Implied Risk-Neutral probability Density functions from options prices : A comparison of estimation methods

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Abstract

This paper compares the goodness-of-fit of eight option-based approaches used to extract risk-neutral probability density functions from a high-frequency CAC 40 index options during a normal and troubled period. Our findings show that the kernel estimator generates a strong volatility smile with respect to the moneyness, and the kernel smiles shape varies with the chosen time to maturity. The mixture of log-normals, Edgeworth expansion, hermite polynomials, jump diffusion and Heston models are more in line and have heavier tails than the log-normal distribution. Moreover, according to the goodness of fit criteria we compute, the jump diffusion model provides a much better fit than the other models on the period just-before the crisis for relatively short maturities. However, during this same period, the mixture of log-normal models performs better for more than three month maturity. Furthermore, in the troubled period and the period just-after the crisis, we find that semi-parametric models are the methods with the best accuracy in fitting observed option prices for all maturities with a minimal difference towards the mixture of log-normals model.

JEL classification : C02; C14; C65; G13.

Keywords : Risk-neutral density, mixture of log-normal distributions, Edgeworth expansions, Hermite polynomials, tree-based methods, kernel regression, Heston’s stochastic volatility model, jump diffusion model.

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1 Introduction

The Risk-Neutral Density (RND) is an interesting tool for an investor who seeks to measure how market expectations evolve over time. A huge literature arose from the early 1990s on the most appropriate way to estimate this density. At the origin of all the methods, we find the famous work of Breeden and Litzenberger (1978), who are the pioneers who determined a relationship between option prices and the RND.

Malik and Thomas (1997) have developed a nonparametric method to estimate the American option prices. This technique has been adopted by Bahra (1996) and Schernick, Garcia and Tirupattur (1996) on European options. It is a representation of RND based on a linear combination of log-normal density functions. These authors estimated the RND of the oil price during the Gulf crisis and found that the estimated density is different from the density obtained by standard methods. Söderlind and Svensson (1997) showed how this method can be applied to different financial assets with emphasis on its use, like monetary policy. Abadir and Rockinger (1997) determined farm formulas of option prices using Kummer functions (hypergeometric) as a basis for RND. Backus, Foresi, Li and Wu (1997) have approached the conditional distribution of underlying assets prices by Gram-Charlier series expansion.

Another approach has been proposed by Jarrow and Rudd (1982). They have developed a method of option valuation under the assumption that the underlying asset does not follow a log-normal distribution. They showed that the RND can be obtained by an Edgeworth expansion. Carrodo and Su (1996) used the Jarrow and Rudd approach to determine the skewness and kurtosis of the price of S&P500 options. A similar approach is that of Madan and Milne (1994), Abken, Madan and Ramamurti (1996) and Coutant (1999) who determined the RND from a Hermite polynomial approximation. Similarly, El Hassan and Kucera (1998) used the Fourier-Hermite development in order to evaluate European and American index options.


Galati and Melick (1999) estimated the moments using a mixture of log-normal laws in order to understand how the central bank interventions are perceived by foreign exchange markets traders regarding JPY / USD options between 1993 and 1996. Weinberger (2001) and Anagnar, Bedendo, Hodges and Tompkins (2002) found a typical form of the implied S&P500 risk neutral distribution after the crisis. Panigirtzoglou and Skiadapoulos (2004) studied implied distributions dynamics and provided algorithms that make their results applicable to the options and risk management.

\footnote{A contingent asset is an asset that pays one unit of good consumed when a world state takes place and nothing else.}
framework. Mark, Feike and Lo (2005) showed that the RND is a function of both the underlying asset returns and volatility. Enzo, Handel and Härdle (2006) estimated the RND using the same approach developed by Aït-Sahalia and Lo (1998). Finally, Jondeau, Ser-Huang Poon and Rockinger (2007) compared the performance of various parametric and semi-parametric methods to extract RND.

Some authors argued that the RND corresponds to real world density only if investors are risk neutral. Thus, the difference between the two densities stems from the existence of a risk premium. Consequently, it is interesting to study the ability of the estimated RND extracted from options prices to assess the real density, its forecasting ability as well as the choice of the suitable RND estimation technique.

In this paper, we estimate parametrically, semi-parametrically and nonparametrically the RND functions of CAC 40 index options between January 1st 2007 and December 31st 2007, and we compare the goodness-of-fit of eight option-based approaches during a normal and troubled period. To our knowledge, this is the first application dealing with European options using structural and non-structural approaches of RND functions estimation. Specially, we use nonparametric estimation methods of the RND, that is the kernel and the tree-based methods as well as six parametric and semi-parametric option-based approaches: (i) the numerical approximation of the RND based on the second derivative of option prices with respect to the strike price, as suggested by Breeden and Litzenberger (1978), (ii) the mixture of log-normal distributions following Melick and Thomas (1997), (iii) the Edgeworth expansion around the log-normal distribution of Jarrow and Rudd (1982), (iv) the Hermite polynomials, suggested by Madan and Mline (1994), (v) Heston’s stochastic volatility model (1993) and (vi) the jump diffusion model following Bates (1991). To our best knowledge, our study is the first that combines the three approaches: parametric, semi-parametric and nonparametric methods of RND estimation.

The paper is organized as follows. Section 2 discusses our methodological non-structural approaches of RND functions estimation. Section 3 presents the structural methods we use to extract the RND functions. Section 4 describes our data, and contains the estimation results as well as related comments regarding the comparison of the various RND estimation methods. Section 5 concludes.

2 Non-structural approaches of RND functions estimation

2.1 The Breeden and Litzenberger relation

Breeden and Litzenberger (1978) were the first to derive the RND using the following price of a call option formula:

\[
C(S_t, t) = e^{-rt} \mathbb{E}^* \left[ \max (S_T - K, 0) | S_t, t \right] \quad (1)
\]

\[
= e^{-rt} \int_0^\infty \max (S_T - K, 0) q(S_T | S_t, t) dS_T. \quad (1)
\]

Where for date t and maturity date T, we denote C the call price, r is the risk-free interest rate, S is the underlying asset price, K is the strike price and \( q(\cdot) \) is the undiscounted RND.

Differentiating this equation with respect to the exercise price K yields the discounted cdf

\[
\frac{\partial C}{\partial K} = -e^{-rt} \int_K^{\infty} q(S_T) dS_T. \quad (2)
\]
and differentiating twice yields the discounted pdf $^{[2]}$

$$\frac{\partial^2 C}{\partial K^2}\bigg|_{K=S_T} = e^{-r\tau} q(S_T).$$

These equations show that the second derivative of the call price yields the discounted RND $^{[3]}$. This suggests that a first method to extract RND is to approximate it numerically applying the finite difference approach to (3). Nevertheless, this method relies on the hypothesis that there exist traded option prices for many strikes. This is not likely to be the case in practice. Also, it has been shown that RNDs estimated in this manner are very unstable. In fact, differentiating twice exacerbates even tiny errors in the prices and may be difficult $^{[4]}$. That is why it is necessary to extract RND using alternative methods that put more structure on the option prices. Before describing such methods, we briefly show how the parameters of these models are estimated.

### 2.2 RND parameters estimation

Suppose that we have to estimate a given model with respect to a set of parameters $\theta$. Assume that for horizon $\tau$, we have $N^c_\tau$ strike prices for which we have call options and $N^p_\tau$ strike prices for which we have put options. For date $t$ and horizon $\tau$, observed call and put option prices are denoted $C_{t,\tau,j}$ for $i = 1, \ldots, N^c_\tau$ and $P_{t,\tau,i}$ for $i = 1, \ldots, N^p_\tau$. Theoretical call and put option prices, implied by the assumed model, are denoted $C_t(K,\tau,\theta)$ and $P_t(K,\tau,\theta)$, respectively, for strike price $K$ and horizon $\tau$. Then, the parameter vector $\theta \in \Theta$ is typically estimated by non-linear least squares, by minimizing for each day and each maturity

$$\min_{\theta \in \Theta} \sum_{i=1}^{N^c_\tau} w^c_i (C_{t,\tau,j} - C_t(K_i,\tau,\theta))^2 + \sum_{i=1}^{N^p_\tau} w^p_i (P_{t,\tau,i} - P_t(K_i,\tau,\theta))^2.$$  

where $w^c_i$ and $w^p_i$ are weights associated with option $i$ and $\Theta$ is the domain of $\theta$. These weights could be given by a measure of liquidity of a given option.

### 2.3 Parametric methods

#### 2.3.1 Black & Scholes Model

The price dynamics of an underlying asset of a Black & Scholes Model (BSM) is a Geometric Brownian motion. Under the log-normal assumption, the volatility is constant across both the exercise price and horizons. The BSM underlying asset price is given by:

$$dS_t = (\mu + \frac{\sigma^2}{2})S_t dt + \sigma S_t dW_t$$

Breeden and Litzenberger (1978) suggest to evaluate the RND using this approximation

$$\frac{\partial^2 C}{\partial K^2} \approx e^{-r\tau} \frac{C(K_{i+1}) - 2C(K_i) + C(K_{i-1})}{(\Delta K)^2}.$$  

where $\Delta K = 50$. See also Roncalli (1997).

