Liquidity Constrained Competing Auctions*

Richard Dutu† University of Waikato

Benoit Julien‡ University of New South Wales

Ian King§ University of Melbourne

July 2008

Abstract

We study the effect of inflation in economies where goods are allocated via competing auctions. To do that we extend the competing auction framework (McAfee, 1993; Peters and Severinov, 1997) in several ways: we allow buyers to choose how much money they bring to an auction; the quantities produced and traded are divisible; we let sellers charge a fee, either positive or negative, to buyers participating to their auction. Two different specifications of the model are considered. In the first model, sellers post a quantity they wish to sell, a reserve price and a fee, and allow the price to be determined by auctions. In the second model, sellers post a price, a reserve quantity and a fee and allow the quantity to be determined by auctions.

When buyers bid prices, the existence of a monetary equilibrium requires that money growth not be too high. Marginal increments in money growth decrease the equilibrium posted quantity and buyers' participation. Sellers charge a positive fee when inflation is low but subsidize buyers when inflation is high. Symmetric efficiency is attained at the Friedman rule where sellers post the efficient quantity and charge no fee to buyers.

When buyers bid quantities, existence requires that money growth is not too high. Marginal increments in money growth decrease the posted price and the quantities traded. Sellers subsidize buyers when inflation is low but charge a positive fee when inflation is high. Symmetric efficiency is attained at the Friedman rule where sellers subsidize buyers, agents trade an inefficiently low quantity in multilateral matches and an inefficiently high quantity in pairwise matches.

Keywords: Competing auctions, Money, Entry fee, Inflation.
JEL Classification: D44; E40

*We thank James Chapman, Stella Huangfu, Guillaume Rocheteau, John Tressler and Randy Wright for very helpful discussions. We also thank the seminar participants at the Econometric Society, the Cleveland Fed Money, Banking, Payments and Finance Workshop, the Society for Advancement of Economic Theory Meeting, the Canadian Economics Association Meetings, the Workshop on Macroeconomic Dynamics, the Australian Workshop on Macroeconomic Dynamics, Georgetown University, the Bank of Canada, and the Universities of Victoria in Wellington and Waikato. All errors are ours.

†Email: rdutu@waikato.ac.nz

‡Email: benoitj@agsm.edu.au

§Email: ipking@unimelb.edu.au

1 Introduction

In the standard competing auctions model (McAfee 1993, Peters and Severinov 1997), many sellers compete to sell a single good by offering auctions to buyers. In the first stage of the game sellers compete by offering auctions. In the second stage buyers select among sellers and finally place their bid. Both the resources available to buyers and the quantity of the good at each auction are exogenously given. In this paper we endogenize both. We first make the model a monetary one by allowing buyers to choose the amount of money they bring to an auction, trading off the cost of holding money with the increase in the expected surplus from participating in an auction. Second, we allow sellers to choose how much of their production good they want to put on auction, trading off the production cost of the advertised quantity against the expected number of potential buyers. Finally, we offer sellers to charge each participating buyer with a fee, either positive or negative, trading off the additional revenue with the number of buyers taking part into their auction. To conduct this exercise we embed the competing auctions framework into the Lagos and Wright (2005) model of monetary exchange. This model is in the tradition of Kiyotaki and Wright (1991)'s environment in which a role for fiat money is determined endogenously from the frictions of the trading environment. That is, money is *essential* (Kocherlakota, 1998; Wallace, 2001). In this model agents have a periodic access to a centralized market in which they can rebalance their money holdings after each round of frictional trading. We use our model to study how monetary policy affects the equilibrium allocation of a competing auctions economy and derive recommendations for optimal monetary policy.

Working with large markets, the equilibrium concept that we use builds on the limit equilibrium concept developed by Peters and Severinov (1997). It begins with a finite number of buyers and sellers, characterizes the posted contracts and the payoffs, and then take the limit of these payoffs in the infinite game. This limit equilibrium enables to exploit the convergence properties of a competitive matching economy, especially that the deviation by one seller will not affect the payoff buyers can get by visiting him. This corresponds to the *market utility property*

(Peters, 2000) by which the buyer's utility in competitive matching economies is determined by the market and is taken as given by sellers. We extend this limit equilibrium concept to the context of competing auctions with monetary exchange and assume rational expectations so that sellers believe that their payoff functions satisfy the market utility property.

That buyers are now constrained in their bidding strategy by the amount of money they have implies that the distribution of money holdings, rather than the distribution of private of valuations, will be the key to determine which buyer wins the auction. Part of the exercise will be to characterize this distribution of money holdings as a function of the posted terms of trade and other equilibrium decisions. We consider two variants of the model, which imply different equilibrium outcomes. In the first, sellers post quantities for sale, a reserve price and a fee, but allow the dollar price of goods to be determined, *ex post*, by an auction. In the second, sellers post a dollar price, a reserve quantity and a fee, but allow the quantity sold to be determined through the auction.

In the first version we assume that prices are formed via second-price auctions. Building on Galenianos and Kircher (2008), we show in that case that as soon as the nominal interest rate is strictly positive, the demand for money follows a continuous non-degenerate distribution *even though* buyers have identical preferences for the good. We use the model to study how changes in monetary policy, measured by changes in the money growth rate, affect this distribution, the terms of trade posted sellers and entry by buyers. We prove that existence of a monetary equilibrium requires that money growth not be too high. Marginal increments in money growth decrease both the equilibrium posted quantity and buyers' participation. Sellers charge a positive fee when inflation is low but subsidize buyers when inflation is high. Symmetric efficiency is attained at the Friedman rule where sellers post the efficient quantity and charge no fee to buyers.

In the second version we study the mirror case in which sellers post a dollar price, a reserve quantity and a fee, and allow the quantity sold to be determined through the auction. This protocol shares similarities with industrial procurement auctions. The difference is that in procurement auctions the bidders are sellers and the bid-taker is a buyer. In this case, we prove that existence requires that money growth is not too high and that the distribution of money holdings is degenerate and equal to the posted price. Marginal increments in money growth decrease the posted price and the quantities traded. Sellers subsidize buyers when inflation is low but charge a positive fee when inflation is high. Symmetric efficiency is attained at the Friedman rule where sellers subsidize buyers, agents trade an inefficiently low quantity in multilateral matches and an inefficiently high quantity in pairwise matches. This second model is similar in spirit to monetary models with divisible goods but indivisible money (Shi (1995), Trejos and Wright (1995), Kultti and Riipinen (2003) and Julien, Kennes, and King (2008)).

Auctions combined with monetary exchange have already been studied. Especially Kultti and Riipinen (2003) and Julien, Kennes, and King (2008) introduce competing auctions in the so-called second generation of monetary search models (Shi (1995), Trejos and Wright (1995). Since money is indivisible in these models buyers can compete only through adjustments in quantity. By way of contrast, here, both money and goods are fully divisible. Galenianos and Kircher (2008) consider second-price auctions with divisible money and indivisible goods. A key difference with this paper is that the auctions in their model are not competitive in the sense that the quantity traded and the matching function are exogenous. Here we allow sellers to post either quantities or prices, and we allow buyers to decide which seller to approach. That is, there is competition between sellers and directed search on the part of buyers.[1] Other papers in the competing auctions literature are Julien (1997), Burguet and Sákovics (1999), Schmitz (2003) and Hernando-Veciana (2005) which consider environments with finite numbers of buyers and sellers.[2] Moldovanu, Sela and Shi (2008) have recently constructed a model in which the supply side, made of two competing auctioneers, can choose the supply of their good as we do here. Their focus, however, is on oligopolistic competition and on the coexistence of

[1] Directed search, using posted prices has received a lot of attention in the labor literature. See, for instance, Montgomery (1991), Moen (1997), Acemoglu and Shimer (1999a,b), Burdett, Shi, and Wright (2001) and the corresponding sections in the surveys by King (2003) and Rogerson, Shimer and Wright (2005). It has been used in monetary models by Rocheteau and Wright (2005), Faig and Jerez (2005, 2006) and Berentsen Menzio and Wright (2008). Competition with auctions has been applied to the labor market by Julien, Kennes, and King (2000).

[2] See also Peters (1997) and the literature on internet auctions (Peters and Severinov 2006).

two competing auction sites.

Finally this paper contributes to the literature on the micro-foundations of money by showing that the Lagos and Wright (2005) model is flexible enough to accommodate competing auctions. In contrast to the bargaining, price-taking, and competitive search pricing mechanisms examined in Rocheteau and Wright (2005), however, auctions generate terms of trade dispersion. Combined with the divisibility of goods and the fee charged by sellers, this produces interesting trade-offs for both sellers and buyers that have not been previously studied.

The article is organized as follows. Section 2 lays out the general environment. Section 3 characterizes the equilibrium and optimal monetary policy when sellers advertise quantities and buyers bid prices. Section 4 studies the mirror economy in which sellers post a price and buyers bid quantities. Section 5 concludes.

