announce an element outside of the set of cooperative announcements. If this happens, we assume that there is no exchange, i.e., there is zero production and no transfers of money.

The previous paragraph set the stage for a redefinition of the notion of cooperation in the presence of monetary transactions. We say that a private information agent cooperates if and only if (i) upon meeting an agent without private information, he announces y^* and an element of y_{ab}; (ii) upon meeting another agent with private information, he announces $y^{\pi,\phi}$; (iii) whenever he meets with an agent without private information that has announced $(y^*, y^{\pi,\phi}, 0)$, if he had announced $(y^*, y^*, 1)$, he must have one unit of money. In turn, we say that an agent without private information cooperates if and only if (i) upon meeting another agent without private information, he announces y^*; (ii) upon meeting an agent with private information, he announces an element of y_{ba}; (iii) whenever he meets an agent with private information and announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$, he must have one unit of money if (a) he is assigned as a consumer, (b) he is assigned as a producer and the private information agent has announced $(y^*, y^*, 1)$. Notice that our definition of cooperation allows an agent to lie about his money holdings when he has one unit of money.

A "money-based social norm" is a norm where agents behave exactly like in section 4 but the definition of cooperation is as above. Proposition 3 proves the main result of this section, i.e., that there exists a money-based social norm equilibrium that improves on the set of allocations of any social norm equilibrium that does not use money. The proof is in the Appendix C.

Proposition 3 *Let* $\pi u(y^{\pi,\phi}) + (1-\pi)\,\phi u\left(y^{\pi,\phi}\right) > y^*$, *and* $\phi < \frac{1-m}{[(1-m)\pi+m]\left[u(y^*)-u\left(y^{\pi,\phi}\right)\right]}$. *For all* $\delta > 0$, *there exists a money-based social norm that implements a better allocation than any social norm equilibrium without money.*

6 Conclusion

A common requisite within the so-called fundamental models of money is that money should be essential as a medium of exchange. In this article, we take this dictum as a benchmark and explore its implications for alternative explanations for the role of money. We frame our analysis in a search model of money with decentralized trade and no commitment regarding actions in future meetings. Our first result is that money is inessential to achieve desirable allocations if agents are sufficiently patient and there is public information regarding the past history of actions, no matter how precarious is this information. Our second result is that money regains its essentiality when there is private information about preferences. These results suggest that private information about fundamentals plays a key role in the unveiling of the reasons for the essentiality of money.

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7 Appendix A - Computation of $V_j - V_{pub}$

Let $j \in \mathbb{N}$. If all agents behave according to the social norm, we must have

$$V_j - V_{pub} = (1-\beta^k)\sigma u(y^*) + \beta^k(1-\varepsilon)^{2^k j}V_{2^k j} + \beta^k \left[1-(1-\varepsilon)^{2^k j}\right] V_{pub} - V_{pub}. \tag{1a}$$

We can rewrite (1a) as

$$V_j - V_{pub} = (1-\beta^k)\underbrace{[\sigma u(y^*) - V_{pub}]}_{u} + \beta^k(1-\varepsilon)^{2^k j}\left(V_{2^k j} - V_{pub}\right). \tag{2a}$$

A similar procedure implies

$$V_{2^k j} - V_{pub} = (1-\beta^k)u + \beta^k(1-\varepsilon)^{2^{2k} j}\left(V_{2^{2k} j} - V_{pub}\right). \tag{3a}$$

If we substitute (3a) in (2a) we obtain

$$V_j - V_{pub} = (1-\beta^k)u\left[1+\beta^k(1-\varepsilon)^{2^k j}\right] + \beta^{2k}(1-\varepsilon)^{j\left(2^k+2^{2k}\right)}\left(V_{2^{2k} j} - V_{pub}\right) \tag{4a}$$

where

$$V_{2^{2k} j} - V_{pub} = (1-\beta^k)u + \beta^k(1-\varepsilon)^{2^{3k} j}\left(V_{2^{3k} j} - V_{pub}\right). \tag{5a}$$

By substituting (5a) into (4a), we obtain

$$V_j - V_{pub} = (1-\beta^k)u\left[1+\beta^k(1-\varepsilon)^{2^k j} + \beta^{2k}(1-\varepsilon)^{j\left(2^k+2^{2k}\right)}\right] + \beta^{3k}(1-\varepsilon)^{j\left(2^k+2^{2k}+2^{3k}\right)}\left(V_{2^{3k} j} - V_{pub}\right). \tag{6a}$$

Proceeding in this fashion for all future periods, we have

$$V_j - V_{pub} = (1-\beta^k)\left[\sigma u(y^*) - V_{pub}\right]\sum_{s=0}^{\infty}\beta^{sk}(1-\varepsilon)^{\frac{j2^k(2^{sk}-1)}{2^k-1}}. \tag{7a}$$

The following Lemma is a straightforward implication of (7a).