It should be mentioned that $q(.)$ is the undiscounted RND whereas $e^{-r\tau}q(.)$ represents an Arrow-Debreu state price, which is referred to as the RND.

For instance, if option prices suffer from non-synchronicity bias (that is, the underlying asset price is not observed at the same time as the option price), if the option price is fudged because of some microstructure reason (for instance, due to the bid-ask spread).
where $\mu$ and $\sigma$ represent respectively the instantaneous return assets expectations and volatility. $W_t$ is a Brownian motion under the real probability measure $\mathbb{P}$.

When markets are complete, Harrison and Pliska (1981) show that there exists a risk-neutral transformation that leads to the following formula:

$$dS_t = rS_t dt + \sigma S_t dW^Q_t$$

where $W^Q_t$ is a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$.

The call option price $C_{BS}$ maturing at date $T = t + \tau$, with strike price $K$ and dividend yield $\delta_t$, is given by:

$$C_{BS}(S_t, K, \tau, r, \delta_t, \sigma) = e^{-r\tau} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} f_{BS}(S_T) dS_T$$

$$= S_t \Phi(d_1) - Ke^{-r\tau} \Phi(d_2).$$

Where $\Phi(.)$ is the standard cumulative normal distribution function and

$$d_1 = \frac{\log(S_t/K) + (r - \delta_t + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}$$

$$d_2 = d_1 - \sigma \sqrt{\tau}.$$  

Consequently, the RND is a log-normal density with a mean $m = (\ln(S_t) + (r - \delta_t) - \frac{\sigma^2}{2})\tau$ and variance $s^2 = \sigma^2\tau$:

$$f_{BS}(S_T) = e^{r\tau} \frac{\partial^2 C_{BS}}{\partial K^2} \bigg|_{K=S_T}$$

$$= \frac{1}{S_T \sqrt{2\pi s^2 \tau}} \exp \left( -\frac{1}{2} \frac{(\log(S_T) - m)^2}{s^2} \right).$$

All Black & Scholes parameters are directly observable except the volatility, which is difficult to estimate. Nevertheless, the BSM hypothesis, according to which the returns are homoscedastic and normally distributed, does not fit with empirical evidence showing that densities are skewed and leptokurtic.

### 2.3.2 Mixture of log-normal distributions

Bahra (1996), Melick and Thomas (1997) and Södernlind (1997) are among the first to describe the RND as a mixture of distributions. For option pricing, the most well-known distribution studied in the literature is the mixture of log-normal densities. The reason is that it appears as a trivial extension of the Black & Scholes model that involves the single log-normal density function.

$$l(S_T, m, \sigma) = \frac{1}{S_T \sqrt{2\pi s^2}} \exp \left( -\frac{1}{2} \left( \frac{\log(S_T) - m}{s} \right)^2 \right).$$

A mixture of such densities yields

$$q(S_T; \theta) = \sum_{i=1}^{M} \alpha_i l(S_T, m_i, s_i).$$

5
where \( \theta \) regroups all the unknown parameters \( \alpha_i, m_i \) and \( s_i \) for \( i = 1, \ldots, M \), and \( M \) denotes the number of mixtures describing the RND. Obviously, to guarantee that \( q \) is a density, we must have \( \alpha_i > 0 \) for all \( i = 1, \ldots, M \), and \( \alpha_1 + \ldots + \alpha_M = 1 \). In other words, \( q \) should be a convex combination of various log-normal densities.

The option price for such a mixture of log-normal distributions is, for a given strike \( K \) and time to maturity \( \tau = T - t \)

\[
C_{LN}(K, \theta) = e^{-r \tau} \int_K^{+\infty} (S_T - K) q(S_T; \theta) dS_T
\]

\[
= e^{-r \tau} \int_K^{+\infty} (S_T - K) \sum_{i=1}^{M} \alpha_i l(S_T, m_i, s_i) dS_T
\]

\[
= e^{-r \tau} \sum_{i=1}^{M} \alpha_i \int_K^{+\infty} (S_T - K) l(S_T, m_i, s_i) dS_T.
\]

where we define the volatility over the horizon of the option as \( s_i = \sigma_i \sqrt{\tau} \), to simplify notations. The last equality is obtained by simply inverting the sum and integral operators. There are various ways to evaluate the integral. For instance, we have \(^5\)

\[
\int_K^{+\infty} (S_T - K) l(S_T, \mu, s) dS_T = \left[ E[S_T | S_T > K] - K \right] Pr[S_T | S_T > K]
\]

\[
= \exp\left(m + \frac{1}{2}s^2\right) \left[ 1 - \Phi\left(\frac{\log(K) - m - s^2}{s}\right)\right] - K \left[ 1 - \Phi\left(\frac{\log(K) - \mu}{s}\right)\right]
\]

Finally, the option price is given by

\[
C_{LN}(K; \theta) = e^{-r \tau} \sum_{i=1}^{M} \alpha_i \left\{ \exp\left(\mu_i + \frac{1}{2}s_i^2\right) \left[ 1 - \Phi\left(\frac{\log(K) - m_i - s_i^2}{s_i}\right)\right] - K \left[ 1 - \Phi\left(\frac{\log(K) - m_i}{s_i}\right)\right] \right\}
\]

\[
= e^{-r \tau} \sum_{i=1}^{M} \alpha_i \exp\left(\mu_i + \frac{1}{2}s_i^2\right) \Phi\left(-\log(K) + m_i + s_i^2\right) - e^{-r \tau} K \sum_{i=1}^{M} \alpha_i \Phi\left(-\log(K) - m_i\right)
\]

Under the risk-neutral probability, we have to impose the martingale condition stating that the current price \( S_t \) under the RND is equal to the expected discounted price of the underlying asset \( e^{-r \tau} E[S_T] \), so that \(^6\)

\[
S_t = e^{-r \tau} E[S_T] = e^{-r \tau} \sum_{j=1}^{M} \alpha_j \exp\left(m_j + \frac{1}{2}s_j^2\right).
\]

\(^5\)This result builds on Johnson, Kotz and Balakrishnan (1994) who indicate that if \( S \) follows a log-normal distribution with mean \( \mu \) and variance \( \sigma^2 \), then

\[
E[S | S > K] = \exp\left(\mu + \frac{1}{2} \sigma^2\right) \frac{1 - \Phi(U - \sigma)}{1 - \Phi(U)}
\]

where \( U = \frac{\log(K) - \mu}{\sigma} \)

\(^6\)In order to estimate the parameters \( \alpha_i, m_i \) and \( s_i \), we use this minimization program

\[
m_{1,2,3,4,5} \min_{\alpha_i, m_i, s_i} \sum_{i=1}^{N} \left( C_{i,1} - C_{i}(K_i, \tau, \theta) \right)^2 + \sum_{i=1}^{N} \left( P_{i,1} - P_{i}(K_i, \tau, \theta) \right)^2 + \left[ e^{-r \tau} \sum_{j=1}^{M} \alpha_j \exp\left(m_j + \frac{1}{2}s_j^2\right) \right]^2.
\]
As an alternative, we can directly use the BSM formula and set, in a similar spirit to what precedes,

\[ C^{LN}(K, \theta) = \sum_{i=1}^{M} \alpha_i \left[ S_i \Phi(d_{1i}) - Ke^{-rT} \Phi(d_{2i}) \right] \] (16)

where \( d_{1i} = \left[ \log \left( \frac{S_i}{K} \right) + \left( m_i \tau + \frac{1}{2} s_i^2 \right) \right] / s_i \) and \( d_{2i} = d_{1i} - s_i \), the martingale condition may be imposed as follows

\[ S_i = C^{LN}(0, \theta) \] (17)

This condition just means that, for a strike price of 0, the option will always get exercised and, hence, at maturity we will always get the underlying asset.

In practice, imposing \( K = 0 \) in the BSM formula is not a good idea because \( \log(0) \) is not defined. It is possible to approach this limit case by setting \( K \) equal to some very small value. There are several advantages of using this technique. First, the implementation is straightforward, because it suffices to slightly modify the usual BSM formula by introducing the additional parameter \( \mu \). Second, the implementation via the BSM formula introduces the current price of the underlying asset, \( S_t \). The parameter \( \mu \) will therefore be of the magnitude of an annualized asset return. This means that its value can be easily bounded, which is a welcome feature for numerical purposes. Third, the time to maturity \( \tau \) is explicitly taken into account. This means that \( \mu \), will be an annualized number which implies that parameters extracted for different maturities are directly comparable.

The mixture of distribution approach comes, however, at a cost. A first drawback in fitting a mixture of distributions is the symmetry between the densities. To illustrate this point, assume which implies that parameters extracted for different maturities are directly comparable.