2 The Environment

Time is discrete and goes on forever. Each period is divided into two trading subperiods. In the first subperiod agents participate in a centralized Walrasian market where they can produce and consume any quantity of a single, homogenous consumption good. Then they enter a second frictional market where quantities of the same good are allocated via auctions. There is a continuum of anonymous infinitely lived agents who, following Rocheteau and Wright (2005), differ in terms of when they produce and consume the good. In the first subperiod, i.e. in the centralized market, all agents can produce and consume the good. In the second subperiod, i.e. during the auction market, agents are divided into buyers who want to consume the good but cannot produce it, and sellers who want to produce the good but cannot consume it. This assumption generates a temporal double coincidence problem. Combined with the assumption that the good is perishable (no commodity money) and that agents are anonymous (no credit), this ensures money is essential (Kocherlakota (1998), Wallace, (2001)). The number of sellers in this economy is fixed and equal to s while the number of buyers, noted b, is endogenous and determined by a free-entry condition with outside option k. This k can be interpreted as the opportunity cost of gathering information about the good for sale and the auctions organized by sellers. Since we consider large markets, both b and s are infinite, but the ratio $\theta = b/s$ is finite.

Money in this economy is a perfectly divisible and storable object whose value relies on its use as a medium of exchange. It available in quantity M_t at time t, and can be stored in any non negative quantity m_t by any agent. New money is injected or withdrawn via lump-sum transfers by the central bank at rate τ such that $M_{t+1} = (1 + \tau) M_t$ and only buyers receive this transfer. In these models, inflation is perfectly forecast and both the quantity theory and the Fisher effect apply: if the money growth rate increases at rate μ, so does inflation and the nominal interest rate given by $i = r + \mu$ where r is the real interest rate. The effect of increasing the money supply is then primarily to reduce agents real money holdings via the inflation tax.

Although the timing of events is discussed in detail for each model, a typical trading round will be the following. In the model with quantity posting and money price bidding, each seller advertise a quantity q of his production goods that he wishes to sell. The final price will be determined by an auction. In the model with price posting and quantity bidding, each seller advertise a price d for his production good and let the quantity produced and traded be determined by an auction. In both cases sellers advertise a fee δ, either positive or negative, that applies to each buyer taking part into his auction. If the fee is positive, the buyer must pay to bid at this auction. If it is negative, the seller pays δ to each buyer participating to his auction. Observing all posted auctions, buyers decide which auction to participate and how much money to bring to the auction market. Finally, all agents proceed to the auction market where buyers place their bid and the winner consumes the good. At the end of the auction market, all agents proceed to the next period Walrasian market where sellers spend any money earned in the auction market, and where buyers produce and sell the good in exchange for the money needed for the next auction market.[3]

[3] There are various types of ascending-bid auctions. We will use second-price auctions because they imply a unique optimal bidding strategy for buyers (Riley and Samuelson, 1981) and are therefore easier to work with. In second-price auctions the seller sells the good to the buyer who makes the final and highest bid and the winner pays a price that corresponds to the second highest bid. Note that our model differs from the multi-unit auction

Whether buyers bid prices or quantities, we assume that there exists an auction house, such as an internet auction website, that reports all the information posted by sellers. Given the information advertised by the sellers, once a buyer has decided which auction site he is going to participate, he pays the corresponding amount of money via the website, which transfers it to the seller. If sellers advertise a negative fee–that is sellers subsidize buyers participating to their auction–then sellers pay the fee to the auction house which transfers it to buyers. We assume that the payment of the auction fees are via the website are organized at the end of the first market, that is *before* the auction market opens.[4] Finally we assume that any buyer who got paid by sellers via the auction house but did not go to his chosen auction is banned permanently from the economy–hence no commitment issues.

Using β to denote the discount factor between the Walrasian and the auction market, a buyer maximizes $\sum_{t=0}^{\infty} U_t^b$ where the per period utility of a buyer is given by

$$U_t^b = x_t + \beta u(q_t).$$

The quantity x_t corresponds to the net utility of consuming and producing x_t units of the good in the Walrasian market and $u(q_t)$ is the utility of consuming q_t units of the good in the auction market.[5] Similarly a seller maximizes U_t^s where the per period utility of a seller is given by

$$U_t^s = x_t - \beta c(q_t)$$

where $c(q_t)$ is the disutility of producing q_t units of the good in the auction market. Note that, in the auction market, buyers have no production costs and sellers do not enjoy any utility. Both

studied by Hansen (1988). Here it is the size (or quality) of the unique good that is chosen by the seller.

[4] An alternative to the auction house would be to allow buyers and sellers to pay the auction fees at the opening or the closing of each auction. This is interesting because it opens the possibility for sellers to carry cash in case they post a negative fee. This creates several difficulties however. For instance, since the number of buyers per seller is stochastic, some sellers may not be able to pay the advertised fee to all the buyers showing up if they did not bring enough money. Note also that because fees are settled on the Walrasian market, this means that money is essential with regards to the working of the auction market, but it is not with regards to the payment of fees.

[5] To eliminate the relevance of trading histories, all we need is quasi-linearity in either production costs or utility in the Walrasian market (cf. Lagos and Wright 2005). Here we assume linearity in both via x_t without loss of generality. Models in which trading history matters can be solved numerically (Molico, 2006; Dressler, 2008) but are not necessary for the issues examined here.

u and c are common knowledge and identical across agents. That is, buyers are homogenous in preferences (they all have the same valuation for the good) and sellers are homogenous in their production costs. In the centralized market the nominal price of the good is normalized to \$1 and it is the price of money ϕ_t (1 unit of money buys ϕ_t units of the general good) that will adjust to market conditions. We make standard concavity and convexity assumptions for u and c, and let q^* denote the quantity that maximizes the trade surplus in a frictionless market, that is: $u'(q^*) = c'(q^*)$. We also denote $\hat{q} > 0$ as the quantity such that $u(\hat{q}) = c(\hat{q})$. Also, as is standard, we assume that $c'(0) = 0$ and $u'(0) > 0$.

3 Quantity Posting and Money Price Bidding

In this section sellers post quantities and buyers bid using money prices. The sequence of events is as follows: first, all buyers receive the money injection from the central bank, regardless of whether they participate in this economy or not. Then buyers make their entry decisions, given the outside option k. Once sellers have observed the number of buyers, the Walrasian market opens and sellers publicly announce a quantity q to be auctioned in the coming auction market, a corresponding reserve price r and a fee δ for participating to their auction. On the basis of the posted terms of trade, buyers decide how much money they will bring to the auction market and which seller to visit. Then, depending on whether the fee is positive or negative, buyers (resp. sellers) pay the auctions fees to the auction house which transfers them to sellers (resp. buyers) who spend it on the Walrasian market. As counterpart buyers receive an entrance ticket which enables them to bid at the coming auction. Finally, buyers and sellers proceed to the auction market where buyers go to the auction they have selected using their entrance ticket. Buyers submit their bids and the good goes to the buyer that bids the most, who pays the price of the second-highest bid. If a buyer is alone at one seller's auction post, he pays a price equal to the seller's reservation value. At the end of the auction market, buyers and sellers proceed to the next period Walrasian market.[6]

[6] Except for the fee and the monetary side, this sequence is similar to the second model studied by Peters and Severinov (1997): buyers learn their valuations *before* they choose among available auctions, and buyers make

A strategy for a seller is a posted q, a reserve price r and a fee δ he will post for each level of entry by buyers. A strategy for a (participating) buyer is a rule that specifies his money holding and the probability with which he chooses a particular seller as a function of the quantity, the reserve price and the fee (q, r, δ) posted by sellers. We focus on symmetric equilibria: sellers post the same expected terms of trade and buyers follow the same decision rule. (Symmetry is a natural outcome in large markets with the realistic properties that each seller receives a random number of buyers and that buyers are indifferent between all sellers.)[7]

Rational expectations play an important role in this economy. Buyers have to correctly forecast the quantity, reserve price and fee posted by sellers, and the resulting number of buyers who will enter the economy. Sellers have to correctly forecast how entry by buyers will react to changes in the advertised auction. Last but not least, both buyers and sellers have to correctly anticipate the distribution of money holdings that will result from the posted terms of trade and entry by buyers.

Before proceeding to building the value functions, we answer a few questions that may help understand the changes brought by money into the competing auction model. First, what does the distribution of money holdings look like? Buyers have the same valuation for the good–they have the same utility function $u(q)$, so what matters for the auction is how much money they hold. Consider the following scenario: suppose all buyers bring the same amount of money to the auction market. If one agent deviates and brings an additional dollar, he wins the auction with probability 1 at a negligible marginal cost. Because each buyer anticipates this, there is no focal point for buyers when it comes to deciding on their money holdings. The trade-off between the additional marginal cost and the discrete yet stochastic increase in gains from trade makes the distribution of money holdings across buyers non-degenerate (Galenianos and

their entry decision *before* they learn their valuation, which in our model corresponds effectively to the amount of money they bring to the auction market. Many interesting variations of this model can be explored, such as having buyers pick their money holding after they choose among sellers as in Peters and Severinov's first model, or allowing entry on the seller's side rather than the buyer's side, as in more standard in the money literature (e.g. Rocheteau and Wright, 2005).

[7] Asymmetric Nash equilibria with directed search can be constructed in small markets. See, for example, Burdett, Shi, and Wright (2001) or Coles and Eeckhout (2001).

Kircher, 2008).[8] We characterize this distribution in section 3.2 as a function of the posted quantity q, the buyer-seller ratio θ, and the nominal interest rate i.

