Balanced-Budget Rule, distortionary taxes and Aggregate Instability: A Comment.

Aurélien Saidi

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Abstract

Sous les hypothèses de concurrence parfaite et de rendements d'échelle constants, Schmitt-Grohe et Uribe [1997] montrent qu'un modèle de croissance à un secteur peut laisser transparaître des phénomènes d'indétermination locale lorsque les taux d'imposition sur les revenus du travail sont établis de manière endogène et guidés par une règle de budget équilibré dans laquelle les dépenses publiques sont fixes. Cet article démontre que l'instabilité agrégée évoquée par les deux auteurs découle moins de l'indétermination locale d'un équilibre particulier que de la multiplicité des équilibres stationnaires. Ainsi, l'indétermination persiste alors même que chacun de ces équilibres apparaît localement déterminé. En passant d'une analyse locale à une analyse globale, nous étendons les conclusions de l'article originel à une plus grande variété de cas, parmi lesquels figurent la coexistence de deux équilibres de point-selle ou encore la possible connexion entre deux équilibres stationnaires au moyen d'une orbite hétéroclinique.

It has been shown that under perfect competition and constant returns-to-scale, a one-sector growth model may exhibit local indeterminacy when income tax rates are endogenously determined by a balanced-budget rule while government expenditures are fixed. This paper shows that the associated aggregate instability does not ensue from the local indeterminacy of a specific stationary equilibrium but from the multiplicity of the stationary equilibria and persists under local determinacy of all of them. We provide a global analysis of the Schmitt-Grohe and Uribe model [1997] and study specific cases that were not investigated in the original paper, when aggregate instability is inherited from the coexistence of two saddle-path equilibria on one hand and from the connection of the two steady states on the other hand.

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1 Introduction

It has been argued by Schmitt-Grohe and Uribe [1997] that a balanced-budget fiscal policy rule may induce aggregate instability (especially indeterminacy) by making the expectations of higher tax rates self-fulfilling if the fiscal authority relies heavily on changes in labor income taxes to eliminate short-run fiscal imbalances. In expansion periods, when agents expect an increase of their after-tax labor incomes they decide to work harder, forcing the fiscal authority to lower the tax rates (to maintain the same amount of government expenditures) as total output rises, fulfilling agents' initial optimistic expectations. Their conclusions, derived from a local analysis, are based on the topological stability of one of the two stationary equilibria, that seems to be consistent with the empirical estimates found by Mendoza et al. [1994].

Although the topological stability of the steady state (and then the multiplicity of equilibrium trajectories) relies on the specific form of the balanced-budget rule embodying a predetermined (and fixed) level of government expenditures, [1] the Schmitt-Grohe and Uribe formulation induces fiscal increasing returns to scale and matches stylized facts already suggested by some empirical works in the literature (Blanchard and Summers [1987], for instance).

This paper extends Schmitt-Grohe and Uribe's analysis by studying the global economic dynamics of their model. While they assume that the economy remains in the neighborhood of one of the two stationary equilibria and state that indeterminacy occurs for plausible values of the labor income tax rate, we do not affect to the capital stock any predetermined value. We then investigate the global dynamics of the system, including cases for which the stationary equilibrium they consider is not indeterminate. We prove that aggregate instability may occur also in such cases. Two relevant configurations must be noticed. From one hand, local indeterminacy of one stationary equilibrium may induce global indeterminacy, even though the other one appears locally determined. This is especially the case when both steady states are connected by an (*heteroclinic*) equilibrium orbit, diverging from the latter but converging to the former. From the other hand, the coexistence of two saddle-path equilibria for the same values of the capital stock is a source of indeterminacy while each of them exhibits local determinacy. These two specific configurations will be exposed in the paper.

Aggregate instability is then an intrinsic property of the specific form of the balanced-budget rule defined by Schmitt-Grohe and Uribe. This aggregate

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[1] Guo and Harrison [2004] show that the topological stability of the steady state disappears if the government finances endogenous public expenditures with fixed income tax rates. Under this type of balanced-budget formulation, the economy exhibits saddle-path stability.

instability may be caused by local indeterminacy of one of the stationary equilibria as suggested by the authors. However, the property is robust to wild changes in the parameter values, even when they do not lay within the range of local indeterminacy. In that sense, the configuration studied by Schmitt-Grohe and Uribe is a particular case of a more general pattern that we propose to investigate more extensively.

Furthermore, following the estimates of Basu and Fernald [1995, 1997], Burnside, Eichenbaum and Rebelo [1995] or Burnside [1997], we slightly modify the original model by integrating a small level of aggregate increasing returns to scale in the social production function, as in Benhabib and Farmer [1994]. This change does not affect the conclusions of the model. However, it will be shown that the higher the degree of increasing returns, the more likely the apparition of a heteroclinic orbit.

The remainder of the paper is organized as follows. Section 2 describes the model setup and equilibrium conditions. Section 3 constructs the phase diagram of the model. Section 4 considers the plausibility of heteroclinic orbits while section 5 investigates the cases in which two saddle-paths equilibria coexist. Finally, section 6 concludes.