Another difficulty lies in testing the number of distributions that are involved in the RND. One may be tempted to use a likelihood-ratio type test to check if the \( i \)th density should be included in the mixture by testing if \( \alpha_i = 0 \). However, if \( \alpha_i = 0 \), the parameters \( m_i \) and \( s_i \) associated with the \( i \)th density are unidentified. Stated differently \( m_i \) and \( s_i \) could take any value, because they would not play a role in the density \( q \). Sometimes, such parameters are called down and specific tests need to be developed. This type of problems appears also in switching regressions. There is no obvious solution to it. Critical values may be obtained by simulations. In practice, one may add distributions up to the point where adding them yields to no further improvement.

There exist various solutions to help the optimizer converge in terms of MSE. In the event of two densities, \( M = 2 \), we start with a grid for \( \alpha \). Since it is known that \( 0 \leq \alpha \leq 1 \), one may subdivide the interval \([0, 1]\) into equally spaced points, where \( \alpha_i = i \Delta \) for \( i = 0, ..., N \). In practice, \( \Delta = 0.1 \) often yields very good results as Jondeau, Ser-Huang Poon and Rockinger (2007) find. Furthermore, to avoid the problem of symmetry mentioned above, it is possible to impose that the densities remain in a given order. One possibility that appears to give satisfactory results is to impose that \( s_1 > s_2 \).
The first density will then have a larger standard deviation than the second one. Similar extensions may be given for the case where $M = 3$. The $a$ parameters may be taken over a simplex.\footnote{For further details, see Jondeau, Ser-Huang Poon and Rockinger (2007).}

### 2.4 Semi-parametric methods

#### 2.4.1 Edgeworth expansions

This method is developed by Jarrow and Rudd (1982) for which a numerical application can be found in Corrado and Su (1996). The idea is to capture deviations from log-normality by an Edgeworth expansion of RND $q(S_T | \theta)$ around the log-normal density. The construction of an Edgeworth expansion is conceptually similar to computing a Taylor expansion but applies to functions in general and to densities in particular. In a conventional Taylor expansion, the function is approximated by a simple polynomial around a given point. Here, the RND is approximated by an expansion around a log-normal distribution. A clear advantage of the expansion approach is the approximation, by involving parameters that can be varied, allows generating more general functions. Flamouris and Giamouridis (2002) and Beber and Brandt (2003) studied also the Edgeworth expansions. Let $Q$ be the cdf of a random variable $S_T$ and $q$ its density. Let’s define the characteristic function of $S$ as $\phi_Q(u) = \int e^{iuq(s)}ds$. If moments of $S_T$ exist up to order $n$, then there exist cumulants of the distribution $Q$, denoted $\kappa_{Q,j}$ and implicitly defined by the expansion

$$
\log(\phi_Q(u)) = \sum_{j=1}^{n-1} \frac{\kappa_{Q,j} (iu)^j}{j!} + o(u^{n-1}) \quad (19)
$$

Thus, if the characteristic function $\phi_Q(.)$ is known, by taking an expansion of its logarithm around $u = 0$, it is possible to obtain the cumulants. We have these relationships between the cumulants and moments up to the fourth order $\kappa_{Q,1} = E[S_T]$, $\kappa_{Q,2} = V[S_T]$, $\kappa_{Q,3} = E[(S_T - E[S_T])^3]$, $\kappa_{Q,4} = E[(S_T - E[S_T])^4] - 3V[S_T]^2$.

Jarrow and Rudd show that an Edgeworth expansion of the fourth order for the true probability distribution $Q$ around the log-normal cdf $L$ can be written, after imposing that the first moment of the approximating density and the true density are equal, ($\kappa_{Q,1} = \kappa_{L,1}$). The density $q(s)$ is

$$
q(s) = l(s) + \frac{\kappa_{Q,2} - \kappa_{L,2}}{2!} \frac{d^2 l(s)}{ds^2} - \frac{\kappa_{Q,3} - \kappa_{L,3}}{3!} \frac{d^3 l(s)}{ds^3} + \frac{\left(\kappa_{Q,4} - \kappa_{L,4}\right)}{4!} \frac{d^4 l(s)}{ds^4} + \epsilon(s) \quad (20)
$$

where $\epsilon(s)$ captures terms neglected in the expansion. The various terms in the expansion correspond to adjustments of the variance, skewness, and kurtosis. The interpretation of this expansion is similar to a Taylor expansion.

Jarrow and Rudd further show that, under the approximated density, the price of a European call option with strike $K$ can be approximated as

$$
C(Q) = e^{-rt} \int_{K}^{\infty} (S_T - K) q(S_T) dS_T
$$

$$
\approx e^{-rt} \int_{K}^{\infty} (S_T - K) l(S_T) dS_T + e^{-rt} \frac{\kappa_{Q,2} - \kappa_{L,2}}{2!} \int_{K}^{\infty} (S_T - K) \frac{d^2 l(S_T)}{ds^2} dS_T
$$

$$
- e^{-rt} \frac{\kappa_{Q,3} - \kappa_{L,3}}{3!} \int_{K}^{\infty} (S_T - K) \frac{d^3 l(S_T)}{ds^3} dS_T
$$

$$
+ e^{-rt} \frac{\left(\kappa_{Q,4} - \kappa_{L,4}\right)}{4!} \int_{K}^{\infty} (S_T - K) \frac{d^4 l(S_T)}{ds^4} dS_T + \epsilon(T, K)
$$
\[-e^{-rt} \frac{\kappa_{Q,3} - \kappa_{L,3}}{3!} \int_K^\infty (S_T - K) \frac{d^3 l(S_T)}{dS_T^3} dS_T + e^{-rt} \frac{(\kappa_{Q,4} - \kappa_{L,4}) + 3(\kappa_{Q,2} - \kappa_{L,2})^2}{4!} \int_K^\infty (S_T - K) \frac{d^4 l(S_T)}{dS_T^4} dS_T.\]

Notice that the first term is simply the Black-Scholes formula. In addition, the log-normal distribution has the following property

\[\int_K^\infty (S_T - K) \frac{d^j l(S_T)}{dS_T^j} dS_T = \frac{d^{j-2} l(S_T)}{dS_T^{j-2}} \bigg|_{S=K}, \quad \text{for } j \geq 2.\]

We deduce for the call option price

\[C(Q) \approx C(L) + e^{-rt} \frac{\kappa_{Q,2} - \kappa_{L,2}}{2!} l(K) - e^{-rt} \frac{\kappa_{Q,3} - \kappa_{L,3}}{3!} dl(K) + e^{-rt} \frac{(\kappa_{Q,4} - \kappa_{L,4}) + 3(\kappa_{Q,2} - \kappa_{L,2})^2}{4!} \frac{d^2 l(K)}{dS_T^2}. \tag{23}\]

For the log-normal density, the first cumulants are given by

\[\kappa_{L,1} = S_t e^r,\]

\[\kappa_{L,2} = \left[\kappa_{L,1} \delta\right]^2,\]

\[\kappa_{L,3} = \left[\kappa_{L,1} \delta\right]^3 \left(3 \delta + \delta^3\right),\]

\[\kappa_{L,4} = \left[\kappa_{L,1} \delta\right]^4 \left(16 \delta^2 + 15 \delta^4 + 6 \delta^6 + \delta^8\right). \tag{24}\]

where \(\delta = \left(e^{rT} - 1\right)^{1/2}\) and where the first relation follows from risk-neutral valuation. Jarrow and Rudd suggest identifying the second moment by imposing \(\kappa_{L,2} = \kappa_{Q,2}\). This argument is also justified on numerical grounds by Corrado and Su (1996) who notice that without this condition there exists a problem of multicollinearity between the second and the fourth moments. Rather than estimating the remaining cumulants, \(\kappa_{Q,3}\) and \(\kappa_{Q,4}\), Corrado and Su (1996) estimate standardized skewness and kurtosis (written respectively \(\gamma_{Q,1}\) and \(\gamma_{Q,2}\)), which are defined through the relationships:

\[\gamma_{Q,1} = \frac{\kappa_{Q,3}}{(\kappa_{Q,2})^{3/2}} = 3 \delta + \delta^3,\]

\[\gamma_{Q,2} = \frac{\kappa_{Q,4}}{(\kappa_{Q,2})^2} = 16 \delta^2 + 15 \delta^4 + 6 \delta^6 + \delta^8. \tag{25}\]

These expressions also hold for the log-normal density. The skewness and kurtosis of the log-normal density can therefore be derived easily from the cumulants above.