Lemma 2 *For all $j \in \mathbb{N}$, $V_j > V_{pub}$. Moreover, $\lim_{\beta \to 1}(V_j - V_{pub}) = 0$, for all $\varepsilon > 0$, and $k \in \mathbb{N}$.*

8 Appendix B - Social Norms and Group Meetings

In this Appendix, we replace the assumption that histories of public meetings are updated with a lag by the assumption that agents meet in groups of size $k \geq 2$. Moreover, in each group and for each agent, there is a probability σ that the agent will be assigned as a consumer, and a probability σ that he will be assigned as a producer. We let $\sigma \in (0, \frac{1}{2})$, so that there is a positive probability that the agent will be neither a consumer or a producer. Given this assumption, consider the following social norm: "All agents start cooperating in period 1. In period t, if cooperation is the only outcome in all meetings that the agent has observed up to that period, he cooperates in period $t+1$. If a defection has occurred in his private meetings but cooperation was the outcome in all public meetings, he defects in period $t+1$. If a deviation has occurred in a public meeting, the agent defects for T periods. After that, he resumes behavior as in period 1."

A private agent does not deviate along the path of play if and only if

$$-(1-\beta)\sigma y^* + \beta\sigma\left[u(y^*) - y^*\right] \geq \beta V_k, \tag{1b}$$

where V_k is the expected payoff of a deviation when there are k defectors among the private agents. Notice that we are already taking into account that the incentives to deviate are stronger when the private agent meets another private agent. Consider now the incentives to follow the social norm out of the equilibrium path. There are two main contingencies that we need to take into account. First, there are histories that include a public observation of a deviation. In this case, agents enter a punishment phase that lasts T periods. Clearly, as long as (1b) is satisfied and an agent wants to cooperate when all other agents are also cooperating, he follows the rules of the social norm after a public defection. The more interesting case arises after histories of deviations in private meetings but with no public observation of a deviation. The reason is that, after observing only private deviations, an agent may be tempted to cooperate in order to slow down the spread of non-cooperative behavior throughout the economy. Consider then the decision of a private agent that holds a belief that there are n deviators among the private agents, including himself. He follows the social norm and deviates if and only if

$$\beta V_{pub} \geq -(1-\beta)\sigma y^* + \beta\widetilde{\varepsilon}^{n-1} V_{1+(n-1)k} + \beta(1-\widetilde{\varepsilon}^{n-1})V_{pub}, \tag{2b}$$

where $V_{pub} = \beta^T\sigma\left[u(y^*) - y^*\right]$, and $\widetilde{\varepsilon}$ is the probability that an agent only meets with private agents in a given group, i.e., $\widetilde{\varepsilon} = (1-\varepsilon)^{k-1}$. If the agent does not follow the social norm and chooses to cooperate, there is a probability $\widetilde{\varepsilon}^{n-1}$ that all other agents that are not cooperating will only meet with private agents, in which case the number of deviators in the next period becomes $1+(n-1)k$. There is a complementary probability that at least one agent that defects will meet with a public agent, and the expected payoff next period will be equal to V_{pub}. Notice that we are already taking into account that the incentives to slow down the spread of the non-cooperative behavior are stronger when a private agent meets a group with a pubic agent. Moreover, since the behavior of a public agent is observed by all agents in the economy, his decision problem out of the equilibrium path is given by (2a).