Second, what are the variables under the seller's control when he designs an auction? In the standard non-monetary competing auction model, sellers extract surplus from buyers by changing the reserve price. A strategy for a seller is a then rule that specifies the reserve price for each level of entry. Increasing the reserve price rules out low bids at the cost of decreasing the expected number of buyers. In the context of infinitely many buyers and sellers, as we have here, when a seller raises his reserve price, he loses buyers with low valuations, yet he can no longer extract more surplus from the remaining buyers now that buyers can choose among many other sellers. That is, the reserve price is driven down to the production cost in limit economies (McAfee, 1993; Peters and Severinov 1997). This means that in our context sellers will effectively compete by ways of quantity announcements q, entry levels θ and fees δ only. Once the seller has posted his contract, buyers can infer the reserve price via the seller's cost function (which is common knowledge). The posted q, θ and δ are sufficient information for buyers to compute their probability to win the auction and therefore their expected trade surplus from participating in this economy. In equilibrium sellers must be able to foresee those relationships correctly.

Finally, what role does the buyer-seller ratio play in our economy? In the competitive search monetary equilibrium studied by Rocheteau and Wright (2005), sellers attract buyers by means of a posted *expected* surplus. When advertising a quantity q and a price d to which they commit, sellers also consider the probability with which a buyer will effectively trade at (q, d) which is why the queue length is a choice variable for sellers. In our model auctioneers also advertise expected terms, the difference being that the price has still to be determined by the auction. Sellers just need to pick the optimal combination of (q, θ, δ) that maximizes their expected payoff subject to the condition that buyers' expected surplus from trade is no smaller than their outside option.

[8] In standard directed search or bargaining models as in Rocheteau and Wright (2005) there is only one possible price expected by buyers and therefore there is a dominant strategy when it comes to choosing how much cash to hold.

3.1 The Value Functions

Let $W^b(m)$ and $V^b(m)$ be the value functions for a buyer holding m units of money in the centralized and auction markets respectively. We have

$$\begin{aligned} W^b(m) &= \max_{x,\hat{m}} \left\{ x + \beta V^b(\hat{m}) \right\} \\ \text{s.t.} \quad \phi\hat{m} + x &= \phi\,(m + T + \delta) \end{aligned}$$

where $\hat{m}$ corresponds to the money carried from the centralized market to the auction market, x is the net consumption of the good in the centralized market and δ is the participation fee paid to the auction house. In this program ϕ corresponds to the value of money in terms of the good and T corresponds to how many units of money per buyer are injected by the central bank each period. This program says that when choosing a quantity of the good to consume and produce in the Walrasian market, x, and a quantity of money to bring to the auction market, $\hat{m}$, buyers take into account that the combined real value of these two quantities must be equal to what they brought to this market, received from the central bank and received from (or paid to) the seller via the auction house. Substituting out x, this program can be rewritten

$$W^b(m) = \phi\,(m + T + \delta) + \max_{\hat{m}} \left\{ -\phi\hat{m} + \beta V^b(\hat{m}) \right\}. \tag{1}$$

Sellers, on the other hand, have no reason to bring money to the auction market. Since, by assumption, they do not receive any transfer of money from the central bank, their program is

$$\begin{aligned} W^s(m) &= \max_{x,q,r,\theta,\delta} \left\{ x + \beta V^s \right\} \\ \text{s.t.} \quad x + \phi\Phi\,(\delta) &= \phi m \end{aligned}$$

The seller's problem is to choose net consumption x in the Walrasian market, a posted quantity q, a reserve price r, a queue length θ and a participation fee δ for the auction market that maximizes his payoff. The budget constraint above says that the money collected from the previous auction market must cover his consumption of general good x and the expected payment to the auction house measure in real terms, $\phi\Phi\,(\delta)$. In large markets, when buyers learn their valuation before choosing an auction as they do here, the reserve price is driven to the production cost (McAfee 1993, Peters and Severinov 1997, Hernando-Veciana 2005). Using this and substituting out x we obtain

$$W^s(m) = \phi m + \max_{q,\theta,\delta} \left\{ -\phi\Phi\left(\delta\right) + \beta V^s \right\}. \tag{2}$$

We use $V^b\left(m\right)$ to denote the value function of a buyer at the opening of the auction market, holding m units of money each worth ϕ_{+1} units of the good in the next period Walrasian market. Letting $V_n^b(m)$ be the same value function when the buyer faces exactly n competitors, we have

$$V^b(m) = \sum_{n\in\mathbb{N}} P\left[X = n\right] V_n^b\left(m\right) - k \tag{3}$$

where k is the buyer's outside option, X is the random variable equal to the number of competing buyers showing up at the seller's shop and $P\left[X = n\right]$ is the probability measure associated with the event $X = n$. The variable X takes values into $\mathbb{N}$ and follows a Poisson process with parameter $\theta = b/s$ so that

$$P\left[X = n\right] = \frac{\theta^n}{n!} e^{-\theta}$$

and

$$\sum_{n\in\mathbb{N}} P\left[X = n\right] = 1.$$

We use μ to denote the random variable equal to how many units of money are held by one competitor, and $F(m) = P\left[\mu < m\right]$ to denote the probability measure associated with the event $\mu < m$. Let $f(m)$ be the corresponding density so that

$$\int_{m\in S'} f(m)dm = 1,$$

the support of which is noted $S' = [\underline{m}, \bar{m}]$ which we define shortly. Finally we note $S = [\underline{m}, m] \subseteq S'$ and assume that F is continuous (this will become clear shortly).

A buyer holding m units of money facing n competitors wins the auction if he holds the highest money holding, which is distributed according to $F^n(m)$ with density $nF^{n-1}\left(m\right) f(m)$; with probability $1 - F^n(m)$ he does not win the auction. We can now compute $V_n^b(m)$ the value function of a buyer holding m units of money, bidding for q units of goods and meeting n competitors. Using z to denote the number of units of money spent if he wins the auctions, this value function is given by

$$V_n^b(m) = \int_{z \in S} \left\{ u(q) + W_{+1}^b \left(m - z\right) \right\} dF^n(z) + \left[1 - F^n(m)\right] W_{+1}^b(m). \tag{4}$$

The first term corresponds to expected payoff to winning the auction. It is equal to the probability that all n competitors have less money than he has, multiplied by the payoff to consuming q units of the good and moving to the centralized market with $m - z$ units of money; we then sum over the quantity of money spent, z, which takes value from the lowest money holding $\underline{m}$ and a quantity of money marginally smaller than the buyer's own money holding m, denoted $m - \varepsilon$. Since F is continuous by assumption, it is continuous to the left with $\lim_{\varepsilon \to 0} F(m - \varepsilon) = F(m)$ so that z takes values in S. The second term corresponds to the probability of not winning the auction, multiplied by the value of entering the centralized market with an unchanged amount of money m.

Using $\phi_{-1} = (1 + \tau)\, \phi$, letting $i = (1 - \beta + \tau) / \beta$ be the nominal interest rate and $\gamma = \phi\delta/\beta$ be the discounted real auction fee, inserting (3) and (4) into (1), eliminating non relevant constant terms and dividing by β, the buyer's program becomes

$$\max_m \chi(m) = -i\phi m + \gamma + \sum_{n \in \mathbb{N}} P\left[X = n\right] \left\{ u(q) F^n(m) - \phi \int_{z \in S} z dF^n(z) \right\} \tag{5}$$

Equation (5) says that the buyer chooses his money holdings m in order to maximize the sum of the auction fee, γ, plus the expected net gain of winning the auction, minus the cost of carrying that amount of money, $i\phi m$. This expected net gain is composed of the utility of consuming q multiplied by the probability of winning minus the expected payment associated with holding m units of money.

As for sellers, because the winner pays the amount of the second richest bidder, we need first to characterize the distribution of the second highest money holding among the n buyers whose cash holding is distributed according to F. Noting $x_{(k)}$ the k^{th} order statistic, its density is given by

$$f_{x_{(k)}}(m) = n\binom{k-1}{n-1}F^{k-1}(m)\left[1-F(m)\right]^{n-k}f(m)$$

where $\binom{k-1}{n-1}$ corresponds to the number of $(k-1)$-combinations from $n-1$ elements. Setting $k = n-1$ in the above formula and remembering that sellers do not hold any money, the value function for the seller posting q and taking the value of money ϕ as given is

$$V^s = \sum_{n\in\mathbb{N}^*} P\left[X = n\right] \int\limits_{z\in S'} \left\{-c(q) + W^s_{+1}(z)\right\} f_{x_{(n-1)}}(z)dz. \tag{6}$$

Inserting (6) into (2), eliminating constant terms and dividing by β, the seller's objective can be rewritten

$$\max_{q,\theta,\delta} -\left(1-e^{-\theta}\right)c(q) - \Phi\left(\gamma\right) + \phi_{+1}\sum_{n\in\mathbb{N}^*} P\left[X = n\right]\int\limits_{z\in S'} zf_{x_{(n-1)}}(z)dz. \tag{7}$$

A seller (or auctioneer) maximizes the difference between the cost of producing the posted q (in all cases but when there is no buyer, that is production is on demand) and the expected return of selling this q via a second-price auction. Since the chosen q affects the distribution of money holdings by buyers F, the choice of q affects the right-hand side of (7) via $f_{x_{(n-1)}}(z)$.