2 Model Setup

The setup presented in this section merges the baseline models of Schmitt-Grohe and Uribe [1997] on one hand and Benhabib and Farmer [1994] on the other hand, combining the balanced-budget rule of the former with the aggregate increasing returns to scale of the latter in a simple one-sector model. At the private level, firms face a Cobb-Douglas production function with constant returns to scale:

$$Y(t) = A(t)K(t)^a L(t)^b \qquad 0 < a < 1 \ \text{ and } \ a + b = 1.$$

At time t, the state of technology $A(t)$ corresponds to a production externality depending on both the average economy-wide levels of capital $\bar{K}$ and labor $\bar{L}$ on which the private decisions of agents have no effect:

$$A(t) = \left(\bar{K}(t)^a \bar{L}(t)^b \right)^{\gamma} \qquad \text{with} \quad \gamma > 0.$$

However, in a symmetric equilibrium each firm rents the same amount of capital and hires the same amount of labor: $K(t) = \bar{K}(t)$ and $L(t) = \bar{L}(t)$. This implies increasing returns to scale in the aggregate level. In order to lighten the notations, redefine the parameters as $\alpha \equiv a(1 + \gamma)$, with $\alpha < 1$ by

assumption, and $\beta \equiv b(1 + \gamma)$.[2]

Furthermore, assume as in Schmitt-Grohe and Uribe [1997] that government taxes labor incomes by an amount $G \in \mathbb{R}_+$ to finance a constant government expenditure according to the following balanced-budget rule:

$$G = \tau(t)w(t)L(t), \tag{1}$$

where $w(t)$ is the time t real wage and $\tau(t)$ the time t tax rate. Since government spending is constant the tax rate is endogenously fixed by the labor effort of agents and varies over time.

The economy is populated by a unit measure continuum of identical households, owners of capital and labor. The representative household starts with a positive stock of capital $K(0)$ and chooses a level of consumption C, labor $L \in [0, \hat{L}]$, and capital K, so as to maximize the present discounted value of its lifetime utility:

$$\max \int_0^\infty \left(\ln C(t) - L(t) \right) e^{-\rho t} dt$$

subject to equation (1) and to the budget constraint:

$$\dot{K}(t) = (r(t) - \delta)K(t) + (1 - \tau(t))w(t)L(t) - C(t),$$

where $\delta > 0$ is the capital depreciation rate and $\rho > 0$ is the discount factor.[3] $\hat{L}$ is assumed to be sufficiently large so that the agents are able to finance government spending whatever the initial value of the capital stock $K(0)$.[4]

Let H be the Hamiltonian associated with the household's programme. The first-order conditions for an interior solution are:

$$\frac{\partial H}{\partial C}(C, K, L, \Lambda, t) = 0 \iff C(t) = \frac{1}{\Lambda(t)} \tag{2}$$

$$\frac{\partial H}{\partial L}(C, K, L, \Lambda, t) = 0 \iff (1 - \tau(t))w(t) = C(t)L(t) \tag{3}$$

$$\frac{\partial H}{\partial K}(C, K, L, \Lambda, t) = -\dot{\Lambda}(t) - \rho\Lambda(t) \iff \frac{\dot{\Lambda}(t)}{\Lambda(t)} = \rho + \delta - r(t). \tag{4}$$

and a transversality condition.

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[2] The assumption $\alpha < 1$ is necessary to ensure the existence of a stationary equilibrium.

[3] Since returns to scale are constant from a private viewpoint, it is recalled that profit breaks even at the equilibrium: $Y(t) = r(t)K(t) + w(t)L(t)$.

[4] This assumption is fulfilled when the sufficient condition $\hat{L} > \max\{L, \frac{G}{(1-a)K(0)}\}$ is satisfied, where L is the steady state value of labor at the higher equilibrium.

The aggregated dynamics of this economy can be expressed as a system of two equations in three unknowns:

$$\dot{K}/K = K^{\alpha}L^{\beta}/K - \delta - C/K - G/K \tag{5}$$
$$\dot{C}/C = aK^{\alpha}L^{\beta}/K - \rho - \delta, \tag{6}$$

associated to equation (3) defining a relationship between the three variables K, L and C:

$$(1-a)K^{\alpha}L^{\beta} = CL + G. \tag{7}$$

As already shown by Schmitt-Grohe and Uribe [1997], the model exhibits a Laffer curve: there exists a tax rate $\hat{\tau}$ for which government expenditures are positively correlated with the tax rate if $\tau \in [0, \hat{\tau})$, reach a maximum $\hat{G}$ for $\tau = \hat{\tau}$ and are negatively correlated with the tax rate for $\tau \in (\hat{\tau}, 1]$. For $G < \hat{G}$, the model has two equilibria; for $G = \hat{G}$ the model has a unique equilibrium; above this value the model has no equilibrium.[5]

In the following, we will assume that $G < \hat{G}$.

3 Phase diagram

In this section, we construct the phase diagram of the system formed by equations (5) to (7). In a Cartesian K/C-plot, we investigate the equation of the $\dot{C} = 0$ and $\dot{K} = 0$ loci. It is worth noting that we temporarily exclude the third dimension of the system, represented by the variable L. This dimension will be reintegrated later in the study.

The $\dot{C} = 0$ locus is easily obtained from equations (6) and (7):

$$C = \frac{(1-a)(\rho+\delta)K - aG}{(\rho+\delta)K^{(1-\alpha)/\beta}}.$$

By definition $\alpha + \beta > 1$. Thus, since $1 - (1-\alpha)/\beta < 1$, the graph of $\dot{C} = 0$ is strictly increasing and concave.

According to equation (6) and (7), $\dot{C}/C = \frac{a}{1-a}\left[C\left(\frac{\rho+\delta}{a}K^{1-\alpha}\right)^{1/\beta} + G\right] - \rho - \delta$ is an increasing function of C (when K is held constant). Consequently, above the $\dot{C} = 0$ locus consumption increases while below the $\dot{C} = 0$ locus consumption decreases.

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[5] See appendix 7.1 for more details.

