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Implied Risk-Neutral probability Density functions from options prices : A comparison of estimation methods

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Abstract

This paper compares the goodness-of-fit of eight option-based approaches used to extract risk-neutral probability density functions from a high-frequency CAC 40 index options during a normal and troubled period. Our findings show that the kernel estimator generates a strong volatility smile with respect to the moneyness, and the kernel smiles shape varies with the chosen time to maturity. The mixture of log-normals, Edgeworth expansion, hermite polynomials, jump diffusion and Heston models are more in line and have heavier tails than the log-normal distribution. Moreover, according to the goodness of fit criteria we compute, the jump diffusion model provides a much better fit than the other models on the period just-before the crisis for relatively short maturities. However, during this same period, the mixture of log-normal models performs better for more than three month maturity. Furthermore, in the troubled period and the period just-after the crisis, we find that semi-parametric models are the methods with the best accuracy in fitting observed option prices for all maturities with a minimal difference towards the mixture of log-normals model.

JEL classification : C02; C14; C65; G13.

Keywords : Risk-neutral density, mixture of log-normal distributions, Edgeworth expansions, Hermite polynomials, tree-based methods, kernel regression, Heston's stochastic volatility model, jump diffusion model.

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1 Introduction

The Risk-Neutral Density (RND) is an interesting tool for an investor who seeks to measure how market expectations evolve over time. A huge literature arose from the early 1990s on the most appropriate way to estimate this density. At the origin of all the methods, we find the famous work of Breeden and Litzenberger (1978), who are the pioneers who determined a relationship between option prices and the RND.

Malik and Thomas (1997) have developed a nonparametric method to estimate the American option prices. This technique has been adopted by Bahra (1996) and Schernick, Garcia and Tirupattur (1996) on European options. It is a representation of RND based on a linear combination of log-normal density fonctions. These authors estimated the RND of the oil price during the Gulf crisis and found that the estimated density is different from the density obtained by standard methods. Söderlind and Svensson (1997) showed how this method can be applied to different financial assets with emphasis on its use, like monetary policy. Abadir and Rockinger (1997) determined farm formulas of option prices using Kummer functions (hypergeometric) as a basis for RND. Backus, Foresi, Li and Wu (1997) have approached the conditional distribution of underlying assets prices by Gram-Charlier series expansion.

Another approach has been proposed by Jarrow and Rudd (1982). They have developed a method of option valuation under the assumption that the underlying asset does not follow a log-normal distribution. They showed that the RND can be obtained by an Edgeworth expansion. Carrodo and Su (1996) used the Jarrow and Rudd approach to determine the skewness and kurtosis of the price of S&P500 options. A similar approach is that of Madan and Milne (1994), Abken, Madan and Ramamurti (1996) and Coutant (1999) who determined the RND from a Hermite polynomial approximation. Similarly, El Hassan and Kucera (1998) used the Fourier-Hermite development in order to evaluate European and American index options.

Among other recent researches, Heston (1993) has provided a quasi-analytical solution for pricing options in a stochastic volatility framework. Breeden and Litzenberger (1978) showed that the option price second derivative over the strike gives the risk-neutral density. Based on this relationship between option prices and RND, Rzepkawski (1996) adopted the Heston method to extract the RND. Similarly, Neuhauss (1995) used the direct relationship of Breeden and Litzenberger (1978) working with the distribution function instead of the risk-neutral density. Rubinstein (1994, 1996) and Jackwerth (1996) suggested a method based on binomial trees. Rubinstein (1994) developed a tree to estimate the implicit prices of contingent assets[1] from prices options. His method is to minimize the gap between the tree implied probabilities and the probabilities determined from the tree of Cox-Ross-Rubinstein (1979). Aït-Sahalia (1996) and Aït-Sahalia and Lo (1998) estimated the S&P500 Risk-Neutral Density by performing Kernel techniques estimation. Campa, Chang and Reider (1998) compared three different RND estimating methods of the underlying asset. These methods are a smoothing of the smile through a degree three polynomial, a Rubinstein implicit tree and a mixture of log-normal distributions.

Galati and Melick (1999) estimated the moments using a mixture of log-normal laws in order to understand how the central bank interventions are perceived by foreign exchange markets traders regarding JPY / USD options between 1993 and 1996. Weinberger (2001) and Anagnar, Bedendo, Hodges and Tompkins (2002) found a typical form of the implied S&P500 risk neutral distribution after the crisis. Panigirtzoglou and Skiadapoulos (2004) studied implied distributions dynamics and provided algorithms that make their results applicable to the options and risk management

[1] A contingent asset is an asset that pays one unit of good consumed when a world state takes place and nothing else.

framework. Mark, Feike and Lo(2005) showed that the RND is a function of both the underlying asset returns and volatility. Enzo, Handel and Härdle (2006) estimated the RND using the same approach developed by Aït-Sahalia and Lo (1998). Finally, Jondeau, Ser-Huang Poon and Rockinger (2007) compared the performance of various parametric and semi-parametric methods to extract RND.

Some authors argued that the RND coresponds to real world density only if investors are risk neutral. Thus, the difference between the two densities stemps from the existence of a risk premium. Consequently, it is interesting to study the ability of the estimated RND extracted from options prices to assess the real density, its forecasting ability as well as the choice of the suitable RND estimation technique.

In this paper, we estimate parametrically, semi-parametrically and nonparametrically the RND functions of CAC 40 index options between January 1^{st} 2007 and December 31^{st} 2007, and we compare the goodness-of-fit of eight option-based approaches during a normal and troubled period. To our knowledge, this is the first application dealing with European options using structural and non-structural approaches of RND functions estimation. Specially, we use nonparametric estimation methods of the RND, that is the kernel and the tree-based methods as well as six parametric and semi-parametric option-based approaches: (i) the numerical approximation of the RND based on the second derivative of option prices with respect to the strike price, as suggested by Breeden and Litzenberger (1978), (ii) the mixture of log-normal distributions following Melick and Thomas (1997), (iii) the Edgeworth expansion around the log-normal distribution of Jarrow and Rudd (1982), (iv) the Hermite polynomials, suggested by Madan and Mline (1994), (v) Heston's stochastic volatility model (1993) and (vi) the jump diffusion model following Bates (1991). To our best knowledge, our study is the first that combines the three approaches : parametric, semi-parametric and nonparametric methods of RND estimation.

The paper is organized as follows. Section 2 discusses our methodological non-structural approaches of RND functions estimation. Section 3 presents the structural methods we use to extract the RND functions. Section 4 describes our data, and contains the estimation results as well as related comments regarding the comparison of the various RND estimation methods. Section 5 concludes.

2 Non-structural approaches of RND functions estimation

2.1 The Breeden and Litzenberger relation

Breeden and Litzenberger (1978) were the first to derive the *RND* using the following price of a call option formula :

$$C(S_t, t) = e^{-r\tau} E^* \left[max \left(S_T - K, 0 \right) | S_t, t \right] (1)$$

$$= e^{-r\tau} \int_0^\infty max \left(S_T - K, 0 \right) q \left(S_T | S_t, t \right) dS_T.((1))$$

Where for date t and maturity date T, we denote C the call price, r is the risk-free interest rate, S is the underlying asset price, K is the strike price and $q(.)$ is the undiscounted RND.

Differentiating this equation with respect to the exercise price K yields the discounted cdf

$$\frac{\partial C}{\partial K} = -e^{-r\tau} \int_K^\infty q(S_T) dS_T. \quad (2)$$

and differentiating twice yields the discounted pdf [2]

$$\frac{\partial^2 C}{\partial K^2}\Big|_{K=S_T} = e^{-r\tau} q(S_T).(3)$$

These equations show that the second derivative of the call price yields the discounted RND[3]. This suggests that a first method to extract RND is to approximate it numerically applying the finite difference approach to (3). Nevertheless, this method relies on the hypothesis that there exist traded option prices for many strikes. This is not likely to be the case in practice. Also, it has been shown that *RNDs* estimated in this manner are very unstable. In fact, differentiating twice exacerbates even tiny errors in the prices and may be difficult[4]. That is why it is necessary to extract RND using alternative methods that put more structure on the option prices.Before describing such methods, we briefly show how the parameters of these models are estimated.

2.2 RND parameters estimation

Suppose that we have to estimate a given model with respect to a set of parameters θ. Assume that for horizon τ, we have N_c^τ strike prices for which we have call options and N_p^τ strike prices for which we have put options. For date t and horizon τ, observed call and put options prices are denoted $C_{t,\tau,i}$ $i = 1, ..., N_c^\tau$ and $P_{t,\tau,i}$ $i = 1, ..., N_p^\tau$. Theoretical call and put option prices, implied by the assumed model, are denoted $C_t(K, \tau, \theta)$ and $P_t(K, \tau, \theta)$, respectively, for strike price K and horizon τ. Then, the parameter vector $\theta \in \Theta$ is typically estimated by non-linear least squares, by minimizing for each day and each maturity

$$\overset{min}{\theta \in \Theta} \sum_{i=1}^{N_c^\tau} w_i^c \left(C_{t,\tau,i} - C_t(K_i, \tau, \theta)\right)^2 + \sum_{i=1}^{N_p^\tau} w_i^p \left(P_{t,\tau,i} - P_t(K_i, \tau, \theta)\right)^2. \tag{4}$$

where w_i^c and w_i^p are weights associated with option i and Θ is the domain of θ. These weights could be given by a measure of liquidity of a given option.

2.3 Parametric methods

2.3.1 Black & Scholes Model

The price dynamics of an underlying asset of a Black & Scholes Model (BSM) is a Geometric Brownian motion. Under the log-normal assumption, the volatility is constant across both the exercice price and horizons. The BSM underlying asset price is given by :

$$dS_t = (\mu + \frac{\sigma^2}{2})S_t dt + \sigma S_t dW_t \tag{5}$$

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[2]Breeden and Litzenberger (1978) suggest to evaluate the RND using this approximation

$$\frac{\partial^2 C}{\partial K^2} \approx e^{-r\tau} \frac{C(K_{i+1}) - 2C(K_i) + C(K_{i-1})}{(\Delta K)^2}.$$

where $\Delta K = 50$. See also Roncalli (1997).

[3]It should be mentioned that $q(.)$ is the undiscounted RND whereas $e^{-r\tau}q(.)$ represents an Arrow -Debreu state price, which is referred to as the RND.

[4]For instance, if option prices suffer from non-synchronicity bias (that is, the underlying asset price is not observed at the same time as the option price), if the option price is fudged because of some microstructure reason (for instance, due to the bid-ask spread).

where μ and σ represent respectively the instantaneous return assets expectations and volatility. W_t is a Brownian motion under the real probability measure $\mathbb{P}$.
When markets are complete, Harrison and Pliska (1981) show that there exists a risk-neutral transformation that leads to the following formula :

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{6}$$

where $W_t^{\mathbb{Q}}$ is a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$.

The call option price C_{BS} maturing at date $T = t + \tau$, with strike price K and dividend yield $\delta_{t,\tau}$, is given by :

$$C_{BS}(S_t, K, \tau, r, \delta_{t,\tau}, \sigma) = e^{-r\tau} \int_0^{\infty} max[S_T - K, 0] f_{BS}(S_T) dS_T$$

$$= S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2). \tag{7}$$

Where $\Phi(.)$ is the standard cumulative normal distribution function and

$$d_1 = \frac{log(S_t/K) + (r - \delta_{t,\tau} + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}. \tag{8}$$

Consequently, the RND is a log-normal density with a mean $m = (ln(S_t) + (r - \delta_{t,\tau}) - \frac{\sigma^2}{2})\tau$ and variance $s^2 = \sigma^2\tau$:

$$f_{BS}(S_T) = e^{r\tau} \frac{\partial^2 C_{BS}}{\partial K^2} \Big|_{K=S_T}$$

$$= \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} exp\left[-\frac{[log(S_T) - m]^2}{2s^2} \right]. \tag{9}$$

All Black & Scholes parameters , except the volatility, are directly observable. Nevertheless, the BSM hypothesis, according to which the returns are homoscedastic and normally distributed, does not fit with empirical evidence showing that densities are skewed and leptokurtik.