3.2 The Distribution of Cash Holdings by Buyers

To solve for the amount of money that buyers bring to the auction market, we take the first order condition of the buyer's program in equation (5). Ignoring subscripts and taking q and ϕ as given, we obtain

$$i\phi = \left[u(q) - \phi m\right] f(m)\theta e^{-\theta[1-F(m)]} \tag{8}$$

which equalizes the marginal cost of an additional dollar to its expected marginal return.[9] Rearranging and integrating this expression over $S \subseteq S'$ gives the distribution of money holdings among buyers. It is a function of the price of money ϕ, the quantity q posted by sellers, the

[9] $f(m)\theta e^{-\theta[1-F(m)]}$ is the density of the cdf $\sum\limits_{n\in\mathbb{N}} \frac{\theta^n}{n!}e^{-\theta}F^n(m) \to e^{-\theta[1-F(m)]}$ which gives the probability that a buyer wins the auction in large markets.

buyer-seller ratio θ, the nominal interest rate i and the lower support of the distribution $\underline{m}$:

$$F(m) = \frac{1}{\theta} \ln \left\{ 1 - ie^{\theta} \ln \left[\frac{u(q) - \phi m}{u(q) - \phi \underline{m}} \right] \right\}.$$

To find $\underline{m}$ note that the seller is indifferent between producing q for $\underline{m}$ and doing nothing such that $-c(q) + W^s(\underline{m}) = W^s(0)$ from which we extract $\underline{m} = \frac{c(q)}{\phi}$ using the linearity of W. It is easy to show that $F(\underline{m}) = 0$ and that $F(\bar{m}) = 1$ implies

$$\bar{m} = \frac{u(q) - e^{-\frac{1-e^{-\theta}}{i}} [u(q) - c(q)]}{\phi}$$

so that

$$S^{'} = \frac{1}{\phi} \left[c(q), u(q) - e^{-\frac{1-e^{-\theta}}{i}} [u(q) - c(q)] \right]$$

and

$$F(m) = \frac{1}{\theta} \ln \left\{ 1 - ie^{\theta} \ln \left[\frac{u(q) - \phi m}{u(q) - c(q)} \right] \right\} \tag{9}$$

which is continuous over $S^{'}$.

Lemma 1 *The random variable X with cdf F take value into $\left[\frac{c(q)}{\phi}, \frac{u(q)}{\phi} \right]$. As $i \to 0$, $F(m) \to 0$ for any $m < \bar{m}$ and $\bar{m} \to u(q)/\phi$. As $i \to \infty$, $F(m) \to 1$ for any $m > \underline{m}$ and $\bar{m} \to c(q)/\phi$.*

3.3 Equilibrium

We can now write the representative seller's program. By contrast to the case of price posting with directed search examined in Rocheteau and Wright (2005), the representative buyer's net gain from trade, taken by sellers to be no smaller than his outside option, is now a random variable. We get around this difficulty by taking the expected value of the representative buyer's net gain from trade and impose the condition that it is no smaller than buyers' outside option. Therefore a seller maximizes his expected payoff subject to the buyers' expected payoff in utility terms being at least equal to k, the buyer's outside option. The buyer's payoff before deciding to enter the economy is just the expectation of the payoff that the buyer would get conditional on his money holding if he enters. Recalling from (5) that $\chi(m)$ is the net surplus of a representative buyer, the seller's problem becomes

$$\max_{q>0,\theta>0,\delta\in\mathbb{R}} -\left(1-e^{-\theta}\right)c(q) - \Phi(\gamma) + \phi_{+1}\sum_{n\in\mathbb{N}^*} P[X=n] \int_{z\in S'} z f_{x_{(n-1)}}(z)dz \tag{10}$$

$$\text{s.t.} \quad E[\chi(m)] = \int_{m\in S'} \chi(m)dF(m) \geq k. \tag{11}$$

Denoting $v(z) = \frac{1-F(z)}{f(z)}$ the difference between the first order statistic and the second order statistic, we have the following lemma

Lemma 2 *The seller's program (10)-(11) simplifies into*

$$\max_{q>0,\theta>0,\delta\in\mathbb{R}} -\left(1-e^{-\theta}\right)c(q) - \Phi(\gamma) + \phi_{+1}\int_{z\in S'} [z-v(z)]\,\theta f(z)e^{-\theta[1-F(z)]}dz \tag{12}$$

$$s.t. \quad \gamma - i\phi\int_{m\in S'} mdF(m) + \frac{1-e^{-\theta}}{\theta}u(q) - \phi\int_{z\in S'} [z-v(z)]\,f(z)e^{-\theta[1-F(z)]}dz \geq k, \tag{13}$$

Proof. See Appendix. ■

The seller maximizes the sum of the disutility of producing the posted quantity q when at least one buyer shows up, the fee that he pays to (or receive from) the auction house and the expected payment in real terms coming from the auction. This maximization is subject to the constraint that the average buyer gets at least his outside option. The first term inside the integral in the seller's payoff, $z - v(z)$, corresponds to Myerson's (1981) virtual valuation of a buyer holding z units of money. It is the difference between the money held by the buyer z and the buyer's rent, which is the difference between the first order statistic and the second order statistic $v(z)$. The term $\theta f(z)e^{-\theta[1-F(z)]}$ corresponds to the probability that a buyer wins the auction (see footnote 7). The last thing to do is to sum over all possible z to compute the expected value. The constraint corresponds to the average buyer's indifference condition between his outside option payment k and the his net gain from taking part into the auctions. By average we mean that the distribution of money holdings has been erased via the expectation operator. The first two terms measure the payment from (or to) the auction house and the disutility of holding the average amount of money. The second term corresponds to the utility he gets from consuming the good multiplied by the probability that a buyer gets served in directed search. Indeed, if the distribution of money holdings has been erased–that is, it is equivalent to all buyers holding the same amount of money, then the auction environment is equivalent to price posting with directed search. Finally, note that given the symmetry of the auction rule and the symmetry of the equilibrium, expected seller revenue is just b times the expected payment by buyers divided by the number of sellers s.

Definition 1 *When buyers bid prices, a* ***competing auction monetary equilibrium*** *is a list* $(q, \theta, \delta, \phi) \in Q \times \Theta \times \mathbb{R} \times \mathbb{R}_+$ *and a distribution of money holdings* $F \in \mathcal{F}$ *such that:*

(i) : *[Profit maximization] Sellers maximize (12) subject to (13)*;

(ii) : *[Rational Expectations]*

· *Buyers' and sellers' beliefs about the relationship between the posted* (q, δ) *and buyers' entry is correct*

· $\Phi(\gamma) = \sum_{n \in \mathbb{N}} P[X = n] n\gamma = \theta\gamma$

(iii) : *[Free entry] Equation (13) is binding due to free-entry on the buyer's side.*

Using (13) to substitute γ into (12), the seller's program becomes

$$\max_{q>0,\theta>0} \left(1 - e^{-\theta}\right)[u(q) - c(q)] - \theta \left[k + i\phi \int_{m \in S'} m dF(m)\right] \tag{14}$$

The seller effectively maximizes the net surplus of the auction in an average trade. This net surplus is made of the sum of the total gross surplus from trade in meetings where one or more buyers show up, $\left(1 - e^{-\theta}\right)[u(q) - c(q)]$, minus the sum of the buyer's opportunity cost and cost of holding the average amount of cash, $k + i\phi \int_{m \in S'} m dF(m)$, multiplied by the average number of buyers per auction, θ. Thanks to the fee, sellers can concentrate on maximizing total gains from trade playing with q and θ. They compete against each other by redistributing the surplus via the fee they charge (or the payment they make) to buyers depending on the value of the nominal interest rate (see section 3.4 below).

The following Lemma will help us characterize some properties of the equilibrium.

Lemma 3 *The seller's program is independent of the price of money ϕ and is given by*

$$\max_{q>0,\theta>0} \left(1-e^{-\theta}\right)[u(q)-c(q)] - \theta k - i\theta u(q) - i\,[u(q)-c(q)] \int_1^{e^\theta} e^{\frac{1-v}{i.e^\theta}} \frac{dv}{v} \tag{15}$$

Proof. See Appendix. ■

Despite the complexity of the auction environment, Lemma 2 shows that terms of trade are formed in real terms as in the other pricing mechanisms examined in Rocheteau and Wright (2005). That is, the price of money adjusts in the end via the clearing condition on the money market and does not affect the seller's posting policy. Money is neutral. It is not superneutral as will be clear shortly.

Proposition 1 *There exists an $\bar{\imath}_P$ such that an equilibrium exists for $i < \bar{\imath}_P$. At the Friedman rule sellers post the efficient quantity q^*, charge no fee ($\gamma^* = 0$) and the buyer-seller ratio is $\theta^* = -\ln \frac{k}{u(q^*)-c(q^*)}$. Sellers would post the same (q^*, θ^*) in a model in which sellers are not allowed to charge (or pay) a fee.*

Uniqueness and comparative statics results are obtained via numerical simulation.

3.4 Comparative statics

In this section we perform simple numerical comparative statics to study the impact of inflation on the equilibrium allocation. Especially, we want to know how inflation impacts on the quantity and participation fee posted by sellers, and on entry by buyers. We also compare the allocation with that of an economy in which sellers do not advertise any fee.[10] See Figure 1.