It is straightforward that the expected payoff of an agent that deviates when there are n defectors among the private agents is

$$V_n = V_{pub} + (1-\beta)\left[\sigma u(y^*) - V_{pub}\right] \sum_{t=0}^{\infty} \beta^t \widetilde{\varepsilon}^{nS(k,t)}, \tag{3b}$$

where $S(k,t) = \frac{k^t - 1}{k-1}$. Notice that, similar to the results in Lemma 1, $V_n > V_{pub}$, for all n. Moreover, if agents are sufficiently patient, the expected payoff of a deviation against private agents converges to the expected payoff of a deviation against a public agent. As a result, there exists β'' such that, for all $\beta > \beta''$, a sufficient condition for (1b) is

$$-(1-\beta)\sigma y^* + \beta\sigma\left[u(y^*) - y^*\right] > \beta V_{pub}. \tag{4b}$$

Now choose $T = T(\beta)$, where T is the smallest integer such that

$$\beta^{T+1}\sigma\left[u(y^*) - y^*\right] \leq \sigma\left[u(y^*) - y^*\right] - \mu y^*, \tag{5b}$$

where $\mu \in (0, 1-\sigma)$. Therefore, a sufficient condition for (4b) is

$$-(1-\beta)u(y^*) + \eta y^* > 0. \tag{6b}$$

Clearly, there exists $\beta' \geq \beta''$ such that (4b) holds for any $\beta > \beta'$. It remains to show that agents follow the rules of the social norm out of the equilibrium path. First, we can rewrite (2b) as

$$(1-\beta)\sigma x^* \geq \beta\widetilde{\varepsilon}^{n-1}\left[V_{1+(n-1)k} - V_{pub}\right]. \tag{7b}$$

Notice from (3b) that $V_n - V_{pub}$ is strictly decreasing in n as long as $\beta < 1$. As a result, we only need to evaluate (7b) at $n = k$. We have

$$\sigma x^* \geq \beta\widetilde{\varepsilon}^{k-1}\left[\sigma u(y^*) - V_{pub}\right] \sum_{t=0}^{\infty} \beta^t \widetilde{\varepsilon}^{[1+(k-1)k]S(k,t)}. \tag{8b}$$

By the definition of $T = T(\beta)$, T is the highest integer such that

$$V_{pub} = \beta^T \sigma\left[u(y^*) - y^*\right] \geq \sigma\left[u(y^*) - y^*\right] - \mu y^*. \tag{9b}$$

Therefore, a sufficient condition for (8b) is

$$\frac{\sigma}{\sigma + \mu} \geq \beta\widetilde{\varepsilon}^{k-1} \sum_{t=0}^{\infty} \beta^t \widetilde{\varepsilon}^{[1+(k-1)k]S(k,t)}. \tag{10b}$$

Finally, notice that $S(n,t) \geq t$, for all $t \geq 0$, and (10b) can be replaced by

$$\frac{\sigma}{\sigma+\mu} \geq \beta\widetilde{\varepsilon}^{k-1}\sum_{t=0}^{\infty}\beta^t\widetilde{\varepsilon}^{[1+(k-1)k]t}. \tag{11b}$$

We can express (11b) in terms of a lower bound on the size of group meetings. Precisely, after some computation, we obtain that (11b) is satisfied whenever the size of group meetings is larger than or equal to $k(\varepsilon)$, where $k(\varepsilon)$ satisfies

$$(\sigma+\mu)(1-\varepsilon)^{(k-1)^2} + \sigma(1-\varepsilon)^{(k-1)[1+(k-1)k]} = \sigma. \tag{3e}$$

Proposition 4 summarizes our result.

Proposition 4 *For any $\varepsilon > 0$, there exists β' and $k(\varepsilon)$ such that the social norm is an equilibrium for all $\beta > \beta'$ and $k > k(\varepsilon)$.*

Moreover, corollary 1 shows that the probability that a public agent participates in any given group of size $k(\varepsilon)+1$, given by $1-(1-\varepsilon)^{k(\varepsilon)+1}$, converges to zero when the measure of public agents converges to zero.

Corollary 1 *In any social norm equilibrium, the minimum number of agents in a group meeting is such that the probability that a public agent participates in this meeting converges to zero when the measure of public agents converges to zero.*

Proof. The value $k'(\varepsilon)$ that solves

$$(\sigma+\mu)(1-\varepsilon)^{(k-1)^2} + \sigma(1-\varepsilon)^{(k-1)^2} = \sigma.$$

is given by

$$k'(\varepsilon) = 1 + \left[\frac{\ln\left(\frac{\sigma}{2\sigma+\mu}\right)}{\ln(1-\varepsilon)}\right]^{\frac{1}{2}}.$$