2.3.2 Mixture of log-normal distributions

Bahra (1996), Melick and Thomas (1997) and Södernlind (1997) are among the first to describe the RND as a mixture of distributions. For option pricing, the most well-known distribution studied in the litterature is the mixture of log-normal densities. The reason is that it appears as a trivial extension of the Black & Scholes model that involves the single log-normal density function.

$$l\,(S_T, m, \sigma) = \frac{1}{S_T\sqrt{2\Pi s^2}} exp\left(-\frac{1}{2} \left(\frac{log\,(S_T) - m}{s} \right)^2 \right). \tag{10}$$

A mixture of such densities yields

$$q\,(S_T; \theta) = \sum_{i=1}^{M} \alpha_i l\,(S_T, m_i, s_i)\,. \tag{11}$$

where θ regroups all the unknown parameters α_i, m_i *and* s_i *for* $i = 1, ..., M$, and M denotes the number of mixtures describing the RND. Obviously, to guarantee that q is a density, we must have $\alpha_i > 0$ for all $i = 1, ..., M$, and $\alpha_1 + ... + \alpha_M = 1$. In other words, q should be a convex combinaison of various log-normal densities.

The option price for such a mixture of log-normal distributions is, for a given strike K and time to maturity $\tau = T - t$

$$\begin{aligned} C^{LN}(K, \theta) &= e^{-r\tau} \int_K^{+\infty} (S_T - K)\, q(S_T; \theta)\, dS_T \\ &= e^{-r\tau} \int_K^{+\infty} (S_T - K) \sum_{i=1}^{M} \alpha_i l(S_T, m_i, s_i)\, dS_T \\ &= e^{-r\tau} \sum_{i=1}^{M} \alpha_i \int_K^{+\infty} (S_T - K)\, l(S_T, m_i, s_i)\, dS_T. \end{aligned} \tag{12}$$

where we define the volatility over the horizon of the option as $s_i = \sigma_i \sqrt{\tau}$, to simplify notations. The last equality is obtained by simply inverting the sum and integral operators. There are various ways to evaluate the integral. For instance, we have[5]

$$\begin{aligned} &\int_K^{+\infty} (S_T - K)\, l(S_T, \mu, s)\, dS_T = (E[S_T | S_T > K] - K)\, Pr[S_T | S_T > K] \\ &= exp\left(m + \frac{1}{2}s^2\right)\left[1 - \Phi\left(\frac{log(K) - m - s^2}{s}\right)\right] - K\left[1 - \Phi\left(\frac{log(K) - \mu}{s}\right)\right] \end{aligned} \tag{13}$$

Finally, the option price is given by

$$\begin{aligned} C^{LN}(K; \theta) &= e^{-r\tau} \sum_{i=1}^{M} \alpha_i \left\{ exp\left(\mu_i + \frac{1}{2}s_i^2\right)\left[1 - \Phi\left(\frac{log(K) - m_i - s_i^2}{s_i}\right)\right] - K\left[1 - \Phi\left(\frac{log(K) - m_i}{s_i}\right)\right]\right\} \\ &= e^{-r\tau} \sum_{i=1}^{M} \alpha_i exp\left(\mu_i + \frac{1}{2}s_i^2\right) \Phi\left(\frac{-log(K) + m_i + s_i^2}{s_i}\right) - e^{-r\tau} K \sum_{i=1}^{M} \alpha_i \Phi\left(\frac{-log(K) + m_i}{s_i}\right) \end{aligned} \tag{14}$$

Under the risk-neutral probability, we have to impose the martingale condition stating that the current price S_t under the RND is equal to the expected discounted price of the underlying asset $e^{-r\tau} E[S_T]$, so that[6]

$$S_t = e^{-r\tau} E[S_T] = e^{-r\tau} \sum_{j=1}^{M} \alpha_j exp\left(m_j + \frac{1}{2}s_j^2\right). \tag{15}$$

[5] This result builds on Johnson, Kotz and Balakrishman (1994) who indicate that if S follows a log-normal distribution with mean μ and variance σ^2, then

$$E[S|S > K] = exp\left(\mu + \frac{1}{2}s^2\right) \frac{1 - \Phi(U - \sigma)}{1 - \Phi(U)}$$

where $U = \frac{(log(K) - m)}{\sigma}$

[6] In order to estimate the parameters α_i, m_i *and* s_i, we use this minimization program

$$\min_{m_1, m_2, s_1, s_2, \alpha_1, \alpha_2} \sum_{i=1}^{N} \left(C_{t,\tau,i} - C_t(K_i, \tau, \theta)\right)^2 + \sum_{i=1}^{N} \left(P_{t,\tau,i} - P_t(K_i, \tau, \theta)\right)^2 + [e^{-r\tau} \sum_{j=1}^{2} \alpha_j exp\left(m_j + \frac{1}{2}s_j^2\right)]^2.$$

As an alternative, we can directly use the BSM formula and set, in a similar spirit to what precedes,

$$C^{LN}(K,\theta) = \sum_{i=1}^{M} \alpha_i \left[S_t \Phi(d_{1,i}) - Ke^{-r\tau}\Phi(d_{2,i})\right] \tag{16}$$

where $d_{1,i} = \left[log(S_T/K) + \left(m_i\tau + \frac{1}{2}s_i^2\right)\right]/s_i$, and $d_{2,i} = d_{1,i} - s_i$, the martingale condition may be imposed as follows

$$S_t = CL^{LN}(0,\theta) \tag{17}$$

This condition just means that, for a strike price of 0, the option will always get exercised and, hence, at maturity we will always get the underlying asset.
In practice, imposing $K = 0$ in the BSM formula is not a good idea because $log(0)$ is not defined. It is possible to approach this limit case by setting K equal to some very small value. There are several advantages of using this technique. First,the implementation is straightforward, because it suffices to slightly modify the usual BSM formula by introducing the additional parameter μ. Second, the implementation via the BSM formula introduces the current price of the underlying asset, S_t. The parameter μ, will therefore be of the magnitude of an annualized asset return. This means that its value can be easily bounded, which is a welcome feature for numerical purposes. Third, the time to maturity τ is explicitly taken into account. This means that μ_i will be an annualized number which implies that parameters extracted for different maturities are directly comparable.

The mixture of distribution approach comes, however, at a cost. A first drawback in fitting a mixture of distributions is the symmetry between the densities. To illustrate this point, assume that a given RND can be correctly described by a mixture involving exactly two normal densities. Using obvious notations, we can write

$$q(S_T, \alpha, m_1, m_2, s_1, s_2) = \alpha \ \ l(S_T, m_1, s_1) + (1-\alpha) \ \ l(S_T, m_2, s_2) \tag{18}$$

Obviously, the same density is obtained if we invert the two log-normal distributions, i.e, $q(S_T, 1-\alpha, m_1, m_2, s_2, s_1)$. Clearly, the order of the parameters plays a critical role here. For an optimization program, this means, however, that several parameter vectors are associated with a same density. This in turn could yield numerically unstable programs where the optimizer cycles in an infinite loop.

Another difficulty lies in testing the number of distributions that are involved in the RND. One may be tempted to use a likelihood-ratio type test to check if the ith density should be included in the mixture by testing if $\alpha_i = 0$. However, if $\alpha_i = 0$, the parameters m_i and s_i associated with the *ith* density are unidentified. Stated differently m_i and s_i could take any value, because they would not play a role in the density q. Sometimes, such parameters are called down and specific tests need to be developed. This type of problems appears also in switching regressions. There is no obvious solution to it. Critical values may be obtained by simulations. In partice, one may add distributions up to the point where adding them yields to no further improvement.

There exist various solutions to help the optimizer converge in terms of MSE. In the event of two densities, $M = 2$, we start with a grid for α. Since it is known that $0 \leq \alpha \leq 1$, one may subdivide the interval $[0, 1]$ into equally spaced points, where $\alpha_i = i\Delta$, for $i = 0, ..., N$. In practice, $\Delta = 0.1$ often yields very good results as Jondeau, Ser-Huang Poon and Rockinger (2007) find. Furthermore, to avoid the problem of symmetry mentioned above, it is possible to impose that the densities remain in a given order. One possibility that appears to give satsisfactory results is to impose that $s_1 > s_2$.

The first density will then have a larger standard deviation than the second one. Similar extensions may be given for the case where $M = 3$. The α parameters may be taken over a simplex.[7]

2.4 Semi-parametric methods

2.4.1 Edgeworth expansions

This method is developed by Jarrow and Rudd (1982) for which a numerical application can be found in Corrado and Su (1996). The idea is to capture deviations from log-normality by an Edgeworth expansion of RND $q(S_T|\theta)$ around the log-normal density. The construction of an Edgeworth expansion is conceptually similar to computing a Taylor expansion but applies to functions in general and to densities in particular. In a conventional Taylor expansion, the function is approximated by a simple polynomial around a given point. Here, the RND is approximated by an expansion around a log-normal distribution. A clear advantage of the expansion approach is that the approximation, by involving parameters that can be varied, allows generating more general functions. Flamouris and Giamouridis (2002) and Beber and Brandt (2003) studied also the Edgeworth expansions. Let Q be the cdf of a random variable S_T and q its density. Let's define the characteristic function of S as $\phi_Q(u) = \int e^{isu} q(s)ds$. If moments of S_T exist up to order n, then there exist cumulants of the distribution Q, denoted $\kappa_{Q,j}$ and implicity defined by the expansion

$$log\left(\phi_Q(u)\right) = \sum_{j=1}^{n-1} \kappa_{Q,j} \frac{(iu)^j}{j!} + \circ(u^{n-1}) \tag{19}$$

Thus, if the characteristic function $\phi_Q(.)$ is known, by taking an expansion of its logarithm around $u = 0$, it is possible to obtain the cumulants. We have these relationships between the cumulants and moments up to the fourth order $\kappa_{Q,1} = E[S_T]$, $\kappa_{Q,2} = V[S_T]$, $\kappa_{Q,3} = E\left[(S_T - E[S_T])^3\right]$, $\kappa_{Q,4} = E\left[(S_T - E[S_T])^4\right] - 3V[S_T]^2$.

Jarrow and Rudd show that an Edgeworth expansion of the fourth order for the true probabiliy distribution Q around the log-normal cdf L can be written, after imposing that the first moment of the approximating density and the true density are equal, $\left(\kappa_{Q,1} = \kappa_{L,1}\right)$. The density $q(s)$ is

$$q(s) = l(s) + \frac{\kappa_{Q,2} - \kappa_{L,2}}{2!} \frac{d^2 l(s)}{ds^2} - \frac{\kappa_{Q,3} - \kappa_{L,3}}{3!} \frac{d^3 l(s)}{ds^3} + \frac{\left(\kappa_{Q,4} - \kappa_{L,4}\right) + 3\left(\kappa_{Q,2} - \kappa_{L,2}\right)^2}{4!} \frac{d^4 l(s)}{ds^4} + \epsilon(s) \tag{20}$$

where $\epsilon(s)$ captures terms neglected in the expansion. The various terms in the expansion correspond to adjustments of the variance, skewness, and kurtosis. The interpretation of this expansion is similar to a Taylor expansion.