The $\dot{K} = 0$ locus is deduced from the differentiation of equations (5) and (7):[6]

$$\frac{dK}{dC} = \frac{\beta Y[1 - (1-a)/L] + C}{C[\alpha Y/K - \delta] + \delta(1-a)\beta Y/L}.$$

As shown in appendix 7.3, the sign of the ratio depends only on the sign of the numerator, $dL/dC > 0$ and $dY/dC > 0$. Thus, provided G is chosen such that $L < 1 - a$, the dK/dC ratio is first negative for $C = 0$, increases as C (and then L) increases from zero and becomes positive at least for $L \geq 1 - a$. In a plot where consumption represents the x-axis and capital the y-axis, the $\dot{K} = 0$ locus is first downward then upward sloping. It can also be deduced that it is convex in the same plot. Furthermore, notice that for $C = 0$ the capital at the $\dot{K} = 0$ locus is greater than the capital at the $\dot{C} = 0$ locus: in a plot where capital represents the x-axis and consumption the y-axis, the former locus lies in the right hand side of the latter, where $\dot{C}/C < 0$.

From appendix 7.2, it is straightforward to show that the $\dot{K} = 0$ curve is downward sloping at the lower steady state (for which τ is close to 1). It has been proved (in appendix 7.1) that there are two stationary equilibria: then the $\dot{K} = 0$ locus must cross the $\dot{C} = 0$ locus at least once again when it is upward sloping. The two loci cannot cross more than three times because of the convexity of the curves. When they cross three times, one of the intersection point is not an equilibrium: this appears when a triplet (K, C, L_1) solves equations (5) and (7) while a triplet (K, C, L_2) solves equations (6) and (7) with $L_1 \neq L_2$. This is one of the disadvantages of omitting the third dimension of the phase diagram, namely labor.

Finally, one more element must be taken into account while designing the phase diagram of the model: for high values of C and low values of K there is no real value of L solving equation (7). The phase diagram is then bounded above by a curve whose equation is:

$$C = \left[\left(\frac{\beta}{1 - \beta}G\right)^{\beta - 1} b\beta K^\alpha\right]^{1/\beta}.$$

Call S this curve. This S curve is increasing and concave respect to K.

We plot below an example of phase diagram that can be obtained from specific values of the parameters. The dotted line represents the S curve, the dashed line is the $\dot{C} = 0$ locus and the full line the $\dot{K} = 0$ locus.

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[6] Notice that for $K = 0$ (or close enough to zero) equation (7) has no solution $(C, L) \in \mathbb{R}_+^2$. Then, it is easier to express K respect to C rather than the inverse to describe the behavior of the $\dot{K} = 0$ locus.
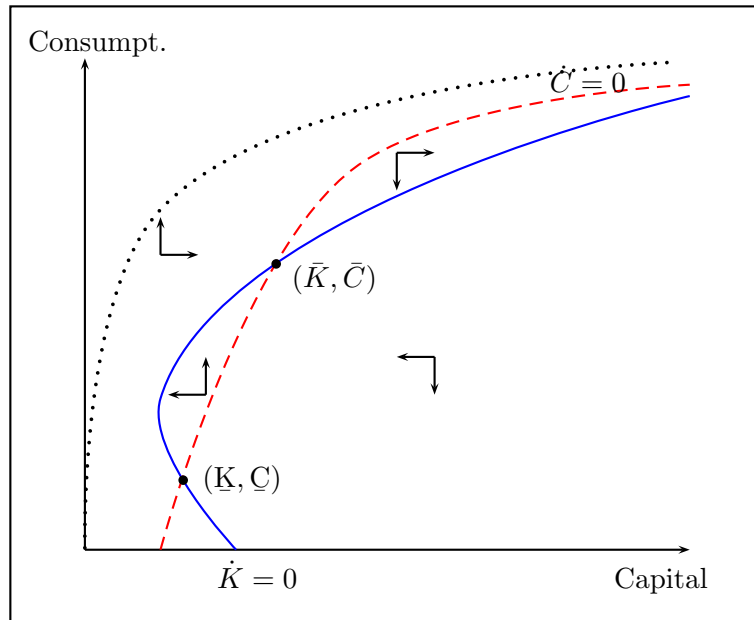
Arrows have been added to picture the direction taken by the different orbits in the phase diagram. The easiest way to derive the sign of $\dot{K}$ consists in evaluating the law of motion of capital when $(K, C) = (0, 0)$. We deduce that $\dot{K}$ is positive in the left-hand side of the $\dot{K} = 0$ locus and negative in the right-hand side.

Now the diagram has been pictured, we are able to stress the main point of this paper: aggregate instability may arise from local indeterminacy of the upper steady state (as in Schmitt-Grohe and Uribe [1997]) however this aggregate instability is a global phenomenon and persists also under determinacy of both steady states. A global study of the model is then required.

Existence of local determinacy for at least one of the two equilibria does not prevent the economy from global indeterminacy, including in the neighborhood of the determined stationary equilibrium. Two cases will be investigated:

(1) If the condition for indeterminacy is satisfied for the upper equilibrium, indeterminacy exists also in the neighborhood of the lower equilibrium which appears locally determined. In fact, it is not sufficient that the equilibrium be a saddle to rule out indeterminacy: orbits diverging from the locally determined steady state but converging to the indeterminate steady state satisfy both the first-order conditions and the transversality condition and must be treated as potential equilibrium paths for the economy. The upper equilibrium is then an attractor and the dynamics is globally indeterminate. In the limit case, all diverging paths converge to the upper steady state: both equilibria are connected by an orbit (the so-called *unstable manifold* of the determined equilibrium) starting from the lower equilibrium and reaching the upper equilibrium as t tends to infinity. Such a situation is called a *heteroclinic orbit*.