Now, since

$$(\sigma+\mu)(1-\varepsilon)^{(k-1)^2} + \sigma(1-\varepsilon)^{(k-1)^2} > (\sigma+\mu)(1-\varepsilon)^{(k-1)^2} + \sigma(1-\varepsilon)^{(k-1)[1+(k-1)k]},$$

it must be the case that $k'(\varepsilon) > k(\varepsilon)$. Finally, since

$$\lim_{\varepsilon\to 0}\left\{1-(1-\varepsilon)^{k'(\varepsilon)+1}\right\} = 0,$$

we have

$$\lim_{\varepsilon\to 0}\left\{1-(1-\varepsilon)^{k(\varepsilon)+1}\right\} = 0.$$

■

Table 1 below provides a numerical example of the relation between the measure of public agents, the discount factor, and the minimum size of group meetings required for the social norm to be equilibrium. The example assumes that $\sigma = 0.3$, $T = 4$, and the utility function is $u(x) = \sqrt{x}$.

Table 1 - Example

ε	$\approx k_{\beta \geq 0.9}(\varepsilon)$	$\approx k_{\beta \geq 0.99}(\varepsilon)$
0.1	3	3
0.01	7	6
0.001	16	13
0.0001	46	32
0.00001	145	75
0.000001	455	180

For instance, we can approximate $\varepsilon = 0.001$ as an economy where for each public agent, there exists nine hundred ninety-nine private agents. In this case, a social norm equilibrium exists for all $\beta \geq 0.9$ as long as agents meet in groups with at least sixteen agents. Note that the discount factor starts to play a key role in reducing the minimum size of groups when the measure of public agents becomes very small.

9 Appendix C

Proof of Proposition 2

Proof. We first show that a social norm equilibrium with $\widetilde{y} = (y, y)$ exists. Consider the social norm as stated in section 4 but where cooperation is redefined to take into account the presence of private information. Along the path of play, a sufficient condition for no deviation to take place is that a private information agent that just realized to be of a low type is willing to produce y^* upon meeting a private agent whose tastes are known. Following the reasoning in (1)-(3) in section 4, cooperation ensues if

$$(1-\beta)\,\sigma\left[\phi u(y^{\pi,\phi}) - y^*\right] + \beta\sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} - (1-\delta)\,y^*\right] \geq (1-\beta^{k-(n-1)})\sigma\widetilde{u}\left(y^{\pi,\phi}\right) + \beta^{k-(n-1)}\widetilde{V}_{pub}, \tag{1c}$$

where $\widetilde{u}\left(y^{\pi,\phi}\right) \equiv \pi u(y^{\pi,\phi}) + (1-\pi)\,\phi u\left(y^{\pi,\phi}\right)$, and $\widetilde{V}_{pub} = \beta^T \sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} - (1-\delta)\,y^*\right]$. Notice that $\widetilde{u}\left(y^{\pi,\phi}\right) > y^*$ is sufficient to make sure that private information agents are willing to participate in the social norm. Since $\widetilde{u}\left(y^{\pi,\phi}\right) > \widetilde{V}_{pub}$, we can restrict attention to

$$(1-\beta)\,\sigma\left[\phi u(y^{\pi,\phi}) - y^*\right] + \beta\sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} - (1-\delta)\,y^*\right] \geq (1-\beta^k)\sigma\widetilde{u}\left(y^{\pi,\phi}\right) + \beta^k\widetilde{V}_{pub}. \tag{2c}$$

For all β, there always exists $T = T'(\beta)$, where $T+1$ is the smallest integer such that

$$\beta\widetilde{V}_{pub} = \beta^{T+1}\sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} - (1-\delta)\,y^*\right] \leq \sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} - (1-\delta)\,y^*\right] - \eta' y^*. \tag{3c}$$

where $\eta' \in (0, 1-\sigma)$. As a result, a sufficient condition is

$$(1-\beta)\left[\phi u(y^{\pi,\phi}) - \widetilde{u}\left(y^{\pi,\phi}\right)\left(1+\beta^{k-1}\right)\right] - \left(1 - \beta^{k-1} - \beta^{k-1}\frac{\eta'}{\sigma}\right) y^* + \beta\delta\left(1-\beta^{k-2}\right)\left(y^* - y^{\pi,\phi}\right) \geq 0. \tag{4c}$$

Clearly, for every k, there exists $\beta'(k) \geq \beta'$ such that (4c) holds for all $\beta > \beta'(k)$. Consider now histories that occur with zero probability along the path of play. A sufficient condition for all agents to follow the social norm is that a private information agent does not want to slow down the spread of non-cooperative behavior upon meeting another private information agent with public histories.