We start with a deflation rate equal to the real rate of interest (assumed equal to 3%) so that the nominal interest rate is zero. This is the Friedman rule. We then increase inflation until the equilibrium unravels. As noted in section 3, the Friedman rule induces sellers to advertise the efficient quantity q^* and they do not charge any fee. As inflation increases, the quantity posted and the buyer-seller ratio decrease. As for the fee, it increases as inflation moves away

[10] We set $u(q) = \sqrt{q}$ and $c(q) = q$. Results are robust to alternative specifications of these two functions. The Mathematica code can be found at:

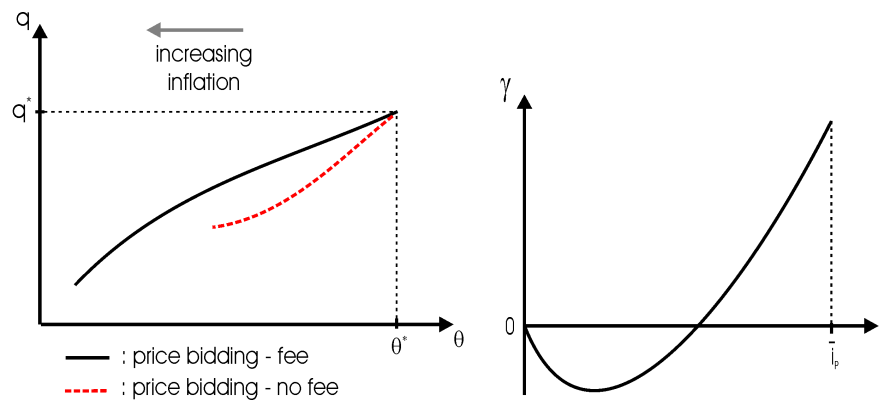

Figure 1: The effect of inflation on q, θ and γ.

from the Friedman rule to reach a maximum from which it decreases to finally become positive as inflation becomes very high.

At the Friedman rule, sellers post the efficient quantity. As inflation increases, buyers bring less real balances due to the now positive nominal interest rate. This forces sellers to reduce the quantity they put on auction reducing entry by buyers. On the other side cash holdings among buyers are now disperse, offering buyers the opportunity to gain some positive surplus in auctions in which they face competitors. This, by contrast, favors entry by buyers. In the end the net effect on entry is negative so that both the posted quantity and the buyer-seller ratio falls as inflation increases. When sellers are allowed to charge a fee, they can limit the fall in total surplus by charging the buyers participating to their auction rather than decreasing entry and the quantity posted. Without a fee sellers would have to reduce further these two variables.

3.5 Optimal monetary policy

Due to the free entry condition on the buyer's side, the buyers' expected payoff is unaffected by changes in monetary policy. When inflation is low the quantity of goods auctioned is higher yet there are more buyers around. In high inflation economies there is less good at each auction but also less buyers on average. That is, the equilibrium is constrained efficient by definition.

In this context a natural test for optimality in monetary policy is whether it maximizes the sellers' profit. This is known as *symmetric efficiency* (Peters and Severinov (1997)).

Proposition 2 *When sellers post quantities and buyers bid prices, the Friedman rule achieves symmetric efficiency where sellers post the efficient quantity whether or not they are allowed to post fees.*

This results is obtained though simulation and is robust to various specification of the utility function, cost function and value attributed to the outside option.

4 Money Price Posting and Quantity Bidding

This section investigates the mirror case in which sellers post dollar prices and buyers bid using quantities. That is, a bid from a buyer is a proposition of a quantity q required in exchange for the posted number of units of money, d. Buyers observe the posted quantities and the auction fee and, given expectations about where other buyers go, decide where to go. Frictions manifest themselves through ex-post market conditions, more buyers at one seller's translating into a smaller quantity for the winning buyer implying terms of trade dispersion as in the previous model.

A key difference with the previous section is that now buyers are no longer cash constrained in their bidding strategy. This has an important consequence. Whenever two or more buyers approach a seller (i.e., in multilateral meetings), because buyers have homogeneous preferences, the outcome of the auction is very simple: through the bidding process, the quantity falls until the terms of trade leave the winning buyer, chosen at random, indifferent between trading or not. We note this quantity q_m. Alternatively, whenever a seller is visited by only one buyer (i.e., in pairwise meetings), the terms of trade will be such that the seller is left indifferent between trading or not. We note this quantity q_p which we name reserve quantity.

The following timing sequence occurs. First, all buyers receive the money injection from the central bank, regardless of whether or not they participate in this economy. Buyers make their entry decisions with outside option k. Sellers announce a dollar price d for their production good to be auctioned, but do not indicate how much of this production good will be sold. They also advertise an auction fee δ and a reserve quantity q_p. Then all buyers observe the posted contracts, decide which auction to visit, and produce, trade, and consume in the centralized market. The auction house organizes the payment of the auction fees and buyers enter the auction market where they go to the auction they have selected and where sellers are bound by their posted dollar prices. Once buyers have allocated themselves to sellers, if two or more buyers compete, a buyer chosen at random wins the auction and receives q_m for d units of money. If the buyer is alone he gets $q_p > q_m$ with probability 1 and pays d.

The procedure for solving in the equilibrium terms of trade is identical to the previous section. We start with finite numbers of buyers and sellers, then turn the economy into a competitive economy by taking the limit of these numbers. The large market hypothesis implies that in the symmetric equilibrium no seller can affect the terms of trade (i.e., the market utility property holds). With rational expectations, buyers believe the market utility property applies and allocate themselves randomly across sellers.

4.1 The Value Functions

Let $W^b(m)$ and $V^b(m)$ be the value functions for a buyer holding m units of money in the centralized and decentralized markets, respectively. We have

$$W^b(m) = \max_{x,\hat{m}} \left\{ x + V^b(\hat{m}) \right\} \tag{16}$$

$$\text{s.t.} \quad \phi\hat{m} + x = \phi(m + T) + \phi\delta \tag{17}$$

as in the previous section. Using standard convergence properties of Binomial distributions, the probability for a seller of a pairwise match is $\xi_p = \theta e^{-\theta}$, of a multilateral match (at least two buyers are present) is $\xi_m = 1 - e^{-\theta} - \theta e^{-\theta}$ and of no buyer showing up is $1 - \xi_p - \xi_m$. Similarly, for a buyer, the probability of a pairwise match is $\psi_p = e^{-\theta}$, the probability of winning the auction in a multilateral match is $\psi_m = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\theta}$ and the probability of not winning the auction is $1-\psi_p-\psi_m$.[11] Using those probabilities, the Bellman equation for a buyer trading d units of money for q units of the special good is given by

$$\begin{aligned} V^b(m) &= \psi_p \left\{ u\left(q_p\right) + W^b_{+1}\left(m-d\right) \right\} + \psi_m \left\{ u\left(q_m\right) + W^b_{+1}\left(m-d\right) \right\} \\ &\quad + \left(1-\psi_p-\psi_m\right) W^b_{+1}(m) - k. \end{aligned} \tag{18}$$

With probability ψ_p a buyer is alone and trades with a seller, in which case he purchases and consumes q_p units of the good and enters tomorrow's centralized market with $m-d$ units of money. With probability ψ_m the buyer meets several other buyers but wins the auction, purchasing and consuming q_m and carrying on to the centralized market with $m-d$ units of money. In all other cases he proceeds to the centralized market with an unchanged amount of money. In all cases buyers have to pay k to participate. Note that by contrast to competition in prices, only the first two marginal buyers influence the terms of trade.

Turning now to sellers, they solve the following program

$$W^s(m) = \max_{x,d,q_p,\theta,\delta} x + \beta V^s \tag{19}$$

$$\text{s.t.} \quad x + \phi\Phi\left(\delta\right) = \phi m. \tag{20}$$

where $\Phi\left(\delta\right)$ is what sellers expect to pay to (or receive from) the auction house when posting a fee δ. Since sellers have no reason to carry money into the auction market, m is not a state variable for sellers in the next submarket and we have

$$\begin{aligned} V^s &= -\Phi\left(\gamma\right) + \xi_p \left\{ -c\left(q_p\right) + W^s_{+1}\left(d\right) \right\} + \xi_m \left\{ -c(q_m) + W^s_{+1}\left(d\right) \right\} \\ &\quad + \left(1-\xi_p-\xi_m\right) W^s_{+1}(0) \end{aligned} \tag{21}$$

with similar interpretation as (18).

4.2 Equilibrium

We note $z=\phi_{+1}d$ the real value of the posted price. We have the following Lemma.