Jarrow and Rudd further show that, under the approximated density, the price of a European call option with strike K can be approximated as

$$\begin{aligned} C(Q) &= e^{-r\tau} \int_K^\infty (S_T - K)\, q(S_T) dS_T \\ &\approx e^{-r\tau} \int_K^\infty (S_T - K)\, l(S_T) dS_T + e^{-r\tau} \frac{\kappa_{Q,2} - \kappa_{L,2}}{2!} \int_K^\infty (S_T - K) \frac{d^2 l(S_T)}{dS_T^2} dS_T \end{aligned}$$

[7] For further details, see Jondeau, Ser-Huang Poon and Rockinger (2007).

$$-e^{-r\tau}\frac{\kappa_{Q,3}-\kappa_{L,3}}{3!}\int_K^\infty (S_T-K)\frac{d^3 l(S_T)}{dS_T^3}dS_T+e^{-r\tau}\frac{\left(\kappa_{Q,4}-\kappa_{L,4}\right)+3\left(\kappa_{Q,2}-\kappa_{L,2}\right)^2}{4!}\int_K^\infty (S_T-K)\frac{d^4 l(S_T)}{dS_T^4}dS_T. \tag{21}$$

Notice that the first term is simply the Black-Scholes formula. In addition, the log-normal distribution has the following property

$$\int_K^\infty (S_T-K)\frac{d^j l(S_T)}{dS_T^j}dS_T=\frac{d^{j-2}l(S_T)}{dS_T^{j-2}}\Big|_{S=K}, \quad for \;\; j\geq 2. \tag{22}$$

We deduce for the call option price

$$C(Q)\approx C(L)+e^{-r\tau}\frac{\kappa_{Q,2}-\kappa_{L,2}}{2!}l(K)-e^{-r\tau}\frac{\kappa_{Q,3}-\kappa_{L,3}}{3!}\frac{dl(K)}{dS_T}+e^{-r\tau}\frac{\left(\kappa_{Q,4}-\kappa_{L,4}\right)+3\left(\kappa_{Q,2}-\kappa_{L,2}\right)^2}{4!}\frac{d^2 l(K)}{dS_T^2}. \tag{23}$$

For the log-normal density, the first cumulants are given by

$$\kappa_{L,1}=S_t e^{r\tau},$$

$$\kappa_{L,2}=\left[\kappa_{L,1}\vartheta\right]^2,$$

$$\kappa_{L,3}=\left[\kappa_{L,1}\vartheta\right]^3\left(3\vartheta+\vartheta^3\right),$$

$$\kappa_{L,4}=\left[\kappa_{L,1}\vartheta\right]^4\left(16\vartheta^2+15\vartheta^4+6\vartheta^6+\vartheta^8\right). \tag{24}$$

where $\vartheta=\left(e^{\sigma^2\tau}-1\right)^{1/2}$ and where the first relation follows from risk-neutral valuation. Jarrow and Rudd suggest identifying the second moment by imposing $\kappa_{L,2}=\kappa_{Q,2}$. This argument is also justified on numerical grounds by Corrado and Su (1996) who notice that without this condition there exists a problem of multicolinearity between the second and the fourth moments. Rather than estimating the remaining cumulants, $\kappa_{Q,3}$ and $\kappa_{Q,4}$, Corrado and Su (1996) estimate standardlized skewness and kurtosis (written respectively $\gamma_{Q,1}$ and $\gamma_{Q,2}$), which are defined through the relationships :

$$\gamma_{Q,1}=\frac{\kappa_{Q,3}}{(\kappa_{Q,2})^{3/2}}=3\vartheta+\vartheta^3,$$

$$\gamma_{Q,2}=\frac{\kappa_{Q,4}}{(\kappa_{Q,2})^2}=16\vartheta^2+15\vartheta^4+6\vartheta^6+\vartheta^8. \tag{25}$$

These expressions also hold for the log-normal density. The skewness and kurtosis of the log-normal density can therefore be derived easily from the cumulants above.

With the assumption of equality of the second cumulants for the approximating and the true distributions, it follows that

$$C(Q)\approx C(L)-e^{-r\tau}\left(\gamma_{Q,1}-\gamma_{L,1}\right)\frac{\kappa_{L,2}^{3/2}}{3!}\frac{dl(K)}{dS_T}+e^{-r\tau}\left(\gamma_{Q,2}-\gamma_{L,2}\right)\frac{\kappa_{L,2}^2}{4!}\frac{d^2 l(K)}{dS_T^2} \tag{26}$$

Using this expression, it is easy to estimate with nonlinear least squares the implied volatility, σ^2, skewness, $\gamma_{Q,1}$, and kurtosis, $\gamma_{Q,2}$. The expression of the RND can be obtained after twice differentiating (26) with respect to K and then evaluation over S_T

$$q(S_T) = l(S_T) - (\gamma_{Q,1} - \gamma_{L,1})\frac{\kappa_{L,2}^{3/2}}{6}\frac{d^3 l(S_T)}{dS_T^3} + (\gamma_{Q,2} - \gamma_{L,2})\frac{\kappa_{L,2}^2}{24}\frac{d^4 l(S_T)}{dS_T^4}. \tag{27}$$

where the partial derivatives can be computed iteratively using

$$\frac{dl(S_T)}{dS_T} = -\left(1 + \frac{log(S_T) - m}{\sigma^2\tau}\right)\frac{l(S_T)}{S_T}. \tag{28}$$

$$\frac{d^2 l(S_T)}{dS_T^2} = -\left(2 + \frac{log(S_T) - m}{\sigma^2\tau}\right)\frac{1}{S_T}\frac{dl(S_T)}{dS_T} - \frac{1}{S_T^2\sigma^2}l(S_T). \tag{29}$$

$$\frac{d^3 l(S_T)}{dS_T^3} = -\left(3 + \frac{log(S_T) - m}{\sigma^2\tau}\right)\frac{1}{S_T}\frac{d^2 l(S_T)}{dS_T^2} - \frac{2}{S_T^2\sigma^2}\frac{dl(S_T)}{dS_T} + \frac{1}{S_{1T}^3\sigma^2}l(S_T), \tag{30}$$

$$\frac{d^4 l(S_T)}{dS_T^4} = -\left(4 + \frac{log(S_T) - m}{\sigma^2\tau}\right)\frac{1}{S_T}\frac{d^3 l(S_T)}{dS_T^3} - \frac{3}{S_T^2\sigma^2}\frac{d^2 l(S_T)}{dS_T^2} + \frac{3}{S_T^3\sigma^2}\frac{dl(S_T)}{dS_T} - \frac{2}{S_T^4\sigma^2}l(S_T), \tag{31}$$

where $m = log(S_t) + (r - \sigma^2/2)\tau$. The RND in the Edgeworth case will be a polynomial whose coefficients directly command the skewness and kurtosis of the RND. We also remark that the RND involves rather complicated terms with derivatives of the log-normal density.

2.4.2 Hermite polynomials

An alternative, yet similar, semi-parametric approach relies on an approximation of the Gaussian density based on Hermite polynomials. The theoretical foundation of this method is elaborated in Madan and Mline (1994) and applied in Abken, Madan, and Ramamurtie (1996) and Coutant, Jondeau, and Rockinger (2001).

For numerical reasons, Madan and Mline consider the map from the actual prices into the space generated by standardized log-returns z. Let's denote the volatility over the horizon of the option as $s = \sigma\sqrt{\tau}$,

$$S_T = S_t exp\left(\mu\tau - \frac{1}{2}s^2 + sz\right) \Rightarrow z = \frac{log(S_T/S_t) - (\mu\tau - \frac{1}{2}s^2)}{s} \tag{32}$$

The standarlized log return z has zero mean and unit variance.

If we focus on a call option, the payoff of such an option, as a function z, is

$$c(S_t, K, \mu, s, \tau) = max\left(S_t exp\left(\mu\tau - \frac{1}{2}s^2 + sz\right) - K, 0\right) = g(z), \tag{33}$$

Hence the price of a call option may be written as

$$C(S_t, K, \mu, s, \tau) = e^{-r\tau}\int_0^{+\infty} c(S_t, K, \mu, s, \tau)q_z(z)dz$$

$$= e^{-r\tau} \int_0^{+\infty} g(z) q_z(z) dz \tag{34}$$

Here, the notation $g(z)$ illustrates that the result holds for all sorts of payoffs. $q_z(.)$ denotes the risk-neutral density of z. We can go from the RND defined in z space to the RND defined in S_T space using the following change of variable

$$q(S_T)dS_T = q_z\left(\frac{log(S_T/S_t) - (\mu\tau - \frac{1}{2}s^2)}{s}\right) \times \frac{1}{s} \times dS_T \tag{35}$$

Now, we assume that any payoff $g(z)$ may be expressed as a function of basis elements. It turns out that a basis for the Gaussian space is given by Hermite polynomials. In other words, there exist real numbers a_k such that any payoff can be written as

$$g(z) = \sum_{k=0}^{\infty} a_k h_k(z), \tag{36}$$

with $a_k = \int_z g(z)h_k(z)\Phi(z)dz$, where $\Phi(z)$ is the Gaussian distribution and $h_k(z)$ denotes Hermite polynomials normalized to unit variance, defined as

$$\begin{aligned} h_0(z) &= 1, \\ h_1(z) &= z, \\ h_2(z) &= \frac{1}{\sqrt{2}}(z^2 - 1), \\ h_3(z) &= \frac{1}{\sqrt{6}}(z^3 - 3z), \\ h_4(z) &= \frac{1}{\sqrt{24}}(z^4 - 6z^2 + 3). \end{aligned} \tag{37}$$

The coefficients a_k can be interpreted as the covariance between the option payoff and the *kth* Hermite polynomial risk. Consequently, the price of the call option becomes

$$C(S_t, K, \mu, s, \tau, r) = e^{-r\tau} \sum_{k=0}^{\infty} a_k \int_z h_k(z) q_z dz. \tag{38}$$

Madan and Milne (1994) assume that the RND $q_z(z)$ can be represented as the product of a change of measure density and a reference measure density

$$q_z(z) = \lambda(z)\Phi(z). \tag{39}$$

Here, the reference measure density $\Phi(z)$ is simply the Gaussian one and the risk-neutral change of measure density $\lambda(z)$ is approximated by a Hermite polynomial expansion

$$\lambda(z) = e^{r\tau} \sum_{l=0}^{\infty} \pi_l h_l(z) \tag{40}$$

Under the reference measure, z is by construction normally distributed with zero mean and unit variance. But more generally, the RND $q_z(z)$ may incorporate some departure from normality. The RND can be rewritten as

$$q_z(z) = \Phi(z)\left(\sum_{l=0}^{\infty} \pi_l h_l(z)\right) \quad (41)$$

The π_l are interpreted as the implicit price of polynomial risk $h_l(z)$. Obviously, since the polynomial components are not traded, these risks can not be traded either. For practical purposes, the infinite sum can be truncated up to the fourth order. Since the Hermite polynomial of order l will depend on the lth moment, we will also refer to π_3 and π_4 as the price of skewness and kurtosis, respectively. It can be easily shown that for $q_z(z)$ to be a density, we need $\pi_0 = e^{-r\tau}$. Besides, since z is the standardized log-return (with zero mean and unit variance by construction), it follows that the shifts for the mean π_1 and the variance π_2 of $q_z(z)$ relative to the reference measure can be set equal to 0. Hence, we estimate the mean μ and the variance σ^2 of log-returns and set $\pi_1 = \pi_2 = 0$. We obtain

$$\begin{aligned} q_z(z) &= \Phi(z)\left(e^{r\tau} \sum_{4}^{l=0} \pi_l h_l(z)\right) = \Phi(z) e^{r\tau}\left(e^{-r\tau} + \pi_3 h_3(z) + \pi_4 h_4(z)\right) \\ &= \Phi(z)\left(1 + \frac{b_3}{\sqrt{6}}(z^3 - 3z) + \frac{b_4}{\sqrt{24}}(z^4 - 6z^2 + 3)\right) \end{aligned} \quad (42)$$

Where the $b_i = e^{r\tau}\pi_i, \quad i = 3, 4$, are the future value of the ith price of risk coefficient. The parameters b_3 and b_4 correspond to the skewness and kurtosis if the reference measure density of z is chosen to be the normal distribution. It is important to emphasize that unlike the Edgeworth case, since a further change of variable from z to S_T has to made, b_3 and b_4 will not correspond to the skewness and kurtosis of the underlying price process S_T but to the skewness and kurtosis of log-returns z. Finally, the skewness and kurtosis of the expansion are given by

$$Skewness\,[z] = \sqrt{6} b_3 \quad (43)$$

$$Kurtosis\,[z] = 3 + \sqrt{24} b_4 \quad (44)$$

The parameters σ , b_3 and b_4 can be obtained using a nonlinear estimation method.
The general expression for the price of a call option is given by

$$C(S_t, K, \mu, s, \tau, r) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_k \pi_l \int_z h_l(z) h_k(z) \Phi(z) dz = \sum_{k=0}^{\infty} a_k \pi_k \quad (45)$$

where the second equality holds because Hermite polynomials form an orthogonal system . When the infinite sum is truncated up to the fouth order, we obtain

$$C(S_t, K, \mu, s, \tau, r) = e^{-r\tau} a_0 + \pi_3 a_3 + \pi_4 a_4 \quad (46)$$