(2) If both steady states are locally determined, there are two simultaneous

saddle-path equilibria converging to two different steady states. It can happen that for the same value of the capital stock, two levels of consumption are compatible with an equilibrium path. And at any period of time, agents can jump from one path to another. Global indeterminacy is then in accordance with local determinacy of the two steady states.

4 Heteroclinic orbits

In our framework, heteroclinic orbits consist of equilibrium paths $(K(t), C(t))$, with $t \in \mathbb{R}$, whose accumulation points are steady state of the system formed by equations (5) to (7):

$$\lim_{t \to +\infty} (K(t), C(t)) = (\bar{K}, \bar{C})$$
$$\lim_{t \to -\infty} (K(t), C(t)) = (\underline{K}, \underline{C}).$$

From a topological point of view, these orbits correspond to the intersection of the *unstable manifold* of the lower steady state with the *stable manifold* of the upper steady state. [7]

Heteroclinic orbits have already been studied within the literature on sunspot equilibria in overlapping generation models by Chiappori and Guesnerie [1991]. They show in which cases expansionary monetary policies may help at reaching a better steady state with a higher level of activity. In a representative agent framework, Benhabib, Schmitt-Grohe and Uribe [2001] show that the popular Taylor rule may create indeterminacy and drive the economy to a liquidity trap where the nominal interest rate is near zero and inflation is possibly negative, even though the monetary policy is active, that is strongly responds to the rate of inflation in setting the nominal interest rate. Both cases of heteroclinic orbits are saddle-source connections: one of the accumulation point is an unstable steady state (a source) while the other is saddle-path stable. However, the dynamics in Chiappori and Guesnerie are robust to wider variations in parameter values than in Benhabib, Schmitt-Grohe and Uribe. In the latter contribution, heteroclinic orbits coincide with the existence of a Bogdanov-Takens bifurcation: far away from the critical parameter values, the heteroclinic orbits vanish. [8] In the former contribution, heteroclinic orbits

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[7] For more details, see Guckenheimer and Holmes [1983].
[8] The existence of a Bogdanov-Takens bifurcation explains also the presence of homoclinic orbits in Benhabib *et al.*, that is orbits connecting the same steady state.

persist as long as the topological stability of the two steady states remains unchanged. It is also the case in our setup but the heteroclinic orbits come from a saddle-sink connection. Actually, there is no general method to establish the existence of heteroclinic orbits: the appropriate method depends on the nature of the problem. In our framework, we adopt the same method as in Chiappori and Guesnerie [1991]. We first determine the local properties of the two steady state then we show that the different orbits of the phase diagram must lie in a closed area around the two steady state so that any path diverging from the lower steady state must converge to the higher.

Following Schmitt-Grohe and Uribe [1997], the local dynamics are derived from equations (2) to (4): the lower steady state $(\underline{K}, \underline{C})$ is always a saddle while the upper steady state $(\bar{K}, \bar{C})$ may be a saddle or a sink. The condition for indeterminacy in our framework is: [9]

$$\beta - 1 + \tau > 0.$$

Straightforwardly, if a heteroclinic orbit exists it must start from the lower steady state up to the higher steady state.

The lower steady state has a stable manifold W^s, corresponding schematically on the figure below to the line segment CD and an unstable manifold W^u starting from area ⓓ: there exists a $t_1 \in \mathbb{R}$ such that for any $t \in (-\infty, t_1)$, $(K(t), C(t)) \in W^u$ is an element of area ⓓ where $\dot{K}(t) < 0$ and $\dot{C}(t) > 0$.

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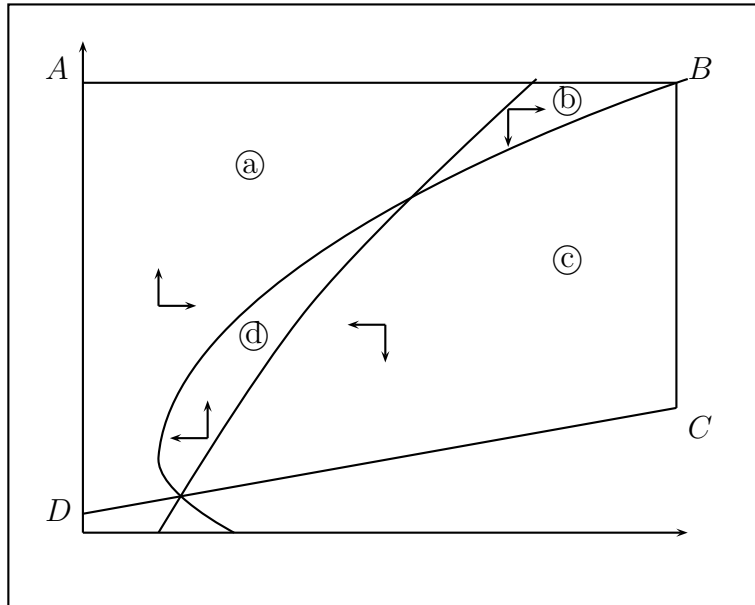
[9] See appendix 7.2 for formal derivations. In absence of increasing returns to scale $\beta = b$ and the condition for indeterminacy collapses to $\tau - a > 0$, as in Schmitt-Grohe and Uribe [1997].

Since the upper part of the $\dot{K} = 0$ locus is upward sloping the unstable manifold cannot remain in area ⓓ and must cross this locus as t increases from $-\infty$, say at $t = t_1$. Then, there exists a $t_2 \in (t_1, \infty)$ such that for any $t \in (t_1, t_2)$, $(K(t), C(t)) \in W^u$ is an element of area ⓐ where $\dot{K}(t) > 0$ and $\dot{C}(t) > 0$.