[11] For instance, the probability for a buyer of getting served in a multilateral match is $\sum_{n\in\mathbb{N}^*} \frac{\theta^n}{n!} e^{-\theta} \frac{1}{n+1} = \frac{1}{\theta} \sum_{n\in\mathbb{N}^*} \frac{\theta^{n+1}}{(n+1)!} e^{-\theta} = \frac{1}{\theta} \sum_{n\geq 2} \frac{\theta}{n!} e^{-\theta} = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\theta}$

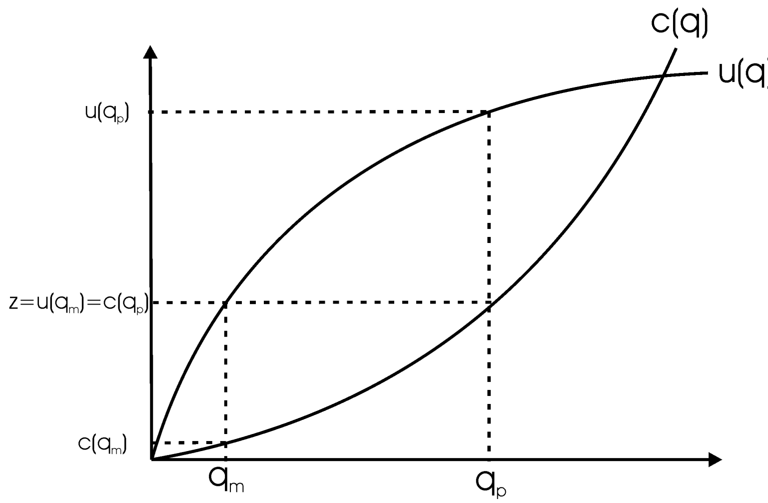

Figure 2: z, q_p and q_m

Lemma 4 *We have $z = c(q_p) = u(q_m)$ and the seller's program is given by*

$$\max_{z\geq 0,\theta\geq 0,\gamma\in\mathbb{R}} -\theta\gamma + \xi_m(\theta)\left\{-c\left[u^{-1}(z)\right] + z\right\} \tag{22}$$

$$s.t. \quad \gamma - iz + \psi_p(\theta)\left\{u\left[c^{-1}(z)\right] - z\right\} \geq k'. \tag{23}$$

where buyers bring exactly $m = d$.

Proof. See Appendix. ■

Effectively, sellers maximize the (expected) sum of the fees they receive from (or pay to) buyers via the auction house plus the surplus they get out of multilateral meetings, subject to the constraint that buyers' net gains from participating to this economy is equal to their outside option. By doing this sellers acknowledge that the real value of the posted price is equal to both their production cost in pairwise meetings and the buyer's utility in multilateral meetings. Note that buyers gain only in pairwise meetings.

Definition 2 *When buyers bid quantities, a* ***competing auction monetary equilibrium*** *is a list $(z, \theta, \delta, \phi) \in (0,1) \times \Theta \times \mathbb{R} \times \mathbb{R}_+$ such that:*

(i) : *[Profit maximization] Sellers maximize (22) subject to (23)*;

(ii) : *[Rational Expectations]*

$\cdot$ *Buyers' and sellers' beliefs about the relationship between the posted* (q, δ) *and buyers' entry is correct*

$\cdot$ $\Phi(\gamma) = \sum_{n \in \mathbb{N}} P[X = n] n\gamma = \theta\gamma$

(iii) : *[Free entry] Equation (23) is binding due to free-entry on the buyer's side.*

Extracting γ from the constraint, the sellers program becomes

$$\max_{z,\theta} \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left\{-c\left[u^{-1}(z)\right] + z\right\} + \theta e^{-\theta}\left\{u\left[c^{-1}(z)\right] - z\right\} - \theta\left[iz + k'\right]. \tag{24}$$

As in the previous model, the seller maximize the net surplus from an average trade. It is made of the gross surplus, i.e. the weighted average of the buyer's gain from trade in a pairwise match $\theta e^{-\theta}\left\{u\left[c^{-1}(z)\right] - z\right\}$ and the seller's gain from trade in a multilateral match $\left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left\{-c\left[u^{-1}(z)\right] + z\right\}$, minus the cost per buyer $iz + k$ multiplied by the average number of buyers at one auction.

Proposition 3 *There exists an* $\bar{\imath}_Q$ *such that an equilibrium exists for* $i < \bar{\imath}_Q$. *At the Friedman rule agents trade* $q_m < q^*$ *in multilateral matches,* $q_p > q^*$ *in pairwise matches and seller subsidize buyers by paying them* $\gamma = \theta e^{-\theta}\left\{[z - c(q_m)] - [u(q_p) - z]\right\} > 0$.

Proof. See Appendix ■

4.3 Comparative statics

Simulation shows that when one increases inflation, the posted price and the buyer-seller ratio decrease. As for the fee, by contrast to the previous model, it is positive at the Friedman rule then decreases to become negative. See Figure 3.

Lemma 5 (i) : *Sellers do not post the same terms of trade at the Friedman rule whether they are allowed to charge a fee or not.*

(ii) : *The use of a fee enables the economy to sustain higher rates of inflation.*

Proof. See Appendix. ■

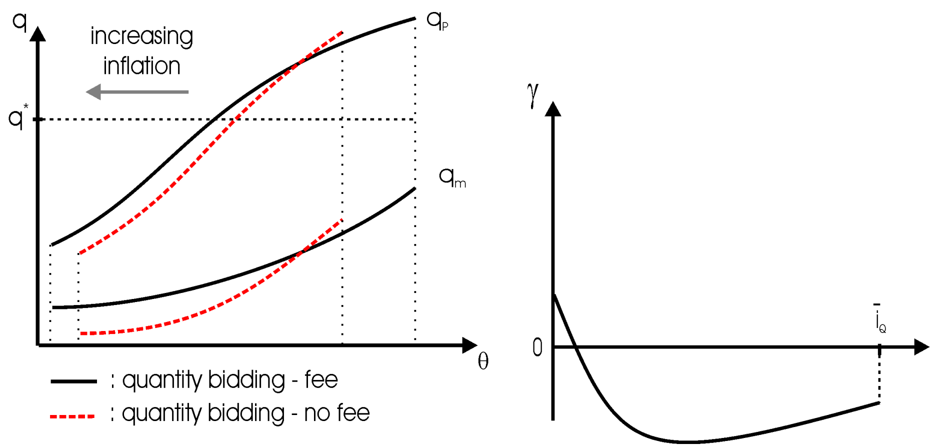

Figure 3: The effect of inflation on q_p, q_m, θ and γ.

4.4 Optimal Monetary Policy

In this section we study how inflation impacts the posted terms of trade and the allocation. We derive the equilibrium fee charged by the seller and statements about optimal monetary policy. The analysis is conducted using numerical simulation.

Proposition 4 *When sellers post prices and buyers bid quantities, the Friedman rule achieves symmetric efficiency.*

5 Conclusion

This paper considers the impact of inflation on markets in which goods are allocated via competing auctions. The economy is monetary: buyers decide how much money to bring to the auction market trading off the cost of holding money against the expected gain from participating to an auction. Search is directed: buyers choose among auctioneers trading off the posted quantity against the probability of winning the good in that auction. Finally the economy is competitive: auctioneers attract buyers trading off the production cost of the advertised quantity against the expected number of potential buyers. We have studied two versions of the model, one in which sellers post a quantity and buyers bid prices, and the mirror case in which sellers post a price and buyers bid quantities.

One interesting by-product of introducing money in the competing auctions framework is that the quantity produced becomes a natural strategic variable. In a non-monetary model the absolute quantity q of the good auctioned plays no role and is conveniently normalized to 1. Now that buyers trade off the cost of real balances with the probability to win the auction, since real balances are proportionate to the advertised quantity, a bigger q will not only increase the seller's production cost but also the cost of holding money for buyers. This can be seen by looking at (9).

In the conclusion of their paper, Peters and Severinov (1997) acknowledge that "it is difficult to think of markets in which sellers compete in reserve prices" in the manner they describe, but hope that "the relatively simple limit equilibrium [they introduce] will make it possible to discover the transaction costs that are missing from [their story]". Our research suggests that these missing transaction costs may be the costs of using money.

Appendix

A1. Proof of Lemma 2

The seller's objective and the buyer's constraint are given by

$$\max_{q>0,\theta>0,\delta\in\mathbb{R}} -\left(1-e^{-\theta}\right)c(q)-\theta\gamma+\phi\sum_{n\in\mathbb{N}^*}P\left[X=n\right]\int\limits_{z\in S'} zf_{x_{(n-1)}}(z)dz$$

$$\text{s.t.}\quad E\left[\chi(m)\right]=\gamma-i\phi\int\limits_{m\in S'} mdF(m)+$$

$$\sum_{n\in\mathbb{N}}P\left[X=n\right]\left\{u(q)\int\limits_{m\in S'}F^n(m)dF(m)-\phi\int\limits_{\underline{m}}^{\bar{m}}\int\limits_{\underline{m}}^{m} zdF^n(z)dF(m)\right\}\geq k,$$

The integral in the second term of the seller's objective corresponds to the expected gross revenue for the seller. Using the definition of $f_{x_{(n-1)}}$ it is equal to

$$\begin{aligned}
&\int\limits_{z\in S'} zn\left(n-1\right)F^{n-2}(z)\left[1-F(z)\right]f(z)dz\\
=\;& n\int\limits_{z\in S'} z\left[1-F(z)\right]dF^{n-1}(z)=n\left[z\left[1-F(z)\right]F^{n-1}(z)\right]_{\underline{m}}^{\bar{m}}-n\int\limits_{z\in S'}\left[1-zf(z)-F(z)\right]F^{n-1}(z)dz\\
=\;& n\int\limits_{z\in S'}\left[zf(z)+F(z)-1\right]F^{n-1}(z)dz.
\end{aligned}$$

Taking the sum over n of the above expression multiplied by $P\left[X=n\right]=\frac{\theta^n}{n!}e^{-\theta}$ we obtain

$$\begin{aligned}
&\int\limits_{z\in S'}\left[zf(z)+F(z)-1\right]\sum_{n\in\mathbb{N}*}\frac{\theta^n}{n!}e^{-\theta}nF^{n-1}(z)dz\\
=\;& e^{-\theta}\theta\int\limits_{z\in S'}\left[zf(z)+F(z)-1\right]\sum_{n\in\mathbb{N}^*}\frac{\theta^{n-1}}{(n-1)!}F^{n-1}(z)dz\\
=\;& \theta\int\limits_{z\in S'}\left[zf(z)+F(z)-1\right]e^{-\theta\left[1-F(z)\right]}dz.
\end{aligned}$$

Factorizing by $f(z)$ yields the expression in (12).