$$\sigma y^{\pi,\phi} \geq \frac{\beta^{k-n}}{(1-\beta)}(1-\varepsilon)^{2^{k-2}-1}\left(\widetilde{V}_{2^{k-2}} - \widetilde{V}_{pub}\right). \tag{5c}$$

Since the spread of non-cooperative behavior under private information is identical to the spread of non-cooperative behavior in the absence of private information, we can apply the results in Appendix A and rewrite (5c) as

$$\sigma y^{\pi,\phi} \geq \beta^{k-n}\frac{1-\beta^k}{1-\beta}(1-\varepsilon)^{2^{k-2}-1}\left[\sigma\widetilde{u}\left(y^{\pi,\phi}\right) - \widetilde{V}_{pub}\right]\sum_{s=0}^{\infty}\beta^{sk}(1-\varepsilon)^{\frac{2^{2k-2}(2^{sk}-1)}{2^k-1}}. \tag{6c}$$

Given the definition of $T'(\beta)$, T is the highest integer that satisfies

$$\widetilde{V}_{pub} = \beta^T\sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} + (1-\delta)\,y^*\right] \geq \sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - \delta y^{\pi,\phi} - (1-\delta)\,y^*\right] - \eta y^*, \tag{7c}$$

As a result, a sufficient condition for (6c) is

$$\frac{\sigma y^{\pi,\phi}}{\sigma\delta y^{\pi,\phi} + \sigma(1-\delta)\,y^* + \eta y^*} \geq \beta^{k-n}\frac{1-\beta^k}{1-\beta}(1-\varepsilon)^{2^{k-2}-1}\sum_{s=0}^{\infty}\beta^{sk}(1-\varepsilon)^{\frac{2^{2k-2}(2^{sk}-1)}{2^k-1}}. \tag{8c}$$

The left-hand side of (8c) is increasing in β, and we can further restrict our attention to

$$\frac{\sigma y^{\pi,\phi}}{\sigma\delta y^{\pi,\phi} + \sigma(1-\delta)\,y^* + \eta y^*} \geq k(1-\varepsilon)^{2^{k-2}-1}\sum_{s=0}^{\infty}(1-\varepsilon)^{\frac{2^{2k-2}(2^{sk}-1)}{2^k-1}}. \tag{9c}$$

Clearly, for any $\varepsilon > 0$, there exists $k'(\varepsilon)$ such that () holds true for all $k \geq k'(\varepsilon)$. Finally, all the reasoning in the proof of Proposition 1 immediately carries out for the setting with private information. Therefore, if $\pi u(y^{\pi,\phi})+(1-\pi)\,\phi u\left(y^{\pi,\phi}\right) > y^*$, for all $\varepsilon > 0$, there exists $k'(\varepsilon)$ such that, for every $k \geq k'(\varepsilon)$, the social norm is an equilibrium for all $\beta > \beta'(k)$. ■

Proof of Proposition 3

Proof. Given the definition of cooperation in the presence of monetary transactions, fix any specific set of announcements that is consistent with this definition. The results of Proposition 1 and Proposition 2 can then be directly applied to prove that, for all $\delta > 0$ and $\varepsilon > 0$, there exists $k''(\varepsilon)$ such that, for every $k \geq k''(\varepsilon)$, the money-based social norm is an equilibrium for all $\beta > \beta''(k)$. Consider now the following set of announcements. A private information agent, upon meeting an agent without private information (i) always announces y^* as the production level that he is committed to produce in case he is assigned as a producer, (ii) always announces $(y^*, y^*, 1)$ whenever his realized type is high and he has a unit of money, (iii) always announces $(\infty, y^{\pi,\phi}, 0)$ if his realized type is low and/or he does not have any money. Moreover, a private information agent always announces $y^{\pi,\phi}$ upon meeting another agent with private information. As for an agent without private information, (i) he always announces y^* upon meeting another agent without private information, (ii) upon meeting an agent with private information, he always announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$ if he has one unit of money, and he always announces $(y^*, y^{\pi,\phi}, 0)$ if he does not have one unit of money. These announcements, together with the implementation scheme defined in section 5, imply that agents without private information will always consume y^* and all private information agents will consume at least $y^{\pi,\phi}$. Moreover, it implies that, conditional on the high type of a private information agent being realized and on this agent holding one unit of money, if he meets an agent without private information that holds zero units of money, and is assigned as a consumer, he will consume the efficient quantity y^*. Hence, as long as we show that an agent has no incentive to deviate from these announcements, there exists a money-based social norm that implements a better allocation that any social norm in the presence of private information that does not use money.