Therefore, it remains to be shown how to obtain the a_k coefficients. Abken, Madan and Ramamurtie (1998) introduce the call option generating function

$$G(u, S_t, K, \mu, s, \tau) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{+\infty} c(u, S_t, K, \mu, s, \tau) exp\left(-\frac{1}{2}(z-u)^2\right) dz, \quad (47)$$

where u is a dummy variable. The coefficients a_k can then be obtained from the generating function as

$$a_k(S_t, K, \mu, s, \tau) = \frac{\partial^k G(u, S_t, K, \mu, s, \tau)}{\partial u^k}|_{u=0} \frac{1}{\sqrt{k!}} \tag{48}$$

The evaluation of $G(u, S_t, K, \mu, s, \tau)$ is shown to be

$$G(u, S_t, K, \mu, s, \tau) = S_t exp\left(\mu\tau + su\right)\Phi(d_1(u)) - K\Phi(d_2(u)) \tag{49}$$

where $d_1(u) = log(S_t/K) + (\mu\tau/s + s/2) + u$ and $d_2(u) = d_1(u) - s$. From there on, it is possible to compute the coefficients a_k. Let's introduce the notation $d_j = d_{j(0)}$, that is we evaluate the $d_{j(0)}$ at zero. We obtain after tedious computations, and introducing the notations $\Phi^{'}, \ \Phi^{''} \ and \Phi^{'''}$, for the first, second, and third derivatives of standard normal density with respect to its argument

$$a_0 = S_t e^{\mu\tau}\Phi(d_1) - K\Phi(d_2)$$

$$a_1 = sS_t e^{\mu\tau}\Phi(d_1) + S_t e^{\mu\tau}\Phi(d_1) - K\Phi(d_2)$$

$$a_2 = \frac{1}{\sqrt{2}}\left[s^2 S_t e^{\mu\tau}\Phi(d_1) + 2sS_t e^{\mu\tau}\Phi(d_1) + S_t e^{\mu\tau}\Phi^{'}(d_1) - K\Phi^{'}(d_2)\right]$$

$$a_3 = \frac{1}{\sqrt{6}}\left[s^3 S_t e^{\mu\tau}\Phi(d_1) + 3s^2 S_t e^{\mu\tau}\Phi(d_1) + 3sS_t e^{\mu\tau}\Phi^{'}(d_1) + S_t e^{\mu\tau}\Phi^{''}(d_1) - K\Phi^{''}(d_2)\right]$$

$$a_4 = \frac{1}{\sqrt{24}}\left[s^4 S_t e^{\mu\tau}\Phi(d_1) + 4s^3 S_t e^{\mu\tau}\Phi(d_1) + 6s^2 S_t e^{\mu\tau}\Phi^{'}(d_1) + 4sS_t e^{\mu\tau}\Phi^{''}(d_1) + S_t e^{\mu\tau}\Phi^{'''}(d_1) - K\Phi^{'''}(d_2)\right] \tag{50}$$

This method has been applied in Coutant, Jondeau and Rockinger (2001) to interest-rate options. Jondeau and Rockinger (2001) have also shown how conditions on b_3 and b_4 could be imposed so that the polynomial approximation remains positive.

2.5 Non-parametric methods

2.5.1 Tree-based methods

The tree-based method has been presented by Rubinstein (1994) and is further extended in Jackwerth and Rubinstein (1996) and Jackwerth (1999). Jackwerth (1997) gives a numerical illustration of the method which is based on two steps. First of all, we take existing options nearest as well as we compute the average of their Black-Scholes implied volatilities. Secondly, we compute the RNDs that would be associated to the binomial tree of Cox, Ross and Rubinstein (1979) and that would be comparable with these implied volatilities.

Formally, let's consider that the tree has N steps. A terminal node, j, of the binomial tree has a probability to realize of $P^{'}_j$. If p' is the risk-neutral probability of an up movement on each node of the tree [8], then

$$P^{'}_j = \frac{N!}{j!(N-j)!} p^{'j}(1-p^{'})^{(N-j)} \tag{51}$$

[8] $p' = \frac{r-d}{u-d}$, where $u = e^{\sigma\sqrt{\tau}}$ and $d = \frac{1}{u}$

In that formula, it is assumed that the nodes are denoted in such a way that a small j corresponds to a low value of the support S_T. Let's denote these values as $S_j,\ j = 1, ...N$. Given the current value of the underlying asset, it is also possible to construct the nodal values of the underlying asset in a risk-neutral environment. Even though by working backwards, we could use the binomial tree to value options, this tree would not be compatible with the observed option smile. The third possibility is to seek terminal probabilities that are close to the tree -based probabilities, yet that also correspond to the given options. Formally, let the risk-neutral probabilities, which are compatible with actual options, be denoted by P_j. These risk -neutral probabilities may then be obtained by solving the following optimization :

$$\min_{P_j} \sum_{j=1}^{N} \left(P_j - P_j^{'}\right)^2 \tag{52}$$

subject to

$$\sum_{j=1}^{N} P_j = 1, \qquad P_j \geq 0, \qquad j = 0, ..., N \tag{53}$$

$$S_t = e^{-r_t(\tau)} \sum_{j=1}^{N} S_j P_j \tag{54}$$

$$C_{\tau,i} = e^{-r\tau} \sum_{j=1}^{N} P_j \left(S_j - K_i\right)^+, \qquad i = 1, ..., N \tag{55}$$

The first constraint provides conditions so that the values P_j define probabilities. The second condition corresponds to the martingale condition. Finally, the last condition states that the probabilities $P_j^{'}$ should be compatible with existing option prices. Various metrics can be used to define the distance between the probabilities P_j and $P_j^{'}$. The idea of using other metrics is expressed in Rubinstein (1994) and tested in Jackwerth and Rubinstein (1996). We could choose the following distances :

$$\min_{P_j} \sum_{j=1}^{N} P_j^{'} - P_j, \tag{56}$$

$$\min_{P_j} \sum_{j=1}^{N} \left(\frac{\left(P_j - P_j^{'}\right)^2}{P_j^{'}} \right) \tag{57}$$

or

$$\max_{P_j} \sum_{j=1}^{N} P_j \log \left(\frac{P_j}{P_j^{'}} \right) \tag{58}$$

The first measure is a metric based on absolute deviations. The second measure is based on percentage deviations. The third measure corresponds to seeking the maximum entropy.

2.5.2 Kernel regression

Aït-Sahalia and Lo (1998) propose to estimate the RND nonparametrically by exploiting Breeden and Litzenberg's (1978) insight that $f_t^*(S_T) = exp(r_{t,\tau})\partial^2 H(.)/\partial K^2$. They suggest using market prices to estimate an option pricing formula $\hat{H}(.)$ nonparametrically, which can then be differentiated twice with respect to K to obtain $\partial^2\hat{H}(.)/\partial K^2$. They use kernel regression to construct $\hat{H}(.)$ and assume that the option-pricing formula H to be estimated is an arbitrary nonlinear function of a vector of option characteristics $y \equiv [F_{t,\tau} \;\; K \;\; \tau \;\; r_{t,\tau}]^{'}$ where $F_{t,\tau}$ is the forward price of the asset.

In practice, they propose to reduce the dimension of the kernel regression by using a semiparametric approach. Assume that the call pricing function is given by the parametric Black-Scholes model except that the implied volatility parameter for that option is a nonparametric function $\sigma(K/F_{t,\tau})$:

$$C(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) = C_{BS}(F_{t,\tau}, K, \tau, r_{t,\tau}, \sigma(K/F_{t,\tau})). \tag{59}$$

We assume that the function C defined by the equation (59) satisfies all required conditions to be a "rational" option-pricing formula in the sense of Merton (1973, 1992)[9]. In this model, we only need to estimate non-parametrically the regression of σ on a subset $\tilde{Y}$ of the vector of explanatory variables Y. The rest of the call pricing functions $C(.)$ is parametric, thereby considerably reducing the sample size n required to achieve the same degree of accuracy as the full nonparametric estimator. Aït-Sahalia and Lo (2000) partition the vector of explanatory variables $y = [\tilde{Y}^{'} F_{t,\tau} r_{t,\tau}]$ where $\tilde{Y}$ contains $\tilde{d}$ non parametric regressors. In their empirical application, they consider $\tilde{Y} \equiv [K/F_{t,\tau}\tau]^{'}(\tilde{d} = 2)$ and form the Nadaraya-Watson Kernel estimator of $E[\sigma|K/F_{t,\tau}\tau]$ as :

$$\hat{\sigma}\left(K/F_{t,\tau}\tau\right) = \frac{\sum_{i=1}^{n} k_{K/F}\left(\frac{K/F_{t,\tau} - K_i/F_{t_i,\tau_i}}{h_{K/F}}\right) k_\tau\left(\frac{\tau-\tau_i}{h_\tau}\right)\sigma_i}{\sum_{i=1}^{n} k_{K/F}\left(\frac{K/F_{t,\tau} - K_i/F_{t_i,\tau_i}}{h_{K/F}}\right) k_\tau\left(\frac{\tau-\tau_i}{h_\tau}\right)} \tag{60}$$

where σ_i is the volatility implied by the option price H_i, and the univariate kernel functions $k_{K/F}$ and k_τ and the bandwidth parameters $h_{K/F}$ and h_τ are chosen to optimize the asymptotic properties of the second derivative of $\tilde{C}(.)$, i.e., of the RND estimator. One estimates the call pricing function as :

$$\hat{C}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) = C_{BS}(F_{t,\tau}, K, \tau, r_{t,\tau}, \hat{\sigma}(K/F_{t,\tau})) \tag{61}$$

The RND estimator follows by taking the second partial derivatives of $\tilde{C}(.)$ with respect to K :

$$\hat{f}^*\left(S_T\right) = e^{r_{i,\tau}\tau}\left[\frac{\partial^2\hat{C}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau})}{\partial K^2}\right]\Big|_{K=S_\tau}. (62)$$

Contrary to the RND estimation approaches previously developed and which are based on the estimation of *RND* for each cross-section of options, the nonparametric kernel regression provides an estimator that is based on both cross-sectional and time-series option prices. Consequently, this approach is consistent over time, but may provide a poor fit for certain dates[10].

[9]In particular , see Merton (1992). These conditions imply that $\sigma(K/F_{t,\tau})$ cannot be an arbitrary function but must yield an $C_{BS}(F_{t,\tau}, K, \tau, r_{t,\tau}, \sigma(K/F_{t,\tau}))$ that satisfies all the conditions of a rational option-pricing formula.

[10]For futher details, see Aït-Sahalia and Lo (1998, 2000) and Jondeau, Ser-Huang Poon and Rockinger (2007).

3 Structural approaches of RND functions estimation

3.1 Jump diffusion model

Modeling the underlying asset with a Black & Scholes model is not consistent. Large values of returns occur too frequently to be consistent with the normality assumption. The rare events may cause brutal assets pricing variations. In order to model such a phenomenon, we should use a Poisson jump process. We assume that the underlying asset price follows a log normal jump diffusion process i.e, the addition of a geometric Brownian motion and a Poisson jump process. This process takes into account the *skewness* and *kurtosis* effects.