Since L is bounded from above and K is increasing it is clear from equation (6) that C must decrease after a while: the unstable manifold cannot remain in area ⓐ and must cross the $\dot{C} = 0$ locus as t increases from t_1, say at $t = t_2$. Then, there exists a $t_3 \in (t_2, \infty)$ such that for any $t \in (t_2, t_3)$, $(K(t), C(t)) \in W^u$ is an element of area ⓑ where $\dot{K}(t) > 0$ and $\dot{C}(t) < 0$. And the unstable manifold is bounded above by the line segment AB.

Since L is bounded from above and C is bounded from below it is clear from equation (5) that $\dot{K} < K^\alpha \bar{L}^\beta - \delta K$ must be negative as K increases: the unstable manifold cannot remain in area ⓑ and must cross the $\dot{K} = 0$ locus as t increases from t_2, say at $t = t_3$. Furthermore, assume there exists $(K(t_3), C(t_3)) \in W^u$ and $(K(t_3), C^s) \in W^s$ such that $C(t_3) > C^s$.[10] Then, there exists a $t_4 \in (t_3, \infty)$ such that for any $t \in (t_3, t_4)$, $(K(t), C(t)) \in W^u$ is an element of area ⓒ where $\dot{K}(t) < 0$ and $\dot{C}(t) < 0$.

In area ⓒ, the unstable manifold is bounded to the right by the line segment BC and from below by the stable manifold CD (in a two-dimension system, two orbits cannot cross each other): the unstable manifold must come back to area ⓓ and remains within the interior of the polygon $ABCD$.

Proposition 1 *Provided the unstable manifold of the lower stationary equilibrium is defined at any period $t \in \mathbb{R}$ and $\beta - 1 + \tau > 0$ there is an equilibrium path connecting the two steady states.*

Proof. Assume the unstable manifold is defined at any period $t \in \mathbb{R}$. Remaining in the polygon $ABCD$, it cannot cross itself and must cross the $\dot{K} = 0$ and the $\dot{C} = 0$ loci several times closer to the upper steady state. When W^u enters in a close neighborhood of the steady state it must satisfy the local dynamics computed in appendix 7.2. Since the upper steady state is a local attractor for $\beta - 1 + \tau > 0$, W^u must converge to it: $\lim_{t \to +\infty} (K(t), C(t)) = (\bar{K}, \bar{C})$. The model exhibits a heteroclinic orbit.

However, if the unstable manifold has a nonempty intersection with the set of elements of the S curve then there exists a $\tilde{t} \in \mathbb{R}$ such that it is not defined for $t \in (\tilde{t}, +\infty)$. The limit as t tends to infinity does not exist and there is no heteroclinic orbit. A necessary condition for the existence of a heteroclinic orbit is that the unstable manifold be defined at any period t. ∎

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[10] We prove this assumption in appendix 7.4.

When proposition 1 holds, the lower equilibrium $(\underline{K}, \underline{C})$ that appears to be locally determined is actually globally indeterminate: in the neighborhood of this stationary equilibrium, there exist diverging paths that converge to the upper equilibrium, satisfying both the first order conditions and the transversality condition. When proposition 1 does not hold (but the upper stationary equilibrium is however locally indeterminate), the equilibrium is locally and globally determined for $K(0) \in (0, \underline{K})$. However, nothing can be generically deduced when $K(0) \in (\underline{K}, +\infty)$, some diverging paths may converge to the upper equilibrium while some others do not. Futhermore, there may be the coexistence for the same stock of capital of two equilibrium paths: the saddle-path equilibrium converging to the lower steady state and the focus converging to the upper steady state. This latter configuration also exhibits indeterminacy and is similar to the case studied in the next section.

5   Coexistence of two saddle-path equilibria

In the previous section, we have seen that local determinacy of a stationary equilibrium does not prevent from global indeterminacy when the other equilibrium is locally indeterminate. Actually, local determinacy of both stationary equilibria does not prevent neither the economy from global indeterminacy.

The proof consists of three steps. First, we show that a solution to the optimal control problem exists for any starting value of the capital stock $K(0)$. Second, we use a continuity argument to prove that the saddle-path of the lower equilibrium must be defined (for a given stock of capital) when the saddle-path of the upper equilibrium is. Finally, we are able to define the range of determinacy and indeterminacy according to the value of the capital stock.

Existence of a solution may be proved by using the Filippov-Cesari theorem (see appendix 7.4). We deduce the following lemma:

Lemma 1 *For any predetermined value of the capital stock, there exists an optimal pair $(K^*(t), u^*(t))$ to the optimization problem defined in section 2, where $u^*(t) = (C^*(t), L^*(t))$ is measurable.*

Proof. See Seierstad and Sydsaeter [1987], theorem 2.8. ∎

The saddle-path of the upper equilibrium may not be defined for any $K \in [K(0), +\infty)$ because as t tens to $-\infty$ the saddle-path is likely to merge with the S curve. Assume it is defined for $K \in [\check{K}, +\infty]$, with $K(0) < \underline{K} < \check{K}$.[11]

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[11] It is worth noting that the condition $\underline{K} \geq \check{K}$ would imply immediately indeterminacy.

According to lemma 1, the saddle-path of the lower equilibrium must be defined at least on $[K(0), \check{K})$.