If agent b announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$ $[(y^*, y^{\pi,\phi}, 0)]$ and is assigned as a consumer, he transfers 1 [0] units of money to agent a. Assume now that agent a is assigned as a consumer. If agent a has announced $(\infty, y^{\pi,\phi}, i)$ and agent b has announced $(y^*, y^{\pi,\phi}, 0)$ or $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$, agent a consumes $y^{\pi,\phi}$ and does not transfer any money. Moreover, If agent a announces $(y^*, y^*, 1)$ and agent b announces $(y^*, y^{\pi,\phi}, 0)$, agent a consumes y^* and gives one unit of money to agent b. Finally, if agent a announces $(y^{*}, y^{*}, 1)$ and agent b announces $(y^{\pi,\phi}, y^{\pi,\phi}, 1)$, agent a consumes $y^{\pi,\phi}$ and agent b transfers one unit of money to agent a. We also need to consider meetings where both agents have private information about their preferences. In this case, we assume that they announce $y^{\pi,\phi}$. Hence, in all such meetings, there is no transfer of money and $y^{\pi,\phi}$ is produced.

Let W_{a0} (W_{a1}) be the expected payoff of a private information agent holding zero (one) units of money along the path of play. When all agents follow the proposed requirements, we obtain (where $\widetilde{u}\left(y^{\pi,\phi}\right) = \pi u\left(y^{\pi,\phi}\right) + (1-\pi)\phi u\left(y^{\pi,\phi}\right)$)

$$(1-\beta) W_{a0} = \beta\sigma m (1-\delta)(W_{a1} - W_{a0}) + \sigma\left[\widetilde{u}\left(y^{\pi,\phi}\right) - y^{\pi,\phi}\right] - \sigma(1-\delta) m \left(y^{*} - y^{\pi,\phi}\right), \tag{10c}$$

and

$$\begin{aligned}(1-\beta) W_{a1} \quad = \quad & -\beta\sigma\pi(1-\delta)(1-m)(W_{a1} - W_{a0}) \\ & +\sigma\pi(1-\delta)(1-m)\left[u(y^{*}) - u\left(y^{\pi,\phi}\right)\right] + \sigma\widetilde{u}\left(y^{\pi,\phi}\right) - \sigma\left[(1-\delta) y^{*} + \delta y^{\pi,\phi}\right].\end{aligned} \tag{11c}$$

As a result,

$$W_{a1} - W_{a0} = \frac{\sigma(1-\delta)(1-m)}{1-\beta+\beta\sigma(1-\delta)\left[(1-m)\pi+m\right]}\left\{\pi\left[u(y^{*}) - u\left(y^{\pi,\phi}\right)\right] - \left(y^{*} - y^{\pi,\phi}\right)\right\} \tag{12c}$$

First, it is clear that in any money-based social norm equilibrium, a private information agent without money is never going to announce $(y^{*}, y^{*}, 1)$. If he does so, there is a positive probability that he will deviate from the equilibrium path in case he is asked to transfer one unit of money. Consider now a private information agent with one unit of money, whose realized type is high. He follows the proposed announcement if and only if

$$\sigma(1-m)\left[u(y^{*}) + \beta W_{a0}\right] + \sigma m\left[u\left(y^{\pi,\phi}\right) + \beta W_{a1}\right] \geq \sigma\left[u\left(y^{\pi,\phi}\right) + \beta W_{a1}\right]. \tag{13c}$$

We can rewrite this condition as

$$u(y^{*}) - u\left(y^{\pi,\phi}\right) \geq \beta(W_{a1} - W_{a0}). \tag{14c}$$

Consider now an agent with one unit of money, whose realized type is low. He follows the proposed announcement if and only if

$$\sigma\left[\phi u\left(y^{\pi,\phi}\right) + \beta W_{a1}\right] \geq \sigma(1-m)\left[\phi u(y^{*}) + \beta W_{a0}\right] + \sigma m\left[\phi u\left(y^{\pi,\phi}\right) + \beta W_{a1}\right]. \tag{15c}$$