The underlying asset price under the risk-neutral world follows the process :

$$dS_t = (r_t - \lambda\, E(k))\, S_t dt + kS_t dW_t + kS_t dq_t \tag{63}$$

where q_t is the Poisson associated probability , λ is the average rate of jump occurence and k is the size of a jump.
Bates (1991) has shown that in case of a diffusion with n jumps, the call price is :

$$C^{JMP}(S_t, t, T, K) = e^{-r_t \tau} \sum_{i=0}^{\infty} P(n\,jumps) E\left[(S_T - K)^+ / n\,jumps \right] \tag{64}$$

Ball and Torous (1985) and Malz (1997) proposed a simplified version of the jump-diffusion model, where there will be at most one jump of a constant size. In this *Bernoulli* version of a jump diffusion model, the equation of the call price is the following :

$$\begin{aligned} C^{JMP}(S_t, t, T, K) &= e^{-r_t \tau} \sum_{i=0}^{1} P(n\,jumps) E\left[(S_T - K)^+ / n\,jumps \right] \\ &(1 - \lambda\,\tau) \left[\frac{S_t}{1 + \lambda\kappa\tau} N\left(d_1 + \sigma\sqrt{\tau}\right) - Ke^{-r_t\tau} N\left(d_1\right) \right] \\ &+ \lambda\tau \left[\frac{S_t}{1 + \lambda\kappa\tau} (1 + \kappa) N\left(d_2 + \sigma\sqrt{\tau}\right) - Ke^{-r_t} N(d_2) \right] \end{aligned} \tag{65}$$

with

$$d_1 = \frac{log\left(\frac{S_t}{K}\right) - log(1 + \lambda\kappa\tau) + \left(r_t - \frac{\sigma^2}{2}\tau\right)}{\sigma\sqrt{\tau}} \tag{66}$$

$$d_2 = \frac{log\left(\frac{S_t}{K}\right) - log(1 + \lambda\kappa\tau) + log(1 + \kappa) + \left(r_t - \frac{\sigma^2}{2}\tau\right)}{\sigma\sqrt{\tau}} \tag{67}$$

where $(1 - \lambda\tau)$ is the probability that no events occur in the option life.

We can easily show that the call price is simply a combinaison of two call prices calculated by Black (1976) model

$$C^{JMP}(S_t, t, T, K|\theta) = (1 - \lambda\tau)\, C^B\left(\frac{S_t}{1 + \lambda\kappa\tau}, \tau, K|\theta\right) + (\lambda\tau)\, C^B\left(\frac{S_t}{1 + \lambda\kappa\tau}(1 + \kappa), \tau, K|\theta\right) \tag{68}$$

This technique is a particular case of a mixture of log-normal distributions. The RND is :

$$q^{JMP}(S_T, S_t, t, T, K|\theta) = (1-\lambda\tau)\, LN(S_T, \alpha, \beta) + \lambda\tau\, LN(S_T, \alpha + log(1+\kappa), \beta) \tag{69}$$

where :

$$LN(x) = \frac{1}{\sqrt{2\pi}\beta\, x} \exp\left[-\frac{(Ln(x)-\alpha)^2}{2\beta^2}\right] \tag{70}$$

$$\alpha = \log(S_t) + (\mu - \frac{1}{2}\sigma)^2(T-t) \tag{71}$$

the $\theta = (\sigma, \lambda, \kappa)'$ vector parameters are estimated using the following program :

$$\min_{\theta} \sum_{i=1}^{n} \left(\widehat{C}_i - C_i^{JMP}\right)^2 \tag{72}$$

3.2 Heston's stochastic volatility model

Most diffusion models of the underlying asset suppose that the volatility of the latter is a constant. Nevertheless, if the underlying asset records strong fluctuations over a short period, it would be necessary to model it with a stochastic volatility process . Heston (1993) proposed the following model :

$$dS_t = \mu_t S_t dt + \sqrt{V_t} S_t dW_t^1 \tag{73}$$

$$dV_t = \kappa(\theta - V_t)\, dt + \sigma\sqrt{V_t} dW_t^2 \tag{74}$$

$$dW_t^1 dW_t^2 = \rho\, dt \tag{75}$$

where $\{S_t\}|_{t\geq 0}$ and $\{V_t\}|_{t\geq 0}$ are the price and volatility process, respectively, and $\left\{W_t^1\right\}\Big|_{t\geq 0}, \left\{W_t^2\right\}|_{t\geq 0}$ are correlated Brownian motion processes (with correlation parameter ρ). $\{V_t\}|_{t\geq 0}$ is a square root mean reverting process, first used by Cox, Ingersoll and Ross (1985), with long-run mean θ, and rate of reversion κ. σ is referred to as the volatility of volatility. All the parameters $\mu, \kappa, \theta, \sigma, \rho$ are time and state homogenous.
There are several economic, empirical and mathematical reasons for choosing a model with such a form [11].

Under the Heston model , the value of any option $\Theta(S_t, V_t, t, T)$ must satisfy the following partial differential equation :

$$\frac{1}{2}VS^2\frac{\partial^2\Theta}{\partial S^2} + \rho\sigma\, VS\frac{\partial^2\Theta}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2\Theta}{\partial V^2} + rS\frac{\partial\Theta}{\partial S} + \left\{\kappa\,[\theta - V] - \Lambda\,(S, V, t)\,\sigma\sqrt{V}\right\}\frac{\partial\Theta}{\partial V} - r\Theta + \frac{\partial\Theta}{\partial t} = 0 \tag{76}$$

where $\Lambda(S, V, t)$ is called the market price of volatility risk. Heston supposes that the market price of volatility risk is propotional to volatility, i.e.

$$\Lambda(S, V, t) = k\sqrt{V} \tag{77}$$

[11] see Cont (2001) for a detailed statistical analysis.

$$\Rightarrow \Lambda(S,V,t)\sigma\sqrt{V} = k\sigma V_t \quad \text{for some constant } k$$
$$= \lambda(S,V,t) \tag{78}$$

Risk neutral valuation is pricing of a contingent claim in an equivalent martingale measure. The price is evaluated as the expected discounted payoff of the contingent claim, under the Risk neutral probability, the option value is

$$Option \;\; value = E^{Q}\left[e^{r(T-t)}H(T)\right] \tag{79}$$

where $H(T)$ is the payoff of the option at time T and r is the risk free of interest over $[t,T]$.

Moving from a real world measure $\mathbb{P}$ to a risk neutral world measure $\mathbb{Q}$ is achieved by Girsanov's theorem. In particular, we have

$$d\widetilde{W}_t^2 = dW_t^2 + \Lambda(S,V,t)dt \tag{80}$$

$$\frac{dQ}{dP} = \exp\left\{-\frac{1}{2}\int_0^t \left(\nu_s^2 + \Lambda(S,V,s)^2\right)ds - \int_0^t \upsilon_s dW_s^1 - \int_0^t \Lambda(S,V,s)dW_s^2\right\} \tag{81}$$

$$\upsilon_t = \frac{\mu - r}{\sqrt{V_t}} \tag{82}$$

where $\left\{\tilde{W}_t^1\right\}|_{t\geq 0}$ and $\left\{\tilde{W}_t^2\right\}|_{t\geq 0}$ are $\mathbb{Q}-$ Brownian motions. Under measure $\mathbb{Q}$, the equations (73), (74) and (75) become,

$$dS_t = r_t S_t dt + \sqrt{V_t} S_t d\widetilde{W_t^1} \tag{83}$$

$$dS_t = \kappa^*\left(\theta^* - V_t\right)dt + \sigma\sqrt{V_t}d\widetilde{W_t^2} \tag{84}$$

$$d\widetilde{W_t^1}d\widetilde{W_t^2} = \rho\, dt \tag{85}$$

and the equation (76) becomes :

$$\frac{1}{2}VS^2\frac{\partial^2\Theta}{\partial S^2} + \rho\sigma\, VS\frac{\partial^2\Theta}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2\Theta}{\partial V^2} + rS\frac{\partial\Theta}{\partial S} + \left\{\kappa^*\left[\theta^* - V\right] - \Lambda\left(S,V,t\right)\sigma\sqrt{V}\right\}\frac{\partial\Theta}{\partial V} - r\Theta + \frac{\partial\Theta}{\partial t} = 0 \tag{86}$$

where,

$$\kappa^* = \kappa + \lambda \tag{87}$$

$$\theta^* = \frac{\kappa\theta}{\lambda + \lambda} \tag{88}$$

In this important result, λ has effectively been eliminated.

The closed-form solution of a European call option on a non-dividend paying asset for the Heston model is :

$$C\left(S_t, V_t, t, T\right) = S_t P_1 - Ke^{-r_t(T-t)}P_2 (89)$$

where

$$P_j\left(x, V_t, K, T\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\log(K)}f_j\left(x, V_t, T, \phi\right)}{i\phi}\right)d\phi \tag{90}$$

$$f_j\left(x, V_t, T, \phi\right) = \exp\left\{C\left(T, \phi\right) + D\left(T, \phi\right) V_t + i\phi\, x\right\} \tag{91}$$

$$C\left(T - t, \phi\right) = r\phi\, i\tau + \frac{\alpha}{\sigma^2}\left[\left(b_j - \rho\sigma\phi\, i + d_j\right)\tau - 2\log\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{92}$$

$$D(T, \phi) = \frac{b_j - \rho\sigma\phi\, i + d}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right] \tag{93}$$

$$g_j = \frac{b_j - \rho\sigma\phi\, i + d}{b_j - \rho\sigma\phi\, i - d} \tag{94}$$

$$d_j = \sqrt{\left((\rho\sigma\phi - b_j)^2 - (2u_j\phi\, i - \phi^2)\right)} \tag{95}$$

$$x = \log\left(S_t\right) \tag{96}$$

for $j = 1, 2$ where, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $\alpha = \kappa\theta$, $b_1 = k^* - \rho\sigma$, $b_2 = \kappa^*$ [12].
The $\theta = (\kappa^*, \theta^*, \rho, V_t, \sigma)'$ vector parameters are then estimated using the following algorithm :

$$\min_{\theta} \sum_{i=1}^{n} \left(\widehat{C_i} - C_i\right)^2 \tag{97}$$

After estimating model parameters, the risk-neutral density is obtained from the numerical calculation of the integral. A method to evaluate formulas in form of (92) has been proposed by Carr and Madan (1999).

4 Application to CAC 40 index options

In this section, we estimate parametrically, semi-parametrically and nonparametrically the RND functions of CAC 40 index options between January 1^{st} 2007 and December 31^{st} 2007. We compare the goodness-of-fit of eight option-based approaches during a normal and troubled period. To our knowledge, this is the first application on the European market dealing with European options using structural and non-structural approaches of RND functions estimation.

4.1 Data

Our data is provided by the SBF-Paris Bourse[13] and includes intraday values of the CAC 40 stock index and intraday transaction prices of CAC 40 options over the period January 1^{st} 2007-December 31^{st} 2007. CAC 40 options are traded on the MONEP, the French derivatives market. Trading covers eight open maturities : three spot months, four quarterly maturities (March, June, September, December) and two half-yearly maturities (March, September). The maturity date is

[12] It is important to note that the interpretations of κ^* and $\sqrt{\theta^*}$ as respectively the rate of reversion and long term volatility are still valid.

[13] SBF-Paris Bourse provides a monthly CD-ROM including intraday values of the CAC 40 stock index and intraday transaction prices of CAC 40 options traded on the MONEP.

the last trading day of each month. Trading takes place on a continuous basis between 9:00 am and 5:30 pm.