With the assumption of equality of the second cumulants for the approximating and the true distributions, it follows that

\[C(Q) \approx C(L) - e^{-rt} \left(\gamma_{Q,1} - \gamma_{L,1}\right) \frac{\kappa_{L,2}^3}{3!} \frac{dl(K)}{dS_T} + e^{-rt} \left(\gamma_{Q,2} - \gamma_{L,2}\right) \frac{\kappa_{L,2}^2}{4!} \frac{d^2 l(K)}{dS_T^2}. \tag{26}\]
Hence the price of a call option may be written as

\[ q(S_T) = l(S_T) - (\gamma_{Q,1} - \gamma_{L,1}) \frac{k_{L,2}^2}{6} \frac{d^3 l(S_T)}{dS_T^3} + (\gamma_{Q,2} - \gamma_{L,2}) \frac{k_{L,2}^2}{24} \frac{d^4 l(S_T)}{dS_T^4}. \]  

(27)

where the partial derivatives can be computed iteratively using

\[ \frac{dl(S_T)}{dS_T} = -\left(1 + \frac{\log(S_T) - m}{\sigma^2} \right) l(S_T). \]  

(28)

\[ \frac{d^2 l(S_T)}{dS_T^2} = -\left(2 + \frac{\log(S_T) - m}{\sigma^2} \right) \frac{1}{S_T} \frac{dl(S_T)}{dS_T} - \frac{1}{S_T^2 \sigma^2} l(S_T). \]  

(29)

\[ \frac{d^3 l(S_T)}{dS_T^3} = -\left(3 + \frac{\log(S_T) - m}{\sigma^2} \right) \frac{1}{S_T^2} \frac{d^2 l(S_T)}{dS_T^2} - \frac{2}{S_T^3 \sigma^2} \frac{dl(S_T)}{dS_T} + \frac{1}{S_T^3 \sigma^2} l(S_T), \]  

(30)

\[ \frac{d^4 l(S_T)}{dS_T^4} = -\left(4 + \frac{\log(S_T) - m}{\sigma^2} \right) \frac{1}{S_T^3} \frac{d^3 l(S_T)}{dS_T^3} - \frac{3}{S_T^4 \sigma^2} \frac{d^2 l(S_T)}{dS_T^2} + \frac{3}{S_T^4 \sigma^2} \frac{dl(S_T)}{dS_T} - \frac{2}{S_T^4 \sigma^2} l(S_T), \]  

(31)

where \( m = \log(S_0) + (r - \sigma^2/2)T \). The RND in the Edgeworth case will be a polynomial whose coefficients directly command the skewness and kurtosis of the RND. We also remark that the RND involves rather complicated terms with derivatives of the log-normal density.

### 2.4.2 Hermite polynomials

An alternative, yet similar, semi-parametric approach relies on an approximation of the Gaussian density based on Hermite polynomials. The theoretical foundation of this method is elaborated in Madan and Mline (1994) and applied in Abken, Madan, and Ramamurtie (1996) and Coutant, Jondeau, and Rockinger (2001).

For numerical reasons, Madan and Mline consider the map from the actual prices into the space generated by standardized log-returns \( z \). Let’s denote the volatility over the horizon of the option as \( s = \sigma \sqrt{T} \),

\[ S_T = S_0 \exp\left(\mu T - \frac{1}{2} s^2 + sz\right) \Rightarrow z = \frac{\log(S_T/S_0) - (\mu T - \frac{1}{2} s^2)}{s} \]  

(32)

The standarlized log return \( z \) has zero mean and unit variance.

If we focus on a call option, the payoff of such an option, as a function \( z \), is

\[ c(S_t, K, \mu, s, \tau) = \max\left(S_t \exp\left(\mu T - \frac{1}{2} s^2 + sz\right) - K, 0\right) = g(z), \]  

(33)

Hence the price of a call option may be written as

\[ C(S_t, K, \mu, s, \tau) = e^{-r \tau} \int_0^{+\infty} c(S_t, K, \mu, s, \tau)q_L(z)dz \]
Here, the notation $g(z)$ illustrates that the result holds for all sorts of payoffs. $q_z(.)$ denotes the risk-neutral density of $z$. We can go from the RND defined in $z$ space to the RND defined in $S_T$ space using the following change of variable

$$q(S_T)dS_T = q_z \left( \log \left( \frac{S_T}{S_t} \right) - (\mu \tau - \frac{1}{2} s^2) \right) \times \frac{1}{s} \times dS_T \quad (35)$$

Now, we assume that any payoff $g(z)$ may be expressed as a function of basis elements. It turns out that a basis for the Gaussian space is given by Hermite polynomials. In other words, there exist real numbers $a_k$ such that any payoff can be written as

$$g(z) = \sum_{k=0}^{\infty} a_k h_k(z), \quad (36)$$

with $a_k = \int g(z) h_k(z) \Phi(z) dz$, where $\Phi(z)$ is the Gaussian distribution and $h_k(z)$ denotes Hermite polynomials normalized to unit variance, defined as

$$h_0(z) = 1,$$
$$h_1(z) = z,$$
$$h_2(z) = \frac{1}{\sqrt{2}} (z^2 - 1),$$
$$h_3(z) = \frac{1}{\sqrt{6}} (z^3 - 3z),$$
$$h_4(z) = \frac{1}{\sqrt{24}} (z^4 - 6z^2 + 3). \quad (37)$$

The coefficients $a_k$ can be interpreted as the covariance between the option payoff and the $kth$ Hermite polynomial risk. Consequently, the price of the call option becomes

$$C(S_t, K, \mu, s, \tau, r) = e^{-r \tau} \sum_{k=0}^{\infty} a_k \int h_k(z) q_z dz. \quad (38)$$

Madan and Milne (1994) assume that the RND $q_z(z)$ can be represented as the product of a change of measure density and a reference measure density

$$q_z(z) = \lambda(z) \Phi(z). \quad (39)$$

Here, the reference measure density $\Phi(z)$ is simply the Gaussian one and the risk-neutral change of measure density $\lambda(z)$ is approximated by a Hermite polynomial expansion

$$\lambda(z) = e^{r \tau} \sum_{l=0}^{\infty} \pi_l h_l(z) \quad (40)$$
Under the reference measure, \( z \) is by construction normally distributed with zero mean and unit variance. But more generally, the RND \( q_z(z) \) may incorporate some departure from normality. The RND can be rewritten as

\[
q_z(z) = \Phi(z) \left( \sum_{l=0}^{\infty} \pi_l h_l(z) \right)
\]

The \( \pi_l \) are interpreted as the implicit price of polynomial risk \( h_l(z) \). Obviously, since the polynomial components are not traded, these risks can not be traded either. For practical purposes, the infinite sum can be truncated up to the fourth order. Since the Hermite polynomial of order \( l \) will depend on the \( l \)th moment, we will also refer to \( \pi_3 \) and \( \pi_4 \) as the price of skewness and kurtosis, respectively. It can be easily shown that for \( q_z(z) \) to be a density, we need \( \pi_0 = e^{-\tau \tau} \). Besides, since \( z \) is the standardized log-return (with zero mean and unit variance by construction), it follows that the shifts for the mean \( \pi_1 \) and the variance \( \pi_2 \) of \( q_z(z) \) relative to the reference measure can be set equal to 0. Hence, we estimate the mean \( \mu \) and the variance \( \sigma^2 \) of log-returns and set \( \pi_1 = \pi_2 = 0 \). We obtain

\[
q_z(z) = \Phi(z) e^{\tau \sum_{l=4}^{\infty} \pi_l h_l(z)} = \Phi(z) e^{\tau (\pi_3 h_3(z) + \pi_4 h_4(z))}
\]

where the \( b_j = e^{\tau} \pi_{i_j} \), \( i = 3, 4 \), are the future value of the \( i \)th price of risk coefficient. The parameters \( b_3 \) and \( b_4 \) correspond to the skewness and kurtosis if the reference measure density of \( z \) is chosen to be the normal distribution. It is important to emphasize that unlike the Edgeworth case, since a further change of variable from \( z \) to \( S_T \) has to made, \( b_3 \) and \( b_4 \) will not correspond to the skewness and kurtosis of the underlying price process \( S_T \) but to the skewness and kurtosis of log-returns \( z \). Finally, the skewness and kurtosis of the expansion are given by

\[
\text{Skewness} [z] = \sqrt{6} b_3
\]

\[
\text{Kurtosis} [z] = 3 + \sqrt{24} b_4
\]

The parameters \( \sigma \), \( b_3 \) and \( b_4 \) can be obtained using a nonlinear estimation method.