The case of global determinacy, pictured on the figure below, can be easy ruled out. It is worth noting first that we can conclude from the phase diagram studied in section 3 that the *intermediate* point of intersection between the $\dot{K} = 0$ and the $\dot{C} = 0$ loci, which is an stationary equilibrium when indeterminacy occurs, is not a good candidate to represent one of the two equilibria in the case of a dual local determinacy. Actually, the topology of this stationary equilibrium is not consistent with the coexistence of both a stable and an unstable manifold: if the stable (resp. unstable) manifold lies in regions ⓐ and ⓒ, the unstable (reps. stable) manifold cannot lie in regions ⓑ and ⓓ. Since it has been proved in appendix 7.1 that there are only two steady states, we conclude that the intermediate point of intersection between the two loci must lie above the S curve, in a region where L is not properly defined.
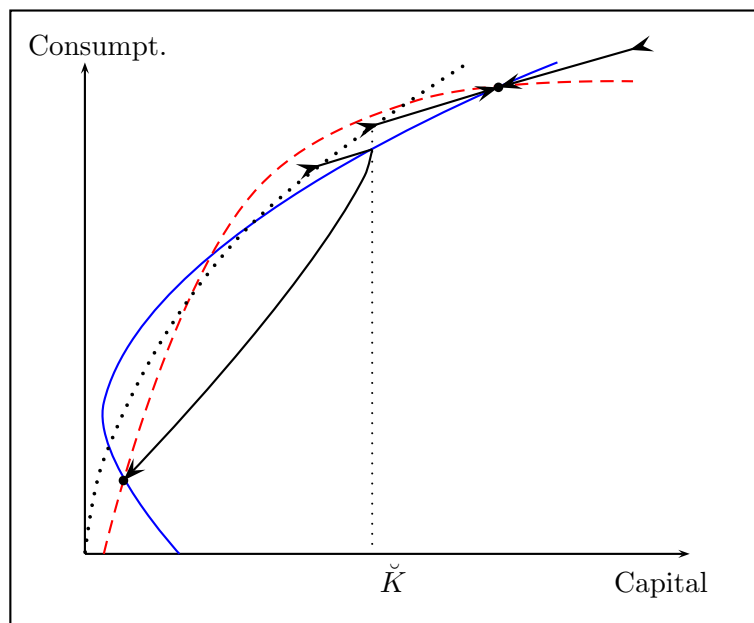
Kamien and Schwartz [1981] have established that in an optimal control problem where the Hamiltonian is concave in the controls, the trajectory of the state variable − if single − must always be monotonic for any given initial point. [12] We can exclude the case of non-monotonicity of the lower saddle-path equilibrium. This equilibrium must then remain in region ⓒ, below the upper saddle-path equilibrium. For any pair (K, C) lying in this region (below the S curve), equation 7 has two real solutions for L. If there exists an optimal path converging to $(\underline{K}, \underline{C})$ and whose state trajectory is defined on $[\underline{K}, \check{K})$ then, by continuity, this trajectory must also be defined on $[\check{K}, +\infty)$ since the

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[12] The result has been extended by Hartl [1987].

first-order conditions (2) to (4) are differentiable. As a consequence:

Proposition 2 *When the condition for indeterminacy $\beta - 1 + \tau > 0$ is not satisfied, the upper equilibrium $(\bar{K}, \bar{C})$ that appears to be locally determined is actually globally indeterminate.*

Aggregate instability occurs as soon as $K(0) \in [\breve{K}, +\infty)$. This is especially the case if we assume from empirical estimates, as Schmitt-Grohe and Uribe do, that the economy lies in the neighborhood of the upper equilibrium. For $K(0) < \breve{K}$, there is a unique saddle-path equilibrium, ruling out the possibility of indeterminacy. However, Schmitt-Grohe and Uribe have already underlined that the high tax rate related to the lower steady state does not fit the empirical estimates established by Mendoza et al. [1994]. It is worth noting that this aggregate instability is robust no matter what the topology of the stationary equilibrium is, whether indeterminate or not.

6   Conclusion

This paper explores the extent to which the Schmitt-Grohe and Uribe balanced-budget rule − embodying fixed government expenditures and fiscal increasing returns − is responsible for aggregate instability. While the two authors deduce their conclusions from a local analysis and empirical plausibility for the parameter values to lay in the range of indeterminacy, we extend the analysis to the second equilibrium, including also parameter values outside the range of (local) indeterminacy.

We prove that indeterminacy may occur not only in the neighborhood of the upper equilibrium studied by Schmitt-Grohe and Uribe but also around the lower one. Furthermore, global indeterminacy is likely to be robust to parameter changes, especially when both equilibria are locally determined for a large range of the capital stock values.

Actually, it appears that the aggregate instability underlined by Schmitt-Grohe and Uribe is a general property of the model rather than dealing with the local indeterminacy of one of two equilibria. Aggregate instability is then an intrinsic characteristic of the balanced-budget rule so defined by the authors, independently of the proximity between the critical values and the empirical estimates. This have important consequences for the way we design economic policy: a balanced-budget rule where the tax rates are exogenously fixed must be favored to balanced-budget rule with exogenous government spending but endogenously determined tax rates to avoid aggregate instability.

From a topological viewpoint, however, configurations are multiple: coexistence of two saddle-path equilibria, coexistence of a saddle and a sink, heteroclinic orbits, etc. The most relevant configurations have been presented in the paper.

7   Appendix

7.1   Existence of a Laffer curve

Combining equations (1) and (2) to (3) at the stationary equilibrium leads to:

$$
G = \frac{(1-a)(\rho+\delta)}{a} K\tau
$$

$$
Y = \frac{\rho+\delta}{a} K
$$

$$
C = \frac{\rho+\delta(1-a)}{a} K - G
$$

$$
L = 1 - \frac{a\rho K}{[\rho+\delta(1-a)]K - aG}
$$

$$
K = \left(\frac{a}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \left(\frac{(1-a)(\rho+\delta)(1-\tau)}{\rho+\delta(1-a)-(1-a)(\rho+\delta)\tau}\right)^{\frac{\beta}{1-\alpha}},
$$

where letters without subscript denote steady state values.