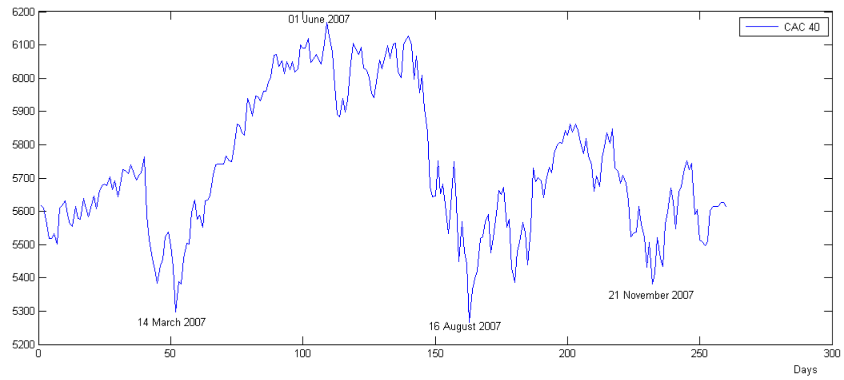

Figure 1: CAC 40 stock index evolution (January 1^{st} 2007- December 31^{st} 2007)

The future underlying asset $F_{t,T}$ cannot be observed exactly at time t, where t denotes a given day and time and T is the maturity . Aït-Sahlia and Lo (1998) suggest extracting an implied future index price from the put-call parity. Given our intraday data, we cannot get contemporaneous trades concerning a call and a put with similar strike price and maturity. Thus, for each maturity, we compute :

$$F_{t,T} = S_t(exp(r_{t,T} - \delta_{t,T})\tau) \tag{98}$$

To circumvent the unobservability of the dividend rate $\delta_{t,T}$, we extract an implied value of the dividend rate between the end of day t and T from the daily closing price of the index, $S_{t_{end}}$ and the setlement future index price, $F_{t_{end}}$, observed at the end of each day. The obtained dividend rate is the dividend rate expected by the market between t_{end} and T [14].

$$\delta_{t_{end},T} = r_{t,T} - \frac{1}{\tau} ln(\frac{F_{t_{end},T}}{S_{t_{end}}}) \tag{99}$$

$F_{t,T}$ is then computed using the dividend rate and the riskfree interest rate proxied by Euribor. Since some CAC 40 options are not traded actively, we need to filter the data carefully. Five filters are applied to the initial data. We omit the quotes of the first and the last 15 minutes each day and the option quotes characterized by a price lower or equal to one tick. We only consider options with a moneyness[15] comprised between 0.85 and 1.15. This procedure eliminates far-away-from the money observations, which are unreliable due to their low volume and low sensitivity towards

[14] The constant dividend hypothesis passes over the clustering of dividends paid by most firms during specific months. However, as far as we are concerned, the bias is not relevant since we use market prices of futures to estimate an implied dividend rate. The constant dividend hypothesis could be replaced by a more realistic actual dividend assumption, leading to the following pricing formula for the futures : $F_{t,T} = S_t exp(r_{t,T}(T-t)) + \sum_{l=1}^{n} D_{l,t_l} exp(r_{t_l,T}(T-t_l))$ where t_l is the payment date of the l^{th} dividend. Nevertheless, this formula requires the prediction of all dividends amounts and payment dates paid by the 40 CAC 40 index shares and the prediction of the forward interest rates.

[15] Strike price divided by the future index level.

volatility. Besides, options quotes violating general no-arbitrage conditions, that is put-call parity, are eliminated as well as we replace the price of all illiquid options, that is in-the-money options, with the price implied by put-call parity at the relevant strike prices. Specially, we replace the price of each in-the-money call option with $P(S_t, K, \tau, r_{t,T}, \delta_{t,T}) + (F_{t,T} - K)e^{-r_{t,T}\tau}$, where, by construction, the put with price $P(S_t, K, \tau, r_{t,T}, \delta_{t,T})$ is out-of-the money and therefore liquid. After this procedure, all the information included in liquid put prices is extracted and resides in corresponding call prices through the put-call parity. Put prices may now be splayed without any loss of reliable information. Since our data include only call options, a single volatility function can be estimated.

We provide in Figure 2 the implied volatility observed on January 1^{st}2007 and Figure 3 reports the volatility smiles for four various maturities observed on July 10^{th} 2007, when the market trend is bear. The implied volatility is computed using the Black & Scholes formula. For a given maturity, we compute the daily average implied volatility for each strike price. For fixed S_t and T, it is shown that the implied volatility function is not constant as assumed in the Black & Scholes model but decreases with respect to the strike price. The implied volatility observed on July 10^{th} 2007 on the French market is then more a "skew" than a "smile". In this case, contrary to what suggests Black-Scholes model, out-the-money and in-the-money options represent increased risk on potentially very large movements in the underlying compared to at-the-money. To compensate for this risk, they tend to be priced higher.

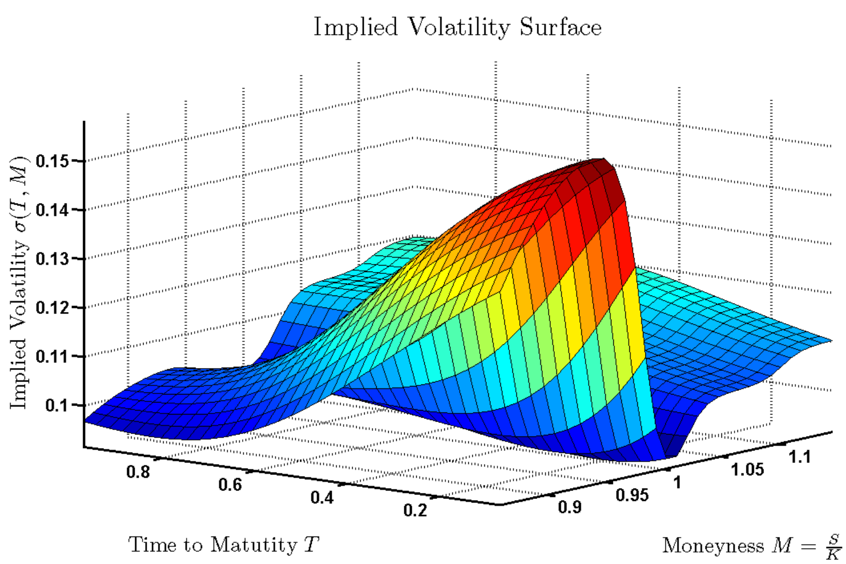

Figure 2: Implied volatility surface on the French market on January 1^{st} 2007

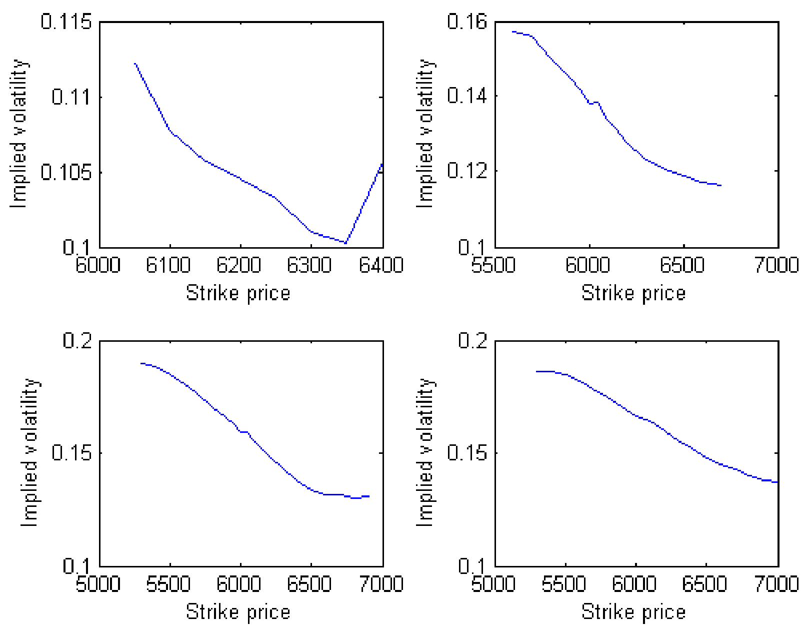

Figure 3: Implied volatility obtained from the Black & Scholes formula on the French market on July 10^{st} 2007

4.2 Methods comparison

4.2.1 Kernel approach estimation

We focus on the nonparametric Kernel approach proposed by Aït-Sahlia and Lo (2001). This method does not require any parametric restriction on either the underlying asset price dynamics, the family of distributions that the RND belongs to or any prior distribution for the RND. We use the whole database in order to estimate nonparametrically the volatility function given in paragraph (2.5.2), with two conditional variables (d=2): the moneyness and the time to maturity τ. We only consider two regressors. We rely on the Aït-Sahlia and Lo (1998)'s procedure to select the kernel functions and the optimal bandwidths. They are chosen to optimize the asymptotic properties of the option pricing function, H. The selected kernels associated with the two regressors are Gaussian :

$$k_{K/F} = \frac{1}{\sqrt{2\pi}} e^{-0.5.(K/F)^2}$$

$$k_\tau = \frac{1}{\sqrt{2\pi}} e^{-0.5.(\tau)^2} \tag{101}$$

For each regressor, we choose the optimal bandwidth h according to the following equations :

$$h_{K/F} = c_{K/F}\sigma_{K/F} n^{-1/(d+2(q_{K/F}+m))} \tag{102}$$

$$h_\tau = c_\tau \sigma_\tau n^{-1/(d+2q_\tau)} \tag{103}$$

where n is the sample size, q_j is the number of existing continuous partial derivatives of the function to be estimated with respect to the j^{th} regressor ($j = K/F, \tau$), m is the order of the partial derivative with respect to the j^{th} regressor that we want to estimate, d is the number of regressors and σ_j is the unconditional standard deviation of the j^{th} regressor. The parameter c_j depends on the choice of the kernel and the function to be estimated. It is typically of the order of one. Aït-Sahlia and Lo (1998) emphasize that small deviations from the exact value have no noticeable effects. We report in Table 2 the values of the coefficients and bandwidths.

Table 2: Selected kernels and bandwidth values

	Kernel	n	q_j	m	d	σ_j	h
K/F	Gaussian	25868	2	2	2	0.0724	0.0262
τ	Gaussian	25868	2	0	2	100.5462	18.4886

Figure 4 shows $\hat{\sigma}\,(K/F_{t,\tau}\tau)$, the Nadaraya-Watson estimator of the volatility parameter with two regressors X/F and τ, for different maturities (20 days , 50 days, 80 days and 170 days i.e 1,2,3 and 6 months). We note that the estimator generates a strong volatility smile with respect to the moneyness. Besides, in confirmity with Aït-Sahlia and Lo (1998) and Bliss and Panigirtzoglou (2002), we observe that the reported smiles shape varies with the chosen time to maturity.

The estimated RND is shown in Figure 5 for 3-month maturity on January 10^{th} 2007. The RND is obtained by computing the second derivative of the call price with respect to the strike price. We note from Figure 5 the limits of the kernel method. Density is unclear because of the low number of observed points. This method would probably be better suited to a study of a large sample or to estimate the returns time series.

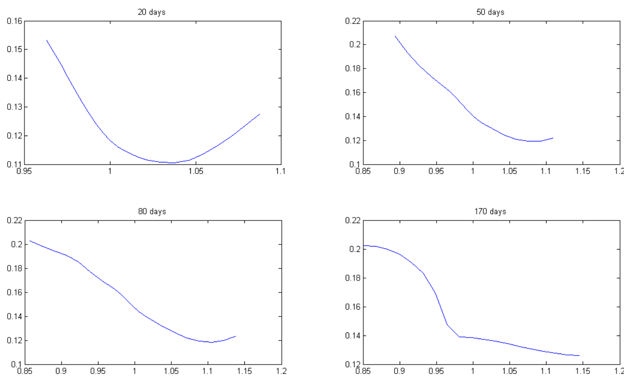

Figure 4: Nonparametric estimators of the volatility for different maturities

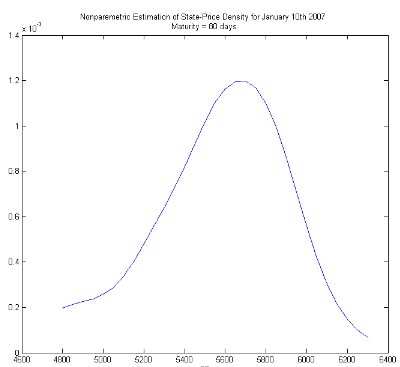

Figure 5: Nonparametric estimator of the state-price density $q(S_T)$ with 3-month maturity on January 10^{th} 2007

4.2.2 Tree-based methods

The estimated RND on a discrete probability framework, as proposed by Jackwerth and Rubinstein (1996), is shown in Figure 6 for 3 periods on January 10^{th} 2007 (normal period), on July 10^{th} 2007 on which the market trend is bear and on October 17^{th} 2007 for various maturities.