The general expression for the price of a call option is given by

\[
C(S_t, K, \mu, s, \tau, r) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_k \pi_l \int z h_l(z) h_k(z) \Phi(z) dz = \sum_{k=0}^{\infty} a_k \pi_k
\]

where the second equality holds because Hermite polynomials form an orthogonal system. When the infinite sum is truncated up to the fourth order, we obtain

\[
C(S_t, K, \mu, s, \tau, r) = e^{-\tau \tau} a_0 + \pi_3 a_3 + \pi_4 a_4
\]

Therefore, it remains to be shown how to obtain the \( a_k \) coefficients. Abken, Madan and Ramamurtie (1998) introduce the call option generating function

\[
G(u, S_t, K, \mu, s, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} c(u, S_t, K, \mu, s, \tau) \exp\left(-\frac{1}{2} (z - u)^2\right) dz,
\]
where \( u \) is a dummy variable. The coefficients \( a_k \) can then be obtained from the generating function as

\[
a_k(S_t, K, \mu, s, \tau) = \frac{\partial^k G(u, S_t, K, \mu, s, \tau)}{\partial u^k} \bigg|_{u=0} \frac{1}{\sqrt{k!}}
\]  

(48)

The evaluation of \( G(u, S_t, K, \mu, s, \tau) \) is shown to be

\[
G(u, S_t, K, \mu, s, \tau) = S_t \exp \left( \mu \tau + su \right) \Phi(d_1(u)) - K \Phi(d_2(u))
\]  

(49)

where \( d_1(u) = \log(S_t/K) + (\mu \tau + s/2) + u \) and \( d_2(u) = d_1(u) - s \). From there on, it is possible to compute the coefficients \( a_k \). Let’s introduce the notation \( d_j = d_{(0)} \) that is we evaluate the \( d_{(0)} \) at zero. We obtain after tedious computations, and introducing the notations \( \Phi’, \Phi”, \text{and} \Phi’’’ \), for the first, second, and third derivatives of standard normal density with respect to its argument

\[
a_0 = S_t e^{u\tau} \Phi(d_1) - K \Phi(d_2)
\]

\[
a_1 = s S_t e^{u\tau} \Phi(d_1) + S_t e^{u\tau} \Phi’(d_1) - K \Phi’(d_2)
\]

\[
a_2 = \frac{1}{\sqrt{2}} \left[ s^2 S_t e^{u\tau} \Phi(d_1) + 2s S_t e^{u\tau} \Phi’(d_1) + S_t e^{u\tau} \Phi’’(d_1) - K \Phi’’(d_2) \right]
\]

\[
a_3 = \frac{1}{\sqrt{6}} \left[ s^3 S_t e^{u\tau} \Phi(d_1) + 3s^2 S_t e^{u\tau} \Phi’(d_1) + 3s S_t e^{u\tau} \Phi’’(d_1) + S_t e^{u\tau} \Phi’’’(d_1) - K \Phi’’’(d_2) \right]
\]

\[
a_4 = \frac{1}{\sqrt{24}} \left[ s^4 S_t e^{u\tau} \Phi(d_1) + 4s^3 S_t e^{u\tau} \Phi’(d_1) + 6s^2 S_t e^{u\tau} \Phi’’(d_1) + 4s S_t e^{u\tau} \Phi’’’(d_1) + S_t e^{u\tau} \Phi’’’’(d_1) - K \Phi’’’’(d_2) \right]
\]  

(50)

This method has been applied in Coutant, Jondeau and Rockinger (2001) to interest-rate options. Jondeau and Rockinger (2001) have also shown how conditions on \( b_3 \) and \( b_4 \) could be imposed so that the polynomial approximation remains positive.

### 2.5 Non-parametric methods

#### 2.5.1 Tree-based methods

The tree-based method has been presented by Rubinstein (1994) and is further extended in Jackwerth and Rubinstein (1996) and Jackwerth (1999). Jackwerth (1997) gives a numerical illustration of the method which is based on two steps. First of all, we take existing options nearest as well as we compute the average of their Black-Scholes implied volatilities. Secondly, we compute the RNDs that would be associated to the binomial tree of Cox, Ross and Rubinstein (1979) and that would be comparable with these implied volatilities.

Formally, let’s consider that the tree has \( N \) steps. A terminal node, \( j \), of the binomial tree has a probability to realize of \( P_j \). If \( p' \) is the risk-neutral probability of an up movement on each node of the tree, then

\[
P_j = \frac{N!}{j!(N-j)!} p'^j (1 - p')^{(N-j)}
\]  

\(^{8}p' = \frac{u^2}{2d^2} \), where \( u = e^{\sqrt{\tau}} \) and \( d = \frac{1}{u} \)
In that formula, it is assumed that the nodes are denoted in such a way that a small \( j \) corresponds to a low value of the support \( S_T \). Let’s denote these values as \( S_j, j = 1,...,N \). Given the current value of the underlying asset, it is also possible to construct the nodal values of the underlying asset in a risk-neutral environment. Even though by working backwards, we could use the binomial tree to value options, this tree would not be compatible with the observed option smile. The third possibility is to seek terminal probabilities that are close to the tree-based probabilities, yet that also correspond to the given options. Formally, let the risk-neutral probabilities, which are compatible with actual options, be denoted by \( P_j \). These risk-neutral probabilities may then be obtained by solving the following optimization:

\[
\min_{P_j} \sum_{j=1}^{N} (P_j - P'_j)^2
\]  

subject to

\[
\sum_{j=1}^{N} P_j = 1, \quad P_j \geq 0, \quad j = 0,...,N
\]  

\[
S_t = e^{-r(t)} \sum_{j=1}^{N} S_j P_j
\]  

\[
C_{t,i} = e^{-r_i} \sum_{j=1}^{N} P_j \left(S_j - K_i\right)^+, \quad i = 1,...,N
\]

The first constraint provides conditions so that the values \( P_j \) define probabilities. The second condition corresponds to the martingale condition. Finally, the last condition states that the probabilities \( P'_j \) should be compatible with existing option prices. Various metrics can be used to define the distance between the probabilities \( P_j \) and \( P'_j \). The idea of using other metrics is expressed in Rubinstein (1994) and tested in Jackwerth and Rubinstein (1996). We could choose the following distances:

\[
\min_{P_j} \sum_{j=1}^{N} P'_j - P_j
\]  

\[
\min_{P_j} \sum_{j=1}^{N} \left(\frac{(P_j - P'_j)^2}{P'_j}\right)
\]  

or

\[
\max_{P_j} \sum_{j=1}^{N} P_j \log \left(\frac{P_j}{P'_j}\right)
\]

The first measure is a metric based on absolute deviations. The second measure is based on percentage deviations. The third measure corresponds to seeking the maximum entropy.
2.5.2 Kernel regression

Aït-Sahalia and Lo (1998) propose to estimate the RND nonparametrically by exploiting Breeden and Litzenberg’s (1978) insight that $f_t(S_t) = \exp(r_t)\hat{H}(\cdot)/\partial K^2$. They suggest using market prices to estimate an option pricing formula $\hat{H}(\cdot)$ nonparametrically, which can then be differentiated twice with respect to $K$ to obtain $\partial^2\hat{H}(\cdot)/\partial K^2$. They use kernel regression to construct $\hat{H}(\cdot)$ and assume that the option-pricing formula $H$ to be estimated is an arbitrary nonlinear function of a vector of option characteristics $y \equiv [F_{t,\tau} \ K \ \tau \ r_{t,\tau}]^T$ where $F_{t,\tau}$ is the forward price of the asset.

In practice, they propose to reduce the dimension of the kernel regression by using a semiparametric approach. Assume that the call pricing function is given by the parametric Black-Scholes approach is consistent over time, but may provide a poor fit for certain dates. Consequently, this estimator that is based on both cross-sectional and time-series option prices. Consequently, this approach is consistent over time, but may provide a poor fit for certain dates.

We assume that the function $C$ defined by the equation (59) satisfies all required conditions to be a "rational" option-pricing formula in the sense of Merton (1973, 1992). In this model, we only need to estimate non-parametrically the regression of $\sigma$ on a subset $\hat{Y}$ of the vector of explanatory variables $\hat{Y}$. The rest of the call pricing functions $C(\cdot)$ is parametric, thereby considerably reducing the sample size $n$ required to achieve the same degree of accuracy as the full nonparametric estimator. Aït-Sahalia and Lo (2000) partition the vector of explanatory variables $y = \{\hat{Y} F_{t,\tau} r_{t,\tau}\}$, where $\hat{Y}$ contains $\delta$ nonparametric regressors. In their empirical application, they consider $\hat{Y} \equiv [K/F_{t,\tau}]$ ($\delta = 2$) and form the Nadaraya-Watson Kernel estimator of $E\{\sigma[K/F_{t,\tau}]\}$ as:

$$\hat{C}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) = C_{BS}(F_{t,\tau}, K, \tau, r_{t,\tau}, \hat{\sigma}(K/F_{t,\tau}))$$

We estimate the call pricing function as:

$$\hat{C}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) = C_{BS}(F_{t,\tau}, K, \tau, r_{t,\tau}, \hat{\sigma}(K/F_{t,\tau}))$$

The RND estimator follows by taking the second partial derivatives of $\hat{C}(\cdot)$ with respect to $K$:

$$\hat{f}_t(S_t) = \hat{C}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) \left. \partial^2 \hat{C}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) \right|_{K=S_t}.$$
3 Structural approaches of RND functions estimation

3.1 Jump diffusion model

Modeling the underlying asset with a Black & Scholes model is not consistent. Large values of returns occur too frequently to be consistent with the normality assumption. The rare events may cause brutal assets pricing variations. In order to model such a phenomenon, we should use a Poisson jump process. We assume that the underlying asset price follows a log normal jump diffusion process i.e, the addition of a geometric Brownian motion and a Poisson jump process. This process takes into account the skewness and kurtosis effects.

The underlying asset price under the risk-neutral world follows the process:

\[
dS_t = (r_t - \lambda E(k)) S_t dt + k S_t dW_t + q_t \lambda dt
\]

where \(q_t\) is the Poisson associated probability, \(\lambda\) is the average rate of jump occurrence and \(k\) is the size of a jump.