Let $s_i \equiv \delta K/Y$ be the steady state investment/output ratio (independent of τ). Using the relationships computed above, one gets:

$$
\frac{\partial G}{\partial \tau} = \frac{G}{\tau} + \frac{\partial K}{\partial \tau} \frac{G}{K}
$$

$$
= \frac{b(1-\alpha)\tau^2 - [(1+\beta-\alpha)(1-s_i) - b\gamma]\tau + (1-\alpha)(1-s_i)}{\tau(1-\alpha)(1-\tau)s_c} G.
$$

The sign of $\frac{\partial G}{\partial \tau}$ depends only on the numerator of the fraction, namely $P(\tau)$. Notice that $P(0) = (1-\alpha)(1-s_i) > 0$, $P(1) = \beta(s_i-a) < 0$ and $lim_{\tau\to\infty}P(\tau) = +\infty$. Thus, $P(\tau)$ has one root between 0 and 1, namely $\hat{\tau}$, and another strictly greater than 1.

For feasible values of τ (i.e. included between 0 and 1), $\frac{\partial G}{\partial \tau}$ is strictly positive for $\tau \in [0, \hat{\tau})$, null for $\tau = \hat{\tau}$ and strictly negative otherwise. Since for $\tau = 0$ and $\tau = 1$ the government expenditures are null, this means that there is a

tax rate $\hat{\tau}$ maximizing government expenditures and an associated amount of government expenditures $\hat{G}$, above which there is no feasible steady state tax rate, below which there are two feasible steady state tax rates, and for which the feasible steady state tax rate is unique (and equal to $\hat{\tau}$).

7.2 Local analysis

Let s_i and s_c be respectively the steady state investment/output and consumption/output ratios and let $x(t)$ denote the log deviation of the variable X from its steady state at time t. From the first-order equations, we can compute the following approximated dynamics around each of the steady state:

$$\begin{bmatrix} \dot{k}(t) \\ \dot{c}(t) \end{bmatrix} = (\rho + \delta) \begin{bmatrix} -\frac{(1+\gamma)(1-\tau)}{\beta-1+\tau} - \frac{\delta}{\rho+\delta} & \frac{\beta(1-\tau)}{a(\beta-1+\tau)} - \frac{s_c}{a} \\ \frac{(1-\alpha)(1-\tau)-\beta}{\beta-1+\tau} & \frac{\beta(1-\tau)}{\beta-1+\tau} \end{bmatrix} \begin{bmatrix} k(t) \\ c(t) \end{bmatrix}$$

The determinant Det depends on the sign of two polynomial, namely $\beta-1+\tau$ and $P(\tau)$ (see appendix 7.1):

$$Det = \frac{(\rho + \delta)^2}{a(\beta - 1 + \tau)} P(\tau).$$

The trace Tr is always strictly negative for standard parameter values if the condition for indeterminacy $\beta - 1 + \tau > 0$ is satisfied :

$$Tr = -\frac{\alpha(1 - \tau)}{\beta - 1 + \tau} - \frac{\delta}{\rho + \delta}.$$

A necessary condition for a Bogdanov-Takens bifurcation is that both eigenvalues be equal to zero, that is $Det = Tr = 0$. Since $Det = 0$ for $\tau = \hat{\tau}$, $\hat{\tau} > b$ and $Tr(\tau) < 0$ for $\tau \in [b, 1]$ this condition cannot be satisfied in our framework.

7.3 Study of the $\dot{K} = 0$ locus

As already noticed:

$$\frac{dK}{dC} = \frac{\beta Y[1 - (1 - a)/L] + C}{C[\alpha Y/K - \delta] + \delta(1 - a)\beta Y/L},$$

and according to the law of motion of capital and the first-order condition (7):

$$Y/K = \delta + C/K + G/K$$
$$G/L = (1 - a)Y/L - C.$$

Let D be the denominator of the fraction:

$$
\begin{aligned}
D &= C[\alpha Y/K - \delta] + \delta(1 - a)\beta Y/L \\
&= C[\alpha\delta + \alpha(C + G)/K - \delta] + \delta(1 - a)\beta Y/L \\
&= \alpha C(C + G)/K - (1 - \alpha)\delta C + \delta(1 - a)\beta Y/L \\
&= \alpha C(C + G)/K + \beta\delta[(1 - a)Y/L - C] + \gamma\delta C \\
&= \alpha C(C + G)/K + \beta\delta G/L + \gamma\delta C > 0.
\end{aligned}
$$

Thus the dK/dC ratio depends only on the sign of the numerator, or equivalently on the sign of $\frac{L}{1-a} - 1 - \frac{C}{1-a}\frac{L}{\beta Y}$.

Furthermore, combining equations (5) and (7) yields:

$$K = \frac{C}{\delta}\left(\frac{L}{1 - a} - 1\right) + \frac{aG}{\delta(1 - a)},$$

that is:

$$\frac{dK}{dC} = \frac{L}{1 - a} - 1 - \frac{C}{1 - a}\frac{dL}{dC}.$$

It is then clear that dL/dC has the same sign as $\frac{L}{\beta Y}$, which is strictly positive. And since $Y = (CL+G)/(1-a)$ we straightforwardly deduce that $dY/dC > 0$.

Finally, one can derive from the previous equation:

$$\frac{d^2 K}{dC^2} = \frac{2}{1 - a}L'(C) + CL''(C),$$

which is unambiguously strictly positive for low enough values of C. This means that the term $\beta\delta G/L$ in the denominator of dK/dC is negligible while the numerator increases respect to C: the denominator increases, at least as long as K decreases whereas C and L increase. The $\dot{K} = 0$ locus is then downward sloping and its slope becomes flatter as C increases. It is then convex as long as K does not increase.