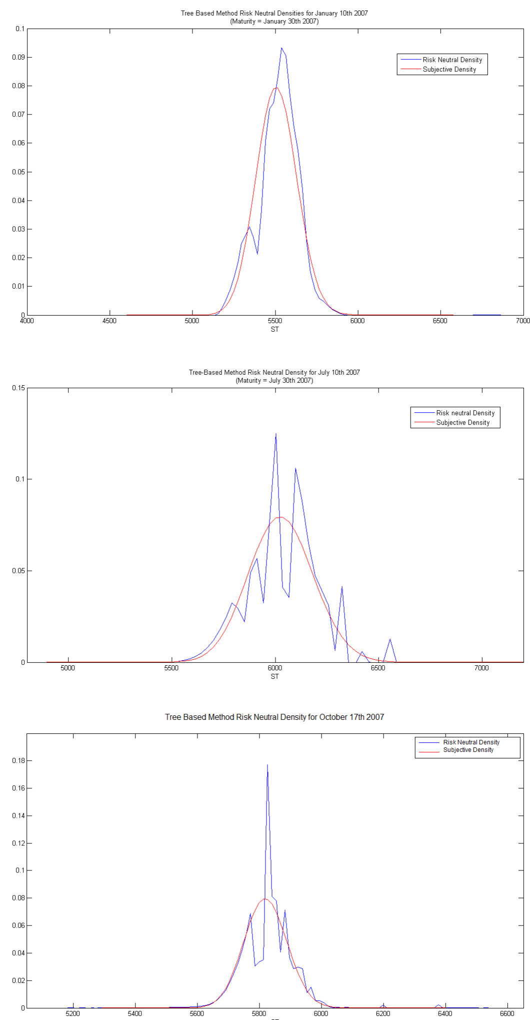

Figure 6: Risk Neutral Densities of the tree based method for different dates and various maturities

We note from Figure 6 that in the troubled period, the RNDs for all the maturities have heavier tails than just before and after the crisis periods. The CAC 40 index options exhibit heavy tail distribution in the troubled period; so in a risk neutral framework, the probability estimated by the tree based method does not undervalue the probability of occurrence of extreme events.

4.2.3 Parametric and semi-parametric RND estimation methods comparison

Several papers compare the performance of various methods to extract RND. Jondeau and Rockinger (2000) compare the mixture of log-normals, the Edgeworth and Gram-Charlier approximation, the single-jump and Heston's stochastic volatility model using exchange rates. They find that up to a certain extent, these methods provide similar RNDs. They remark that during normal periods, a mixture of log-normals provides rather good results. However during a troubled period, the jump diffusion model performs better. Coutant et al. (2001) compare the mixture of log-normals, Gram-Charlier expansion and the entropy-based method regarding speed, estimation robustness and ease of implementation using interest rates. They find that for interest rates, the method based on Gram-Charlier expansion seems to provide rather stable results. Bliss and Panigrizoglou (2002) compare the robustness of a model based on the mixture of two log-normal distributions and a smoothed implied volatility smile model, in the spirit of Shimko (1993). They find that their smoothed spline method performs better than the mixture of log-normal distributions. However, despite its high flexibility, the smoothed spline method does not recover the tails of the RND outside the range of available strike prices.

In addition to nonparametric estimation methods of the RND, i.e. the kernel and the tree-based methods, we now compare six parametric and semi-parametric option-based approaches used to extract RND from CAC 40 index options during a normal and troubled period using the same data as in section 4.1. The first technique is the numerical approximation of the RND based on the second derivative of option prices with respect to the strike price, as suggested by Breeden and Litzenberger (1978). The second method is obtained using a mixture of two log-normal distributions, following Melick and Thomas (1997). The third RND technique is the Edgeworth expansion around the log-normal distribution of Jarrow and Rudd (1982). The fourth method is Hermite polynomials, suggested by Madan and Mline (1994). The fifth RND is based on Heston's stochastic volatility model (1993). Finally, we consider the jump diffusion model following Bates (1991). The results of estimated parameters of different RND models are reported in Tables 3 to 8 [16]. The various resulting RNDs are plotted in Figures 7 to 12[17].

The mixture of log-normals and Edgeworth expansion, Hermite polynomials and jump-diffusion models are more in line with historical data than the numerical approximation of the RND using the Breeden and Litzenberg formula. For this reason, the nonparametric approaches such as kernel and tree-based model ,as described above, were developed. The mixture of log-normals, Edgeworth expansion, hermite polynomials, jump diffusion and Heston models have heavier tails than the log-normal distribution.

The existence of several RND estimation methods from option prices raises the question of the suitable method to be chosen. To answer this question, we provide goodness-of-fit measures, allowing ton investigate how well theoretical options prices calculated with RND estimation approaches fit observed market prices $\hat{C}$. Synthetic measures for errors calculating are used : the Mean Squared Error (MSE) and the Average Relative Error (ARE).

$$MSE = \frac{10^2}{m-n}\sum_{i=1}^{m}(C_i - \hat{C}_i)^2 \tag{104}$$

[16]See Appendix B.
[17]See Appendix A.

$$ARE = \frac{10^4}{m-n}\sum_{i=1}^{m}(\frac{C_i - \hat{C}_i}{C_i})^2 \quad (105)$$

where m is the number of observed options prices and n is the number of method parameters. The best method is the one which has the smallest errors.

In order to compare the various RND models, not only we check the statistical properties of the different RNDs but we also calculate the MSE and the ARE for different dates 01-10-2007, 07-10-2007 (troubled period) and 10-17-2007 and various maturities 1,2 and 3 months. All results are displayed in Tables 9 to 13 [18].

First of all, we compare the volatilities, skewness and kurtosis obtained under different RNDs. Concerning the volatilities (Table 9), the volatility induced by the Black & Scholes model is smaller than the one we obtain using the other methods, and this is true for all periods and all maturities, [19] which explain the bias implied by the log-normality assumption.

Regarding the implicit skewness (Table 10) and the implicit kurtosis (Table 11), the log-normal model is less interesting because it does not allow for asymetry or fat tails. We observe a strongly negative skewness and a larger excess of kurtosis for all methods but the Black & Scholes model. The skewness obtained by the semi-parametric methods (Hermite polynomial and Edgeworth expansion models) are lower than that of the other models on the normal period (01-10-2007). Whereas on the troubled period (07-07-2007), Hermite polynomials and the mixture of log-normals models have the lower skewness, on the period just after the crisis, the jump-diffusion model has the lower skewness from the two month maturities, followed by the Hermite polynomials model. The graphs of the RND corroborate these findings.

Table 12 reports the Mean Squared Error (MSE) for the different methods. According to this criterion, the jump diffusion model on the period just before the crisis for relatively short maturities (1 and 2 months) provides a much better fit than the other models. However, during this same period, the mixture of log-normal models provides the best fit of the data for the 3 month maturity. We notice that in the troubled period (07-10-2007), the Edgeworth expansion model provides the best fit for all maturities. Concerning the period just after the crisis, the jump diffusion model performs well for the two month maturity while the Edgeworth expansion model provides the best fit for the three month maturity.

The Average Relative Error (ARE) results are presented in Table 13. As for the MSE criterion, the jump diffusion model outperforms the other models on the period just before the crisis. On the troubled period, Hermite polynomial method and the mixture of log-normal models have the smallest errors. The difference between the two corresponding ARE is small. Concerning the period just after the crisis, the semi-parametric methods provide the best fit for the one and three month maturities, while the mixture of log-normals approach provides a much better fit than the other models.

The RNDs functions are of paramount importance. Once estimated, it is possible to extract a lot of various information. For instance, we can perform tests and compute confidence intervals around the expected value whose evolution offers the opportunity to investors to measure how markets are thought to evolve through time. Besides, RNDs can provide a measure of expected extreme variations in the underlying asset price, which is an important tool for risk management. Moreover, since the existence of a possibly time-varying risk premium RNDs differs from real probability distributions, the RNDs incorpate information concerning the risk aversion of investors.

[18] See Appendix B.

[19] except on 10-17-2007 for less than one month maturity, the jump diffusion model has the smallest volatility.

5 Conclusion

In this paper, eight option-based approaches are presented and applied to the French derivatives market. Structural and non-structural approaches are used to estimate risk-neutral probability density functions from a high-frequency CAC 40 index options during a normal and troubled period. We use Black & Scholes model, mixture of log-normals, Edgeworth expansions, Hermite polynomials, tree-based methods, kernel regression, Heston's stochastic volatility model and jump diffusion model. We determine nonparametric, semi-parametric, parametric and stochastic implied volatility functions and Risk-Neutral Densities (RND). In order to compare the various RND models, not only we check the statistical properties of the different RNDs, but also we calculate goodness of fit measures. We find that the kernel estimator generates a strong volatility smile with respect to the moneyness, and kernel smiles shape varies with the chosen time to maturity. The mixture of log-normals, Edgeworth expansion, hermite polynomials, jump diffusion and Heston models are more in line with historical data and have heavier tails than the log-normal distribution. Moreover, according to the goodness of fit criteria, the jump diffusion model provides a much better fit than the other models on the period just-before the crisis for relatively short maturities. However, during this same period the mixture of log-normals model performs better for more than three month maturity. Furthermore, in the troubled period and the period just-after the crisis, we find that semi-parametric models are the methods with the best accuracy in fitting observed option prices for all maturities with a minimal difference towards the mixture of log-normals model.

References

[1] **Abadir. K & Rockinger. M (1997)**,*Density-Embedding Functions*, HEC Working Paper.

[2] **Abken.P, D.B. Madan & Ramamurtie. R (1996)**, *Estimation of Risk-Neutral and Statistical Densities by Hermite Polynomial Appriximation: With an Application to Eurodollar Futures Options*, Federal Reserve Bank of Atlanta,Working Paper 96-5.

[3] **Aït-Sahalia Y & Lo, Anderw W. (1998)**, *Non-parametric estimation o state-price densities implied in financial asset prices*,Journal of Finance, vol. 53, pp. 499-517, April.

[4] **Aït-Sahalia Y & Lo Anderw W. (2000)**, *Non-parametric Risk Management and Implied Risk Aversion*, The Journal of Econometrics, 94, 9-51.

[5] **Anagnar, I, Mascia. B, Hodges & Tompkins. T (2002)**, *The relation between implied and realised probabilty density functions*, Working Paper University of Technology Vienna.

[6] **Azzalini.A & Bowman, A. W (1997)**, *Iapplied Smoothing Techniques for Data Analysis*, London :Oxford University Press.

[7] **Bahra, B. (1996)**, *Probability distributions of future asset prices implied by option prices*, Bank of England Quarterly Bulletein: August 1996, 299-311.

[8] **Bates David S.,(1976)**, *The Crash of '87': Was It Expected? The Evidence from Option Markets*, Journal of Finance, Vol. 46, pp 1009-1044,July.

[9] **Black F (1976)**, *The Pricing of Commodity Contracts*, Journal of Financial Economics, Vo. 3, pp 167-179.

[10] **Black F & Scholes M (1973)**, *The pricing of Options and Corporate Liabilities*, Journal of Political Economy, Vol. 81, pp. 637-659, May-June.

[11] **Bliss. R & Panigirtzoglou. N (2002)**, *Testing the Stability of Implied Probability Density Functions*, Journal of Banking and Finance, 26, pp. 381-422, March.

[12] **Bliss. R & Panigirtzoglou. N (2004)**, *Option-implied Risk Aversion Estimates*, Journal of Finance, Vol. 59, No. 1, pp. 407-446.

[13] **Boes. M. J, Drost. F. C & Bas. J. M. W (2005)**, *Nonparametric Risk-Neutral Return and Volatility Distributions*, Tilbury University, Working Paper.

[14] **Breeden, D.T, & R.H. Litzenberger(1978)**, *Prices of state-contingent claims implicit in option prices*, Journal of Business, Vol. 51, pp. 621-651, October.

[15] **Campa.J. M, Chang. P. H. K & Reider. R. L (1998)**, *Implied Exchange Rate Distributions : Evidence from OTC Options Markets*, Journal of Internationational Money and Finance, Vol. 17, No. 1, pp. 117-160.