Bates (1991) has shown that in case of a diffusion with \(n\) jumps, the call price is:

\[
C^{\text{IMP}}(S_t, t, T, K) = e^{-rT} \sum_{i=0}^{\infty} P(n \text{jumps}) E\left[(S_T - K)^+ / n \text{jumps}\right]
\]

Ball and Torous (1985) and Malz (1997) proposed a simplified version of the jump-diffusion model, where there will be at most one jump of a constant size. In this Bernoulli version of a jump diffusion model, the equation of the call price is the following:

\[
C^{\text{IMP}}(S_t, t, T, K) = e^{-rT} \sum_{i=0}^{1} P(n \text{jumps}) E\left[(S_T - K)^+ / n \text{jumps}\right]
\]

\[
(1 - \lambda \tau) \left[ \frac{S_t}{1 + \lambda \kappa \tau} N\left(d_1 + \sigma \sqrt{\tau}\right) - Ke^{-rT} N\left(d_1\right) \right]
\]
\[
+ \lambda \tau \left[ \frac{S_t}{1 + \lambda \kappa \tau} (1 + \kappa) N\left(d_2 + \sigma \sqrt{\tau}\right) - Ke^{-rT} N\left(d_2\right) \right]
\]

with

\[
d_1 = \frac{\log\left(S_t / K\right) - \log(1 + \lambda \kappa \tau) + (r_t - \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}
\]

\[
d_2 = \frac{\log\left(S_t / K\right) - \log(1 + \lambda \kappa \tau) + \log(1 + \kappa) + (r_t - \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}
\]

where \((1 - \lambda \tau)\) is the probability that no events occur in the option life.

We can easily show that the call price is simply a combination of two call prices calculated by Black (1976) model

\[
C^{\text{IMP}}(S_t, t, T, K|\theta) = (1 - \lambda \tau) C^B\left(\frac{S_t}{1 + \lambda \kappa \tau}, \tau, K|\theta\right) + (\lambda \tau) C^B\left(\frac{S_t}{1 + \lambda \kappa \tau} (1 + \kappa), \tau, K|\theta\right)
\]
This technique is a particular case of a mixture of log-normal distributions. The RND is:

\[ q^{MP}(S_T, S_t, t, T | \theta) = (1 - \lambda T) \ln(N(S_T, \alpha, \beta) + \lambda T \ln(N(S_T, \alpha + \log(1 + \kappa), \beta)) \]  

where:

\[ \ln(x) = \frac{1}{\sqrt{2\pi}\beta} x \exp\left[-\frac{(\ln(x) - \alpha)^2}{2\beta^2}\right] \]  

\[ \alpha = \log(S_t) + (\mu - \frac{1}{2}\sigma^2)(T - t) \]  

the \( \theta = (\sigma, \lambda, \kappa)' \) vector parameters are estimated using the following program:

\[ \min_\theta \sum_{i=1}^n (\hat{C}_i - C^{(M)}_i)^2 \]  

3.2 Heston’s stochastic volatility model

Most diffusion models of the underlying asset suppose that the volatility of the latter is a constant. Nevertheless, if the underlying asset records strong fluctuations over a short period, it would be necessary to model it with a stochastic volatility process. Heston (1993) proposed the following model:

\[ dS_t = \mu_t S_t dt + \sqrt{V_t} S_t dW_1^t \]  
\[ dV_t = \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} dW_2^t \]  
\[ dW_1^t dW_2^t = \rho dt \]

where \( \{S_t\}_{t \geq 0} \) and \( \{V_t\}_{t \geq 0} \) are the price and volatility process, respectively, and \( \{W_1^t\}_{t \geq 0} \) is a square root mean reverting process, first used by Cox, Ingersoll and Ross (1985), with long-run mean \( \theta \) and rate of reversion \( \kappa \). \( \sigma \) is referred to as the volatility of volatility. All the parameters \( \mu, \kappa, \theta, \sigma, \rho \) are time and state homogenous.

There are several economic, empirical and mathematical reasons for choosing a model with such a form.\(^{[11]}\)

Under the Heston model, the value of any option \( \Theta(S_t, V_t, t, T) \) must satisfy the following partial differential equation:

\[ \frac{1}{2} V S^2 \frac{\partial^2 \Theta}{\partial S^2} + \rho \sigma V S \frac{\partial^2 \Theta}{\partial S \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial \Theta}{\partial V^2} + r S \frac{\partial \Theta}{\partial S} + \left[ \kappa [\theta - V] - \Lambda(S, V, t) \sigma \sqrt{V} \right] \frac{\partial \Theta}{\partial V} - \partial \Theta + \frac{\partial \Theta}{\partial t} = 0 \]  

where \( \Lambda(S, V, t) \) is called the market price of volatility risk. Heston supposes that the market price of volatility risk is propotional to volatility, i.e.

\[ \Lambda(S, V, t) = k \sqrt{V} \]  

\(^{[11]}\) see Cont (2001) for a detailed statistical analysis.
\[ \Rightarrow \Lambda(S, V, t) \sigma \sqrt{V} = k \sigma V, \text{ for some constant } k \]
\[ = \lambda(S, V, t) \]  
(78)

Risk neutral valuation is pricing of a contingent claim in an equivalent martingale measure. The price is evaluated as the expected discounted payoff of the contingent claim, under the Risk neutral probability, the option value is

\[ \text{Option value} = E^Q [e^{r(T-t)}H(T)] \]  
(79)

where \( H(T) \) is the payoff of the option at time \( T \) and \( r \) is the risk free of interest over \([t, T]\).

Moving from a real world measure \( P \) to a risk neutral world measure \( Q \) is achieved by Girsanov’s theorem. In particular, we have

\[ d\tilde{W}^2_t = dW^2_t + \Lambda(S, V, t) dt \]  
(80)

\[ \frac{dQ}{dP} = \exp \left\{ -\frac{1}{2} \int_0^t (\nu^2 + \Lambda(S, V, s)^2) ds - \int_0^t \nu_s dW_s^1 - \int_0^t \Lambda(S, V, s) dW_s^2 \right\} \]  
(81)

\[ \nu_t = \frac{\mu - r}{\sqrt{V_t}} \]  
(82)

where \( \{\tilde{W}^1_t\}_{t \geq 0} \) and \( \{\tilde{W}^2_t\}_{t \geq 0} \) are \( Q \)– Brownian motions. Under measure \( Q \), the equations (73), (74) and (75) become,

\[ dS_t = r_t S_t dt + \sqrt{V_t} S_t d\tilde{W}^1_t \]  
(83)

\[ dS_t = \kappa^* (\theta^* - V_t) dt + \sigma \sqrt{V_t} d\tilde{W}^2_t \]  
(84)

\[ d\tilde{W}^1_t d\tilde{W}^2_t = \rho dt \]  
(85)

and the equation (76) becomes :

\[ \frac{1}{2} VS^2 \frac{\partial^2 \Theta}{\partial S^2} + \rho \sigma VS \frac{\partial^2 \Theta}{\partial S \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 \Theta}{\partial V^2} + \nu S \frac{\partial \Theta}{\partial S} + \left\{ \kappa^* [\theta^* - V] - \lambda(S, V, t) \sigma \sqrt{V} \right\} \frac{\partial \Theta}{\partial V} - r \Theta + \frac{\partial \Theta}{\partial t} = 0 \]  
(86)

where,

\[ \kappa^* = \kappa + \lambda \]  
(87)

\[ \theta^* = \frac{\kappa \theta}{\lambda + \lambda} \]  
(88)

In this important result, \( \lambda \) has effectively been eliminated.

The closed-form solution of a European call option on a non-dividend paying asset for the Heston model is :

\[ C(S_t, V_t, t, T) = S_t P_1 - K e^{-r(T-t)} P_2 \]  
(89)

where

\[ P_j(x, V_t, K, T) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left\{ \frac{e^{-i\phi \log(K)} f_j (x, V_t, T, \phi)}{i\phi} \right\} d\phi \]  
(90)
\[ f_j(x, V_t, T, \phi) = \exp \left\{ C(T, \phi) + D(T, \phi)V_t + i \phi x \right\} \] (91)

\[ C(T-t, \phi) = r \phi i t + \frac{\alpha}{\sigma^2} \left( \frac{b_j - \rho \sigma \phi i + d_j}{1 - g_j \epsilon^{d_j/t}} \right) \tau - 2 \log \left( \frac{1 - g_j \epsilon^{d_j/t}}{1 - g_j} \right) \] (92)

\[ D(T, \phi) = \frac{b_j - \rho \sigma \phi i + d_j}{\sigma^2} \left[ \frac{1 - \epsilon^{d_j/t}}{1 - g_j \epsilon^{d_j/t}} \right] \] (93)

\[ g_j = \frac{b_j - \rho \sigma \phi i + d_j}{b_j - \rho \sigma \phi i - d_j} \] (94)

\[ d_j = \sqrt{\left( (\rho \sigma \phi - b_j)^2 - (2u_j \phi i - \phi^2) \right)} \] (95)

\[ x = \log(S_t) \] (96)

for \( j = 1, 2 \) where, \( u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, \alpha = \kappa \theta, b_1 = k^+ - \rho \sigma, b_2 = k^- \).