We can rewrite (15c) as

$$\phi\left[u\left(y^{*}\right)-u\left(y^{\pi,\phi}\right)\right]\leq\beta\left(W_{p1}-W_{p0}\right). \tag{16c}$$

Substituting for (12c), we obtain that (14c) and (16c) will be satisfied if and only if

$$\phi\left[u\left(y^{*}\right)-u\left(y^{\pi,\phi}\right)\right]\leq\frac{\beta\sigma\left(1-\delta\right)\left(1-m\right)}{1-\beta+\beta\sigma\left(1-\delta\right)\left[\left(1-m\right)\pi+m\right]}\leq u\left(y^{*}\right)-u\left(y^{\pi,\phi}\right). \tag{17c}$$

When agents are sufficiently patient, condition (17c) holds whenever

$$\phi\left[u\left(y^{*}\right)-u\left(y^{\pi,\phi}\right)\right]<\frac{\left(1-m\right)}{\left[\left(1-m\right)\pi+m\right]}\left\{\pi\left[u\left(y^{*}\right)-u\left(y^{\pi,\phi}\right)\right]-\left(y^{*}-y^{\pi,\phi}\right)\right\}<u\left(y^{*}\right)-u\left(y^{\pi,\phi}\right). \tag{18c}$$

Now let W_{b0} (W_{b1}) be the expected payoff of an agent without private information holding zero (one) units of money along the path of play. If all agents follow the proposed requirements, we obtain

$$\begin{aligned}W_{b0} &= \sigma\left[u\left(y^{*}\right)+\beta W_{b0}\right]+\sigma\left(1-\delta\right)\left(-y^{*}+\beta W_{b0}\right) \\ &\quad +\sigma\delta m\pi\left(-y^{*}+\beta W_{1}\right)+\sigma\delta\left(1-m\pi\right)\left(-y^{\pi,\phi}+\beta W_{b0}\right)+\left(1-2\sigma\right)\beta W_{b0},\end{aligned} \tag{19c}$$

and

$$\begin{aligned}W_{b1} &= \sigma u\left(y^{*}\right)+\sigma\left(1-\delta\right)\beta W_{b1}+\sigma\delta\beta W_{b0}+\sigma\left(1-\delta\right)\left(-y^{*}+\beta W_{b1}\right) \\ &\quad +\sigma\delta m\pi\left(-y^{\pi,\phi}+\beta W_{b0}\right)+\sigma\delta\left(1-m\pi\right)\left(-y^{\pi,\phi}+\beta W_{b1}\right)+\left(1-2\sigma\right)\beta W_{b1}.\end{aligned} \tag{20c}$$

From (19c) and (20c), we obtain

$$W_{b1}-W_{b0}=\frac{\sigma\delta m\pi\left(y^{*}-y^{\pi,\phi}\right)}{1-\beta\left(1-\sigma\delta\right)} \tag{21c}$$

If an agent without private information holds one unit of money and meets an agent with private information he follows the proposed announcement if and only if

$$-y^{\pi,\phi}+m\pi\beta W_{b0}+\left(1-m\pi\right)\beta W_{b1}\geq-m\pi y^{*}-\left(1-m\pi\right)y^{\pi,\phi}+\beta W_{b1}. \tag{22c}$$

We can rewrite (22c) as

$$y^{*}-y^{\pi,\phi}\geq\beta\left(W_{b1}-W_{b0}\right). \tag{23c}$$

From (21c) we have that (22c) holds if and only if

$$y^{*}-y^{\pi,\phi}\geq\frac{\beta\sigma\delta m\pi\left(y^{*}-y^{\pi,\phi}\right)}{1-\beta\left(1-\sigma\delta\right)}. \tag{24c}$$

If agents are sufficiently patient, condition (24c) holds when$1 > m\pi$, which is always true. Finally, if an agent without private information does not hold one unit of money and meets an agent with private information, he follows the proposed announcement if and only if

$$m\pi\left(-y^{*}+\beta W_{b1}\right)+(1-m\pi)\left(-y^{\pi,\phi}+\beta W_{b0}\right) \geq m\pi\left(-y^{*}+\beta W_{b1}\right)-(1-m\pi)\left(-y^{\pi,\phi}+\beta W_{b0}\right). \tag{25c}$$

Clearly, (25c) always holds. ■