As for buyers,the second term simplifies according to:

$$\begin{aligned} & u(q) \sum_{n \in \mathbb{N}} P[X=n] \int_{m \in S'} F^n(m) dF(m) \\ = \; & u(q) \sum_{n \in \mathbb{N}} P[X=n] \left[\frac{F^{n+1}(m)}{n+1} \right]_{\underline{m}}^{\bar{m}} \\ = \; & u(q) \sum_{n \in \mathbb{N}} \frac{\theta^n}{(n+1)!} e^{-\theta} \\ = \; & u(q) \frac{1}{\theta} \sum_{n' \in \mathbb{N}^*} \frac{\theta^{n'}}{n'!} e^{-\theta} = u(q) \frac{1-e^{-\theta}}{\theta}. \end{aligned}$$

Finally, carefully reversing the order of integration, the double integral is the last term can be rewritten

$$\begin{aligned} & \int_{\underline{m}}^{\bar{m}} \int_{z}^{\bar{m}} z dF^n(z) dF(m) \\ = \; & \int_{\underline{m}}^{\bar{m}} z dF^n(z) [1 - F(z)] \\ = \; & \int_{z \in S'} [z f(z) + F(z) - 1] F^n(z) dz. \end{aligned}$$

It is straightforward to see that the sum over n of the above expression multiplied by $P[X=n]$ is equal to

$$\int_{s \in S'} [z f(z) + F(z) - 1] e^{-\theta[1-F(z)]} dz$$

which is the seller's expected gross revenue divided by θ. Factorizing by $f(z)$ yields the expression in (13).

A.2. Proof of Lemma 3

From (9), the density $f(m)$ is given by

$$\frac{\partial F(m)}{\partial m} = \frac{i e^{\theta} \phi}{\theta [u(q) - \phi m] \left[1 - i e^{\theta} \ln \left(\frac{u(q) - \phi m}{u(q) - c(q)} \right) \right]}.$$

We make the following change in variable, $v = 1 - ie^{\theta}\ln\left(\frac{u(q)-\phi m}{u(q)-c(q)}\right)$ so that $dv = \frac{ie^{\theta}\theta}{u(q)-\phi m}dm$ and $m = \frac{u(q)-e^{\frac{1-v}{ie^{\theta}}}[u(q)-c(q)]}{\phi}$. Introducing these values into (14), the last term in (14) transforms into

$$-i\theta u(q) + i\left[u(q)-c(q)\right]\int_{1}^{e^{\theta}} e^{\frac{1-v}{i.e^{\theta}}}\frac{dv}{v}$$

which has no ϕ in it.

A.3. Proof of Proposition 1

To begin, rewrite the seller's program

$$\Psi(i) = \max_{q\in([0,\hat{q}],\theta>0} g(q,\theta;i) \tag{25}$$

with

$$g(q,\theta;i) = \left(1-e^{-\theta}\right)\left[u(q)-c(q)\right] - \theta\left[k + i\phi\int_{m\in S'} mdF(m)\right]$$

Part 1: There is no interior solution for high i. First note that the support of F is $\left[\frac{c(q)}{\phi},\frac{u(q)}{\phi}\right]$. Since $\left(1-e^{-\theta}\right)\left[u(q)-c(q)\right]-\theta k$ is bounded there is no interior solution to (25) for large enough values of i. We note $I = (0,\bar{\imath}_P)$ the set of values for i such that an equilibrium may exist.

Part 2: Existence

Let us note $\alpha(\theta) = 1-e^{-\theta}$ the probability for a buyer of not being alone at an auction. We can invert α to define $\theta = \theta(\alpha)$ so that the seller's problem can be rewritten

$$\Psi(i) = \max_{q\in[0,\hat{q}],\alpha\in[0,1]} g(q,\theta(\alpha);i).$$

Since g is continuous over the finite set $[0,\hat{q}]\times[0,1]$ then for all i such that $\Psi(i)>0$ there exists a solution.

Part 3: The equilibrium at the Friedman rule.

$g(q,\theta;i=0)$ is strictly concave in q and θ so that the first order condition is necessary and sufficient. The first order condition with respect to q implies $u'(q) = c'(q)$ so that sellers post the efficient q^*. The first order condition with respect to θ yields $e^{-\theta}[u(q)-c(q)]=k$ so that

$$\theta^* = -\ln\left[\frac{k}{u(q^*)-c(q^*)}\right].$$

To see that no fee is charged at the Friedman rule, recall that when $i=0$ the distribution of cash holdings is degenerate and equal in real terms to $u(q)$. This implies that a buyer gets some positive surplus only when alone, in which case he pays the reserve price $c(q)$ and enjoys $u(q)$. As for the seller, he pays $c(q)$ only when he meets a buyer and get $c(q)$ in payment if there is only one buyer and $u(q)$ if there is at least two of them.[12] Algebraically the seller solves

$$\max_{q,\theta,\gamma} -\theta\gamma - \left(1-e^{-\theta}\right)c(q) + \theta e^{-\theta}c(q) + \left(1-e^{-\theta}-\theta e^{-\theta}\right)u(q) \tag{26}$$

$$\text{s.t.} \quad \gamma + e^{-\theta}[u(q)-c(q)] = k, \tag{27}$$

which simplifies into

$$\max_{q,\theta,\gamma} -\theta\gamma + \left(1-e^{-\theta}-\theta e^{-\theta}\right)[u(q)-c(q)] \tag{28}$$

$$\text{s.t.} \quad \gamma + e^{-\theta}[u(q)-c(q)] = k. \tag{29}$$

Substituting γ into the seller's objective, the first order condition with respect to θ yields

$$e^{-\theta}[u(q)-c(q)] = k$$

which from (29) implies $\gamma = 0$.

Finally, to see that sellers post the same q and θ whether allowed to charge fees or not, simply set $\gamma = 0$ into (28) and (29) and use (29) to plug θ into (28) he first order condition with respect to q gives $q = q^*$ which once inserted into (**??**) yields again $\theta^* = -\ln\left[\frac{k}{u(q^*)-c(q^*)}\right]$.

A.3. Numerical simulation

In general comparative statics results cannot be derived analytically so we proceed numerically. To do that, we run a numerical maximization algorithm using Mathematica, in which we

[12] This corresponds to Coles and Eeckout (2001)'s directed search model with contingent prices in which buyers extract all the surplus when alone and sellers extract all the surplus when two or more buyers show up.

vary the nominal interest rate from 0% until the equilibrium unravels and plot the outcome in the (q,θ) space. We also compute the value of the fee γ in real terms and plot as a function of the inflation rate it in the (i,γ) space. For each model, we run the algorithm when sellers are allowed to charge a fee and when they are not and compare the outcome.

A few intermediate steps are necessary.

Step 1: Simplifying the payoffs

Noting $\Gamma(a,z)=\int_z^\infty t^{a-1}e^{-t}dt$ the seller's simplified program is given by

$$\left(1-e^{-\theta}\right)[u(q)-c(q)]-\theta k-i\theta u(q)+i\,[u(q)-c(q)]\,e^{\frac{e^{-\theta}}{i}}\left[\Gamma\left(0,\frac{1}{i}\right)-\Gamma\left(0,\frac{e^{-\theta}}{i}\right)\right].$$

Step 2:Use the numerical maximization algorithm available at:

www.mngt.waikato.ac.nz/departments/staff/rdutu/index_files/DJK_algo.nb

Step 3: We have to set manually PrecisionGoal and MaxIteration to obtain sensible results.

A.4. Proof of Lemma 3

First it is easy to check from (16) that a buyer will bring an amount of money just enough to cover the posted price so that the distribution of cash holdings is degenerate and equal to z in real terms. If a buyer faces no competitor he is able to impose terms of trade that leave the seller indifferent between not trading or producing and trading q_p against z, i.e. $z=c(q_p)$. Similarly competition between two or more buyers leads to $z=u(q_m)$. Inserting (21) into (19), using the linearity of W^s and getting rid of constant terms, the seller's objective (19) becomes

$$\max_{z,\theta,\gamma}-\Phi(\gamma)+\beta\xi_p\left[-c(q_p)+z\right]+\xi_m\left[-c(q_m)+z\right]. \tag{30}$$

As for buyers, free entry implies that a buyer in the current period's centralized market is indifferent between saving $\hat{m}$ units of money for the coming auction market, or spending all his money now and proceeding directly to the next period's centralized market without money. Algebraically,

$$-\phi\hat{m}+\beta V^b(\hat{m})=W^b_{+1}(0). \tag{31}$$

Substituting x into (16) using (17), inserting (18) into (16), using the linearity of W^b and focussing on steady-state equilibria where real balances are constant, that is $\phi_{+1}(1+\tau)=\phi$,

(31) transforms into

$$\phi\delta/\beta - iz + \psi_p\left[u(q_p) - z\right] + \psi_m\left[u(q_m) - z\right] = k'. \tag{32}$$

Finally, inserting $z = c(q_p) = u(q_m)$ into (30) and (32) yields (22) and (23).