The \( \theta = (\kappa^+, \theta^+, \rho, V_t, \sigma) \) vector parameters are then estimated using the following algorithm:

\[ \min_{\theta} \sum_{i=1}^{n} (\hat{C}_i - C_i)^2 \] (97)

After estimating model parameters, the risk-neutral density is obtained from the numerical calculation of the integral. A method to evaluate formulas in form of (92) has been proposed by Carr and Madan (1999).

4 Application to CAC 40 index options

In this section, we estimate parametrically, semi-parametrically and nonparametrically the RND functions of CAC 40 index options between January 1st 2007 and December 31st 2007. We compare the goodness-of-fit of eight option-based approaches during a normal and troubled period. To our knowledge, this is the first application on the European market dealing with European options using structural and non-structural approaches of RND functions estimation.

4.1 Data

Our data is provided by the SBF-Paris Bourse\(^{13}\) and includes intraday values of the CAC 40 stock index and intraday transaction prices of CAC 40 options over the period January 1st 2007-December 31st 2007. CAC 40 options are traded on the MONEP, the French derivatives market. Trading covers eight open maturities: three spot months, four quarterly maturities (March, June, September, December) and two half-yearly maturities (March, September). The maturity date is

\(^{12}\)It is important to note that the interpretations of \( \kappa^+ \) and \( \sqrt{\theta^+} \) as respectively the rate of reversion and long term volatility are still valid.

\(^{13}\)SBF-Paris Bourse provides a monthly CD-ROM including intraday values of the CAC 40 stock index and intraday transaction prices of CAC 40 options traded on the MONEP.
the last trading day of each month. Trading takes place on a continuous basis between 9:00 am and 5:30 pm.

Figure 1: CAC 40 stock index evolution (January 1st 2007- December 31st 2007)

The future underlying asset $F_{t,T}$ cannot be observed exactly at time $t$, where $t$ denotes a given day and time and $T$ is the maturity. Aït-Sahalia and Lo (1998) suggest extracting an implied future index price from the put-call parity. Given our intraday data, we cannot get contemporaneous trades concerning a call and a put with similar strike price and maturity. Thus, for each maturity, we compute:

$$F_{t,T} = S_t \left( \exp \left( r_{t,T} - \delta_{t,T} \right) \right)$$ (98)

To circumvent the unobservability of the dividend rate $\delta_{t,T}$, we extract an implied value of the dividend rate between the end of day $t$ and $T$ from the daily closing price of the index, $S_{t_{\text{end}}}$ and the settlement future index price, $F_{t_{\text{end}},T}$ observed at the end of each day. The obtained dividend rate is the dividend rate expected by the market between $t_{\text{end}}$ and $T$:

$$\delta_{t_{\text{end}},T} = r_{t,T} - \frac{1}{\tau} \ln \left( \frac{F_{t_{\text{end}},T}}{S_{t_{\text{end}}}} \right)$$ (99)

$F_{t,T}$ is then computed using the dividend rate and the riskfree interest rate proxied by Euribor. Since some CAC 40 options are not traded actively, we need to filter the data carefully. Five filters are applied to the initial data. We omit the quotes of the first and the last 15 minutes each day and the option quotes characterized by a price lower or equal to one tick. We only consider options with a moneyness \footnote{Strike price divided by the future index level.} comprised between 0.85 and 1.15. This procedure eliminates far-away-from-the-money observations, which are unreliable due to their low volume and low sensitivity towards

\footnote{The constant dividend hypothesis passes over the clustering of dividends paid by most firms during specific months. However, as far as we are concerned, the bias is not relevant since we use market prices of futures to estimate an implied dividend rate. The constant dividend hypothesis could be replaced by a more realistic actual dividend assumption, leading to the following pricing formula for the futures: $F_{t,T} = S_t \exp (r_{t,T}(T-t)) \+ \sum_{i=1}^{n} D_i t \exp (r_{t,T}(T-t_i))$ where $t_i$ is the payment date of the $i^{th}$ dividend. Nevertheless, this formula requires the prediction of all dividends amounts and payment dates paid by the 40 CAC 40 index shares and the prediction of the forward interest rates.}
volatility. Besides, options quotes violating general no-arbitrage conditions, that is put-call parity, are eliminated as well as we replace the price of all illiquid options, that is in-the-money options, with the price implied by put-call parity at the relevant strike prices. Specially, we replace the price of each in-the-money call option with $P(S_t, K, \tau, r_t, \delta_t) + (F_{t,T} - K)e^{-r_t\tau}$, where, by construction, the put with price $P(S_t, K, \tau, r_t, \delta_t)$ is out-of-the money and therefore liquid. After this procedure, all the information included in liquid put prices is extracted and resides in corresponding call prices through the put-call parity. Put prices may now be splayed without any loss of reliable information. Since our data include only call options, a single volatility function can be estimated.

We provide in Figure 2 the implied volatility observed on January 1st 2007 and Figure 3 reports the volatility smiles for four various maturities observed on July 10th 2007, when the market trend is bear. The implied volatility is computed using the Black & Scholes formula. For a given maturity, we compute the daily average implied volatility for each strike price. For fixed $S_t$ and $T$, it is shown that the implied volatility function is not constant as assumed in the Black & Scholes model but decreases with respect to the strike price. The implied volatility observed on July 10th 2007 on the French market is then more a “skew” than a “smile”. In this case, contrary to what suggests Black-Scholes model, out-the-money and in-the-money options represent increased risk on potentially very large movements in the underlying compared to at-the-money. To compensate for this risk, they tend to be priced higher.

Figure 2: Implied volatility surface on the French market on January 1st 2007
Figure 3: Implied volatility obtained from the Black & Scholes formula on the French market on July 10\textsuperscript{th} 2007
4.2 Methods comparison

4.2.1 Kernel approach estimation

We focus on the nonparametric Kernel approach proposed by Aït-Sahalia and Lo (2001). This method does not require any parametric restriction on either the underlying asset price dynamics, the family of distributions that the RND belongs to or any prior distribution for the RND. We use the whole database in order to estimate nonparametrically the volatility function given in paragraph (2.5.2), with two conditional variables ($d=2$): the moneyness and the time to maturity $\tau$. We only consider two regressors. We rely on the Aït-Sahalia and Lo (1998)'s procedure to select the kernel functions and the optimal bandwidths. They are chosen to optimize the asymptotic properties of the option pricing function, $H$. The selected kernels associated with the two regressors are Gaussian:

\[
\begin{align*}
 k_{K/F} &= \frac{1}{\sqrt{2\pi}} e^{-0.5(K/F)^2} \\
 k_\tau &= \frac{1}{\sqrt{2\pi}} e^{-0.5(\tau)^2}
\end{align*}
\]

For each regressor, we choose the optimal bandwidth $h$ according to the following equations:

\[
\begin{align*}
 h_{K/F} &= c_{K/F} \sigma_{K/F} n^{-1/(d+2(q_{K/F}+m))} \\
 h_\tau &= c_\tau \sigma_\tau n^{-1/(d+2q_\tau)}
\end{align*}
\]

where $n$ is the sample size, $q_j$ is the number of existing continuous partial derivatives of the function to be estimated with respect to the $j^{th}$ regressor ($j=K/F, \tau$), $m$ is the order of the partial derivative with respect to the $j^{th}$ regressor that we want to estimate, $d$ is the number of regressors and $\sigma_j$ is the unconditional standard deviation of the $j^{th}$ regressor. The parameter $c_j$ depends on the choice of the kernel and the function to be estimated. It is typically of the order of one. Aït-Sahalia and Lo (1998) emphasize that small deviations from the exact value have no noticeable effects. We report in Table 2 the values of the coefficients and bandwidths.

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Figure 4 shows $\hat{\sigma}(K/F, \tau)$, the Nadaraya-Watson estimator of the volatility parameter with two regressors $X/F$ and $\tau$, for different maturities (20 days, 50 days, 80 days and 170 days i.e 1,2,3 and 6 months). We note that the estimator generates a strong volatility smile with respect to the moneyness. Besides, in confirmity with Aït-Sahalia and Lo (1998) and Bliss and Panigirtzoglou (2002), we observe that the reported smiles shape varies with the chosen time to maturity.

The estimated RND is shown in Figure 5 for 3-month maturity on January 10th 2007. The RND is obtained by computing the second derivative of the call price with respect to the strike price. We note from Figure 5 the limits of the kernel method. Density is unclear because of the low number of observed points. This method would probably be better suited to a study of a large sample or to estimate the returns time series.
Figure 4: Nonparametric estimators of the volatility for different maturities

Figure 5: Nonparametric estimator of the state-price density $q(S_T)$ with 3-month maturity on January 10th 2007
4.2.2 Tree-based methods

The estimated RND on a discrete probability framework, as proposed by Jackwerth and Rubinstein (1996), is shown in Figure 6 for 3 periods on January 10th 2007 (normal period), on July 10th 2007 on which the market trend is bear and on October 17th 2007 for various maturities.