When K starts increasing, the numerator becomes positive (by definition). According to equations (5) and (7), $\hat{y} > \hat{c}$ and $\hat{k} > \hat{c}$, where $\hat{x}$ denotes the growth rate of X. Thus, the numerator must increase at a rate strictly greater than $\hat{c}$ while the denominator must increase (or possibly decrease) at a rate strictly lower than $\hat{c}$. The $\dot{K} = 0$ locus is then upward sloping and its slope becomes steeper as C increases. It is then also convex when K increases.

Assume the stable manifold W^s crosses the $\dot{K} = 0$ locus at (K^s, C^s) with $C^s < C^u$, where $(K^s, C^u) \in W^u$. This would mean that the unstable manifold W^u would cross the $\dot{K} = 0$ locus for a higher value of the capital stock: $K(t_3) > K^s$. Then, the stable manifold would be bounded from above and below by the unstable manifold since they cannot cross each other.

Thus, it has to cross the $\dot{K} = 0$ and the $\dot{C} = 0$ loci several times closer to the upper steady state. According to appendix 7.2, we can rule out the possibility of closed orbit around the upper steady state which requires the trace to be equal to zero. Finally, the stable manifold must enter in a close neighborhood of the upper steady state as t tends to $-\infty$: $\lim_{t\to-\infty}(K^s(t), C^s(t)) = (\bar{K}, \bar{C})$.

But this clearly contradicts the fact that the upper steady state is a local attractor. And we can deduce that $C^u < C^s$.
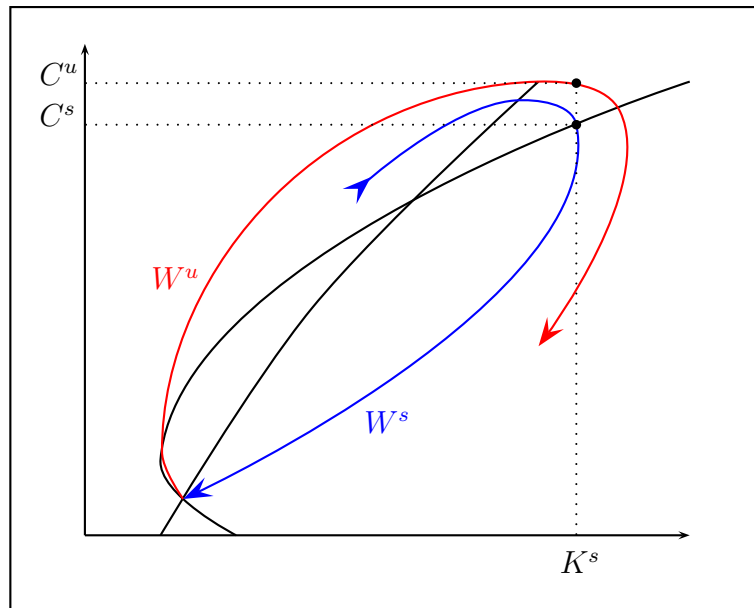
7.5   Proof of lemma 1

Following the Filippov-Cesari theorem, assume that:

  i. there exists an admissible pair $(K(t), u(t))$ with $u(t) = (C(t), L(t)) \in U$,
 ii. for each (K,t) the set $N(K, U, t) \in \mathbb{R}^2$ defined by

$$N(K,V,t) = \{[(\ln C(t) - L(t))\,e^{-\rho t} + \eta,$$
$$(r(t) - \delta)K(t) + (1 - \tau(t))w(t)L(t) - C(t)] : \eta \geq 0, u \in U\}$$

is convex,

iii. U is closed and bounded,

iv. there exists a number $\hat{K}$ such that $K(t) \leq \hat{K}$ for all $t \in \mathbb{R}_+$ and all admissible pairs $(K(t), u(t))$.

there exists an optimal pair $(K^*(t), u^*(t))$ to the optimization problem defined in section 2, where $u^*(t)$ is measurable.

Choosing $C(t)$ such that $K(0)^\alpha \hat{L}^\beta - \delta K(0) - C(t) = 0$ at any period t satisfies conditions i.

The utility function is jointly concave in (C, L) (the propriety can be easily shown since the function is additively separable) while $(r(t) - \delta)K(t) + (1 - \tau(t))w(t)L(t) - C(t) = A(t)K(t)^a L(t)^b - \delta K(t) - C(t) - G$. From the point of view of the firm, returns to scale are constant, the law of motion of capital is then jointly concave in (K, C, L) and $N(K, V, t)$ is a convex set. Condition ii is also satisfied.

By assumption, L is bounded above by $\hat{L}$; C and G are bounded below by zero. Then, $\dot{K}(t) \leq K(t)^\alpha \hat{L}^\beta - \delta K \equiv g(K(t))$, where $g(K(t))$ is positive for $K(t) \in \left[0, \left(\frac{\hat{L}^\beta}{\delta}\right)^{\frac{1}{1-\alpha}}\right]$. Then, K is bounded above by $\hat{K} = \left(\frac{\hat{L}^\beta}{\delta}\right)^{\frac{1}{1-\alpha}}$, which implies condition iv.

By asumption, $L \in [0, \hat{L}]$. The agents cannot consume more than the production and the capital stock: $C \in [0, \hat{K}^\alpha \hat{L}^\beta + \hat{K}]$. We assume that $C = 0$ is feasible: in that case, the utility function takes the value $-\infty$ for any $L \in [0, \hat{L}]$. Thus $U = [0, \hat{K}^\alpha \hat{L}^\beta + \hat{K}] \times [0, \hat{L}]$ is closed and bounded, as required by condition iii.

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