A.5. Proof of Proposition 3

The existence argument is the same as in the case of price bidding. Same for the threshold value on the nominal interest rate: gains from trade are bounded yet the cost of holding money is not so that the monetary equilibrium unravels for too high nominal interest rate. The first order condition with respect to z is

$$-i\theta + \theta e^{-\theta}\left[\frac{u'\left[c^{-1}(z)\right]}{c'\left[c^{-1}(z)\right]} - 1\right] + \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left[1 - \frac{c'\left[u^{-1}(z)\right]}{u'\left[u^{-1}(z)\right]}\right] = 0.$$

Using $u(q_m) = c(q_p) = z$ this gives

$$-i\theta + \theta e^{-\theta}\left[\frac{u'(q_p) - c'(q_p)}{c'(q_p)}\right] = \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left[\frac{c'(q_m) - u'(q_m)}{u'(q_m)}\right]. \tag{33}$$

Setting $i = 0$, the above equality imposes that $u'(q_p) - c'(q_p)$ is of the same sign as $c'(q_m) - u'(q_m)$. Since $q_p > q_m$ they must be negative implying $u'(q_p) < c'(q_p)$ and $c'(q_m) < u'(q_m)$ so that $q_m < q^* < q_p$.

The first order condition with respect to θ yields

$$-iz + e^{-\theta}\left\{u\left[c^{-1}(z)\right] - z\right\} + \theta e^{-\theta}\left\{\left[z - c(q_m)\right] - \left[u(q_p) - z\right]\right\} = k$$

which by identification yields the γ in Proposition 3.

A.6. Proof of Lemma 5

Part (i) : To show that sellers do not post the same price at the Friedman rule when not allowed to charge a fee, set $i = \gamma = 0$ in the seller's program

$$\max_{z \geq 0, \theta \geq 0} \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\left\{-c\left[u^{-1}(z)\right] + z\right\} \tag{34}$$

$$\text{s.t.} \quad e^{-\theta}\left\{u\left[c^{-1}(z)\right] - z\right\} = k'. \tag{35}$$

Using (35) to plug θ into (34), taking the first order condition with respect to z and using $u(q_m) = c(q_p) = z$ yields

$$\theta e^{-\theta}\left[\frac{u'(q_p) - c'(q_p)}{c'(q_p)}\right] = A\left[\frac{c'(q_m) - u'(q_m)}{u'(q_m)}\right]. \tag{36}$$

with

$$A = e^{-\theta}\left[u(q_p) - z\right]\frac{u(q_p) - z - k(1+\theta)}{k\left[z - c(q_m)\right]}.$$

Using (35) to replace $u(q_p) - z$ by $e^{\theta}k$ in the fraction above yields

$$A = e^{-\theta}\left[u(q_p) - z\right]\frac{k\left(e^{\theta} - 1 - \theta\right)}{k\left[z - c(q_m)\right]} \tag{37}$$

$$= \left(1 - e^{-\theta} - \theta e^{-\theta}\right)\frac{u(q_p) - z}{z - c(q_m)} \tag{38}$$

From (33), (36) and (38), for the first-order condition with respect to z to yield the same z with and without a fee requires $u(q_p) - z = z - c(q_m)$. Inserting this equality into the expression of the fee implies that $\gamma = 0$, a contradiction. Therefore sellers do not post the same price whether they are allowed to advertise a fee or not.

Part (ii) : For each pair (z, θ) let us note i_1 the maximum value for the nominal interest rate such that the seller's substituted profit with fee in (24) is non negative. For each pair (z, θ) let us note i_2 the maximum value for the nominal interest rate such that the buyer's participation constraint (23) in which we set $\gamma = 0$ is no smaller that the buyer's outside option k (no fee). A little algebra shows that

$$i_2 - i_1 = \frac{\xi_m(\theta)\left\{-c\left[u^{-1}(z)\right] + z\right\}}{z\theta} > 0.$$

Inflation decrease both z and θ. The inequality above shows that an economy with fee can sustain more inflation.

References

[1] Acemoglu, Daron and Robert Shimer (1999a). "Holdups and Efficiency with Search Frictions". *International Economic Review* 40(4), p. 827-849.

[2] Acemoglu, Daron and Robert Shimer (1999a). "Efficient Unemployment Insurance". *Journal of Political Economy* 107(5), p. 893-928.

[3] Berentsen Aleksander, Guido Menzio and Randall Wright (2008). "Inflation and Unemployment in the Long Run", *NBER* Working Paper #13924.

[4] Burdett, Kenneth, Shouyong Shi and Randall Wright (2001). "Pricing and Matching with Frictions", *Journal of Political Economy* 109 (5), 1060-1085.

[5] Burguet, Roberto and József Sákovics (1999). "Imperfect Competition in Auction Designs", *International Economic Review* 40(1), 231-247.

[6] Coles, Melvyn and Jan Eeckhout (2003). "Indeterminacy and Directed Search", Journal of Economic Theory 111, 265-276.

[7] Dressler, Scott (2008), "The distribution of money holdings in a perfectly competitive market", mimeo.

[8] Galenianos, Manolis and Philipp Kircher, (2008). "Directed Multilateral Matching in a Monetary Economy", *Journal of Monetary Economics*, forthcoming.

[9] Hansen, Robert (1988). "Auctions with Endogenous Quantity", *Rand Journal of Economics*, Spring.

[10] Hernando-Veciana, Ángel (2005). "Competition among auctioneers in large markets", *Journal of Economic Theory*, p. 107-127.

[11] Julien, Benoit, (1997) "Essays in Resource Allocation Under Informational Asymmetries", Unpublished PhD Dissertation, University of Western Ontario.

[12] Julien, Benoit, John Kennes and Ian King (2000). "Bidding for Labor", *Review of Economic Dynamics* 3(4), p. 619-649.

[13] Julien, Benoit, John Kennes and Ian King (2008). "Bidding for Money", *Journal of Economic Theory* (forthcoming)

[14] King, Ian (2003). "A Directed Tour of Search-Theoretic Explanations for Unemployment". *New Zealand Economics Papers* 37, p. 245-267.

[15] Kiyotaki, N. and R. Wright (1991), "A Contribution to the Pure Theory of Money", *Journal of Economic Theory* 53, 215-235.

[16] Kocherlakota, N.R. (1998), "Money is Memory", *Journal of Economic Theory* 81, 232-251.

[17] Kultti Klaus and Toni Riipinen (2003), "Multilateral and Bilateral Meetings with Production Heterogeneity", Finish Economic Papers 16(1), 27-37.

[18] Lagos, Ricardo and Randall Wright (2005). "A Unified Framework for Monetary Theory and Policy Analysis", *Journal of Political Economy* 113 (3), pp. 463-484.

[19] McAfee, Preston R. (1993). "Mechanism Design by Competing Sellers". *Econometrica* 61(6), 1281-1312.

[20] Moen, Espen R. (1997). "Competitive Search Equilibrium". *Journal of Political Economy* 105(2), p.385-411.

[21] Moldovanu, Benny, Aner Sela and Xianwen Shi (2008). "Competing Auctions with Endogenous Quantities", *Journal of Economic Theory* (forthcoming).

[22] Molico, Miguel (2006). "The Distribution of Money and Prices in Search Equilibrium", *International Economic Review* 47(3), p. 701-722.

[23] Montgomery, James (1991). "Equilibrium Wage Dispersion and Interindustry Wage Differentials", *Quarterly Journal of Economics* 106, 163-79.

[24] Peters, Michael (1997). "A Competitive Distribution of Auctions", *Review of Economic Studies* 64, 97-124.

[25] Peters, Michael (2000). "Limits of Exact Equilibria for Capacity Constrained Sellers with Costly Search". *Journal of Economic Theory* 95, 139-168.

[26] Peters Michael and Sergei Severinov (1997). "Competition among Sellers Who Offer Auctions Instead of Prices." *Journal of Economic Theory* 75, 141-179.

[27] Peters Michael and Sergei Severinov (2006). "Internet Auctions with Many Traders", Journal of Economic Theory 130, 220-245.

[28] Riley, J. and W. Samuelson (1981). "Optimal Auctions". *American Economic Review* 71, p. 381-392.

[29] Rocheteau, Guillaume and Randall Wright (2005). "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium", *Econometrica* 73, 175-202.

[30] Rogerson, Richard, Robert Shimer and Randall Wright (2005). "Search-Theoretic Models of the Labour Market: A Survey". *Journal of Economic Literature* 63, p. 959-988.

[31] Schmitz, Patrick W. (2003), "On Second-price Auctions and Imperfect Competition", *Journal of Mathematical Economics* 39, p.901-909.

[32] Shi, Shouyong, (1995) "Money and Prices: A Model of Search and Bargaining", *Journal of Economic Theory*, 67, 467-496.

[33] Stokey, Nancy L. and Robert E. Lucas, with Edward C. Prescott, 1989, *Recursive Methods in Economic Dynamics*. Cambridge, MA: Harvard University Press.

[34] Trejos, Alberto, and Randall Wright (1995) "Search, Bargaining, Money, and Prices", *Journal of Political Economy*, 103, 118-141.

[35] Wallace, Neil, (2001). "Whither Monetary Economics?", *International Economics Review* 42(4), 847-870.