![Figure 6: Risk Neutral Densities of the tree based method for different dates and various maturities](image)

We note from Figure 6 that in the troubled period, the RNDs for all the maturities have heavier tails than just before and after the crisis periods. The CAC 40 index options exhibit heavy tail
distribution in the troubled period; so in a risk neutral framework, the probability estimated by the tree based method does not undervalue the probability of occurrence of extreme events.
4.2.3 Parametric and semi-parametric RND estimation methods comparison

Several papers compare the performance of various methods to extract RND. Jondeau and Rockinger (2000) compare the mixture of log-normals, the Edgeworth and Gram-Charlier approximation, the single-jump and Heston’s stochastic volatility model using exchange rates. They find that up to a certain extent, these methods provide similar RNDs. They remark that during normal periods, a mixture of log-normals provides rather good results. However during a troubled period, the jump diffusion model performs better. Coutant et al. (2001) compare the mixture of log-normals, Gram-Charlier expansion and the entropy-based method regarding speed, estimation robustness and ease of implementation using interest rates. They find that for interest rates, the method based on Gram-Charlier expansion seems to provide rather stable results. Bliss and Panigrizoglou (2002) compare the robustness of a model based on the mixture of two log-normal distributions and a smoothed implied volatility smile model, in the spirit of Shimko (1993). They find that their smoothed spline method performs better than the mixture of log-normal distributions. However, despite its high flexibility, the smoothed spline method does not recover the tails of the RND outside the range of available strike prices.

In addition to nonparametric estimation methods of the RND, i.e. the kernel and the tree-based methods, we now compare six parametric and semi-parametric option-based approaches used to extract RND from CAC 40 index options during a normal and troubled period using the same data as in section 4.1. The first technique is the numerical approximation of the RND based on the second derivative of option prices with respect to the strike price, as suggested by Breeden and Litzenberger (1978). The second method is obtained using a mixture of two log-normal distributions, following Melick and Thomas (1997). The third RND technique is the Edgeworth expansion around the log-normal distribution of Jarrow and Rudd (1982). The fourth method is Hermite polynomials, suggested by Madan and Mline (1994). The fifth RND is based on Heston’s stochastic volatility model (1993). Finally, we consider the jump diffusion model following Bates (1991). The results of estimated parameters of different RND models are reported in Tables 3 to 8. The various resulting RNDs are plotted in Figures 7 to 12. The mixture of log-normals and Edgeworth expansion, Hermite polynomials and jump-diffusion models are more in line with historical data than the numerical approximation of the RND using the Breeden and Litzenberg formula. For this reason, the nonparametric approaches such as kernel and tree-based models, as described above, were developed. The mixture of log-normals, Edgeworth expansion, hermite polynomials, jump diffusion and Heston models have heavier tails than the log-normal distribution.

The existence of several RND estimation methods from option prices raises the question of the suitable method to be chosen. To answer this question, we provide goodness-of-fit measures, allowing to investigate how well theoretical options prices calculated with RND estimation approaches fit observed market prices $\hat{C}$. Synthetic measures for errors calculating are used: the Mean Squared Error (MSE) and the Average Relative Error (ARE).

$$\text{MSE} = \frac{10^2}{m - n} \sum_{i=1}^{m} (C_i - \hat{C}_i)^2$$

16See Appendix B.
17See Appendix A.
\[
ARE = \frac{10^4}{m-n} \sum_{i=1}^{m} \left( \frac{C_i - \hat{C}_i}{C_i} \right)^2
\]

where \(m\) is the number of observed options prices and \(n\) is the number of method parameters. The best method is the one which has the smallest errors.

In order to compare the various RND models, not only we check the statistical properties of the different RNDs but we also calculate the MSE and the ARE for different dates 01-10-2007, 07-10-2007 (troubled period) and 10-17-2007 and various maturities 1,2 and 3 months. All results are displayed in Tables 9 to 13. First of all, we compare the volatilities, skewness and kurtosis obtained under different RNDs. Concerning the volatilities (Table 9), the volatility induced by the Black & Scholes model is smaller than the one we obtain using the other methods, and this is true for all periods and all maturities, which explain the bias implied by the log-normality assumption.

Regarding the implicit skewness (Table 10) and the implicit kurtosis (Table 11), the log-normal model is less interesting because it does not allow for asymmetry or fat tails. We observe a strongly negative skewness and a larger excess of kurtosis for all methods but the Black & Scholes model. The skewness obtained by the semi-parametric methods (Hermite polynomial and Edgeworth expansion models) are lower than that of the other models on the normal period (01-10-2007). Whereas on the troubled period (07-07-2007), Hermite polynomials and the mixture of log-normals models have the lower skewness, on the period just after the crisis, the jump-diffusion model has the lower skewness from the two month maturities, followed by the Hermite polynomials model. The graphs of the RND corroborate these findings.

Table 12 reports the Mean Squared Error (MSE) for the different methods. According to this criterion, the jump diffusion model on the period just before the crisis for relatively short maturities (1 and 2 months) provides a much better fit than the other models. However, during this same period, the mixture of log-normal models provides the best fit of the data for the 3 month maturity. We notice that in the troubled period (07-10-2007), the Edgeworth expansion model provides the best fit for all maturities. Concerning the period just after the crisis, the jump diffusion model performs well for the two month maturity while the Edgeworth expansion model provides the best fit for the three month maturity.

The Average Relative Error (ARE) results are presented in Table 13. As for the MSE criterion, the jump diffusion model outperforms the other models on the period just before the crisis. On the troubled period, Hermite polynomial method and the mixture of log-normal models have the smallest errors. The difference between the two corresponding ARE is small. Concerning the period just after the crisis, the semi-parametric methods provide the best fit for the one and three month maturities, while the mixture of log-normals approach provides a much better fit than the other models.

The RNDs functions are of paramount importance. Once estimated, it is possible to extract a lot of various information. For instance, we can perform tests and compute confidence intervals around the expected value whose evolution offers the opportunity to investors to measure how markets are thought to evolve through time. Besides, RNDs can provide a measure of expected extreme variations in the underlying asset price, which is an important tool for risk management. Moreover, since the existence of a possibly time-varying risk premium RNDs differs from real probability distributions, the RNDs incorporate information concerning the risk aversion of investors.

\[^{18}\text{See Appendix B.}\]
\[^{19}\text{except on 10-17-2007 for less than one month maturity, the jump diffusion model has the smallest volatility.}\]
5 Conclusion

In this paper, eight option-based approaches are presented and applied to the French derivatives market. Structural and non-structural approaches are used to estimate risk-neutral probability density functions from a high-frequency CAC 40 index options during a normal and troubled period. We use Black & Scholes model, mixture of log-normals, Edgeworth expansions, Hermite polynomials, tree-based methods, kernel regression, Heston’s stochastic volatility model and jump diffusion model. We determine nonparametric, semi-parametric, parametric and stochastic implied volatility functions and Risk-Neutral Densities (RND). In order to compare the various RND models, not only we check the statistical properties of the different RNDs, but also we calculate goodness of fit measures. We find that the kernel estimator generates a strong volatility smile with respect to the moneyness, and kernel smiles shape varies with the chosen time to maturity. The mixture of log-normals, Edgeworth expansion, hermite polynomials, jump diffusion and Heston models are more in line with historical data and have heavier tails than the log-normal distribution. Moreover, according to the goodness of fit criteria, the jump diffusion model provides a much better fit than the other models on the period just-before the crisis for relatively short maturities. However, during this same period the mixture of log-normals model performs better for more than three month maturity. Furthermore, in the troubled period and the period just-after the crisis, we find that semi-parametric models are the methods with the best accuracy in fitting observed option prices for all maturities with a minimal difference towards the mixture of log-normals model.
References


6 Appendix

A. Graphs
Figure 7: The Black & Scholes Risk Neutral Densities for different dates and various maturities
Figure 8: Hermite polynomials Risk Neutral Densities for different dates and various maturities
Figure 9: Edgeworth expansions Risk Neutral Densities for different dates and various maturities
Figure 10: Mixture of log-normals Risk Neutral Densities for different dates and various maturities
Figure 11: Jump diffusion Risk Neutral Densities for different dates and various maturities
### Table 3: Estimation of Black & Scholes parameters

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### Table 4: Estimation of Mixture of log-normal parameters

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Table 8: Estimation of Heston’s Stochastic Volatility model parameters

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Table 9: Implicit standard deviation

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Note that $\tau$ is the maturity, B&S is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, and Jump is the jump-diffusion model.

Table 10: Implicit skewness

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Note that $\tau$ is the maturity, B&S is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, and Jump is the jump-diffusion model.

Table 11: Implicit kurtosis

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Note that $\tau$ is the maturity, B&S is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, and Jump is the jump-diffusion model.
Table 12: MSE

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Note that τ is the maturity, B&S is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, Jump is the jump-di ffusion model and Heston is the Heston’s stochastic volatility model.

Table 13: ARE (10⁴)

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Note that τ is the maturity, B&S is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, Jump is the jump-di ffusion model and Heston is the Heston’s stochastic volatility model.