[16] **Coutant S. (1999)**, *Implied risk aversion in option prices using hermite polynomials*, in *Estimating and interpreting probability density function*, Bank for International Settlements.

[17] **Coutant S., Rockinger. M & Jondeau. E (2001)**, *Reading PIBOR Futures Options Smile : The 1997 Snap Election*, Journal of Banking and Finance, Vol. 25, No. 11, pp. 1957-1987.

[18] **Coutant. S (2001)**, *Contenu en information dans le prix d'options: estimation de la densité neutre au risque du sous-jacent et applications*, Thesis, Paris Dauphine University.

[19] **Cox, J.C, & S.A. Ross (1976)**, *The valuation of option for alternative stochastic processus*, Journal of Financial Economics, Volume. 3, Issues 1-2, January-March 1976, pp. 145-166 .

[20] **Enzo.G, Handel.M & Härdle. W. K (2006**, *Time Dependent Relative Risk Aversion*, SFB 649 Discussion Papers SFB649DP2006-020, Sonderforschungsbereich 649, Humboldt University, Berlin, Germany.

[21] **Heston, S.L, (1993)**, *A closed-from solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies, Vol. 6, pp. 327-343.

[22] **Jackworth J.C & M.Rubinstein (1996)**, *Recovering Probability Distributions from option Prices*,Journal of Finance, Vol. 51, pp. 1611-1631.

[23] **Jackworth J. C (2000)**, *Recovering Risk Aversion from Opion Prices and Realized Returns*, The Review of Financial Studies, Vol. 13, pp. 433-451.

[24] **Jarrow, R & A. Rudd (1982)**, *Approximate valuation for arbitrary stochastic processes*, Journal of Financial Economics, Vol. 10, pp. 349-369.

[25] **Jondeau. E & Rockinger. M, (1997)**, *'Contingent claims valued and hedged by pricing and investing in a basis*,Mathematical Finance, 4, 223-245.

[26] **Jondeau. E & Rockinger. M, (2000)**, *Reading the smile: the message conveyed by methods which infer risk neutral densities*, Journal of Internationnal Money and Finance, Vol. 19, pp. 885-915.

[27] **Jondeau. E, Ser-Huang. P and & Rockinger. M, (2007)**, *Financial Modeling Under Non-Gaussian Distributions*, book, Library of Congress Control Number: 2006923899.

[28] **Jondeau. E & Rockinger. M, (2000)**, *Gram-Charlier densities*, Journal of Economic Dyanamics and Control, Vol. 25, pp. 1457-1483.

[29] **Madan, D.B & Milne. F (1994)**, *Contingent claims valued and hedged by pricing and investing in a basis*, Mathematical Finance, Vol. 4, N. pp. 223-245.

[30] **Madan, D.B & Milne. F (2002)**, *Extracting Information from Option Markets: Smiles, State-Price Densities and Risk-Aversion*, European Financial Management, Vol. 8, N. 4, pp. 495-513.

[31] **Malz Allan. M (1997)**, *Estimating the probability distribution of the future exchange rate from options prices*, Journal of Derivatives, pp. 18-36.

[32] **Melik, W.R, & C.P. Thomas (1997)**, *Recovering an asset's implied PDF from options prices: an applications to crude oil during the Gulf crisis*, Journal of Financial and Quantitative Analysis, Vol. 32, pp. 91-116.

[33] **Panigirtzoglou. N & Skiadapoulos. G (2004)**, *A new approach to modelling the dynamics of implied distribution: Theory and Evidence from the S&P 500 Options*,Journal of Banking and Finance, Vol. 28, pp. 1499-1520.

[34] **Rubinstein, M, (1994)**, *Implied binomial trees, Journal of Finance*,Journal of Finance", Vol. 49, pp. 771-818.

[35] **Sherrick B.J, P. Garcia & V. Tirupattur (1996)**, *Recovering probabilitistic information from option markets: tests of distributional assumptions*, Journal of Futures Markets, Vol. 16, pp. 545-560.

[36] **Shimko D. C (1993)**, *Bounds of probability*, Risk, Vol. 6, pp. 33-37.

[37] **Silverman. B. W (1986)**, *Density Estimation for Statistics and Data Analysis*,Monographs on Statistics and Applied Probability, London: Chapman and Hall.

6 Appendix

A.Graphs

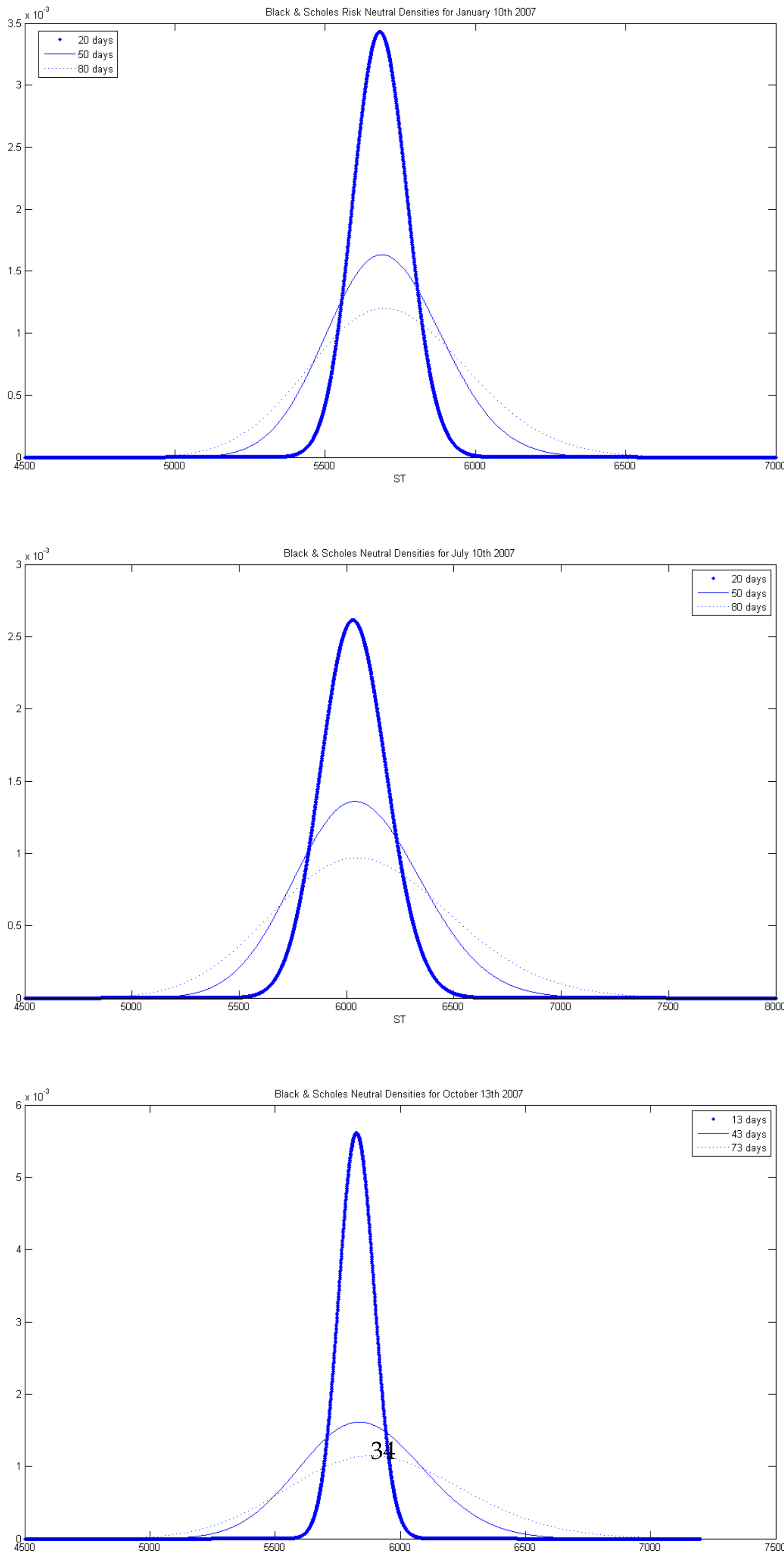

Figure 7: The Black & Scholes Risk Neutral Densities for different dates and various maturities

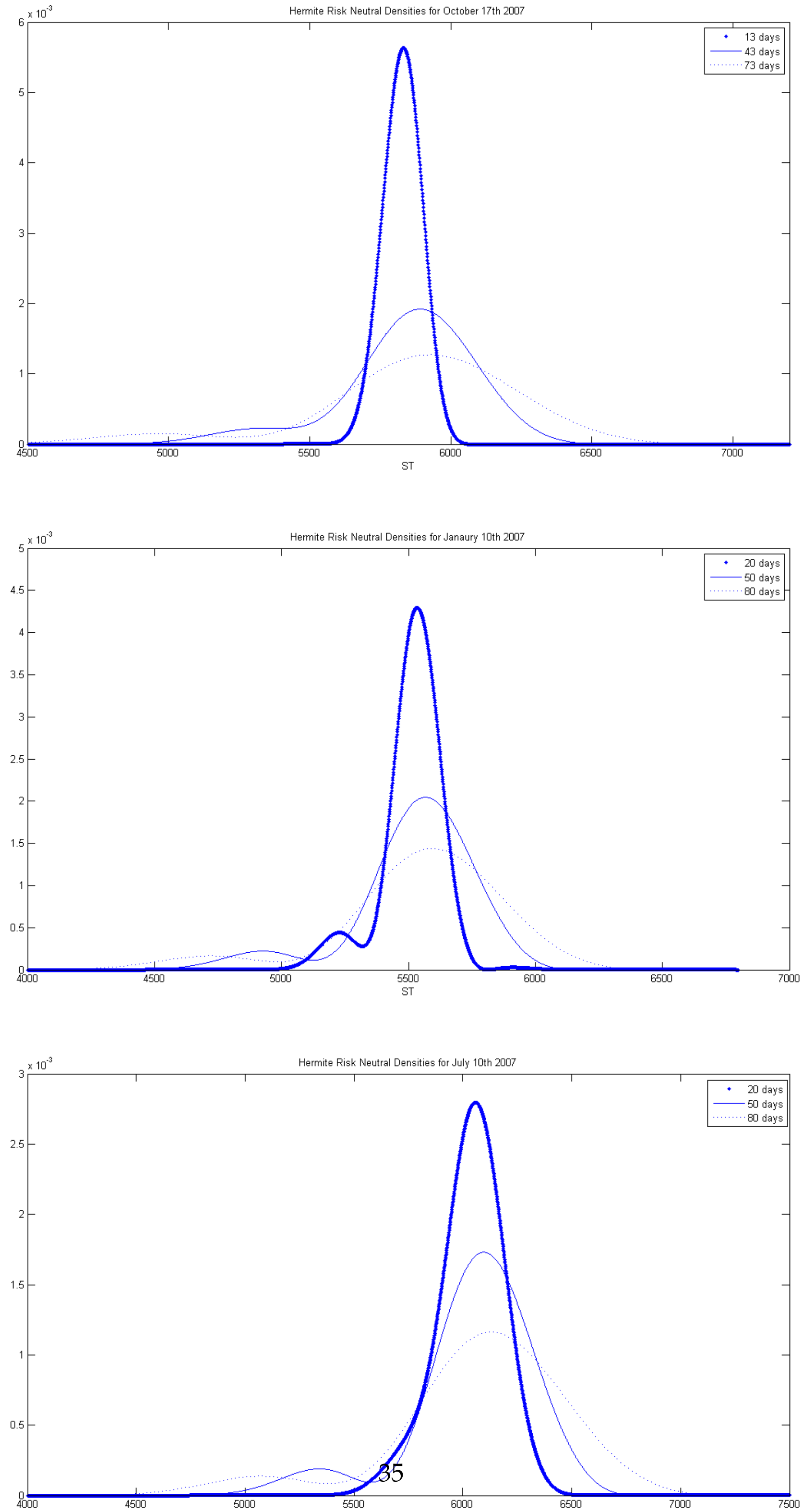

Figure 8: Hermite polynomials Risk Neutral Densities for different dates and various maturities

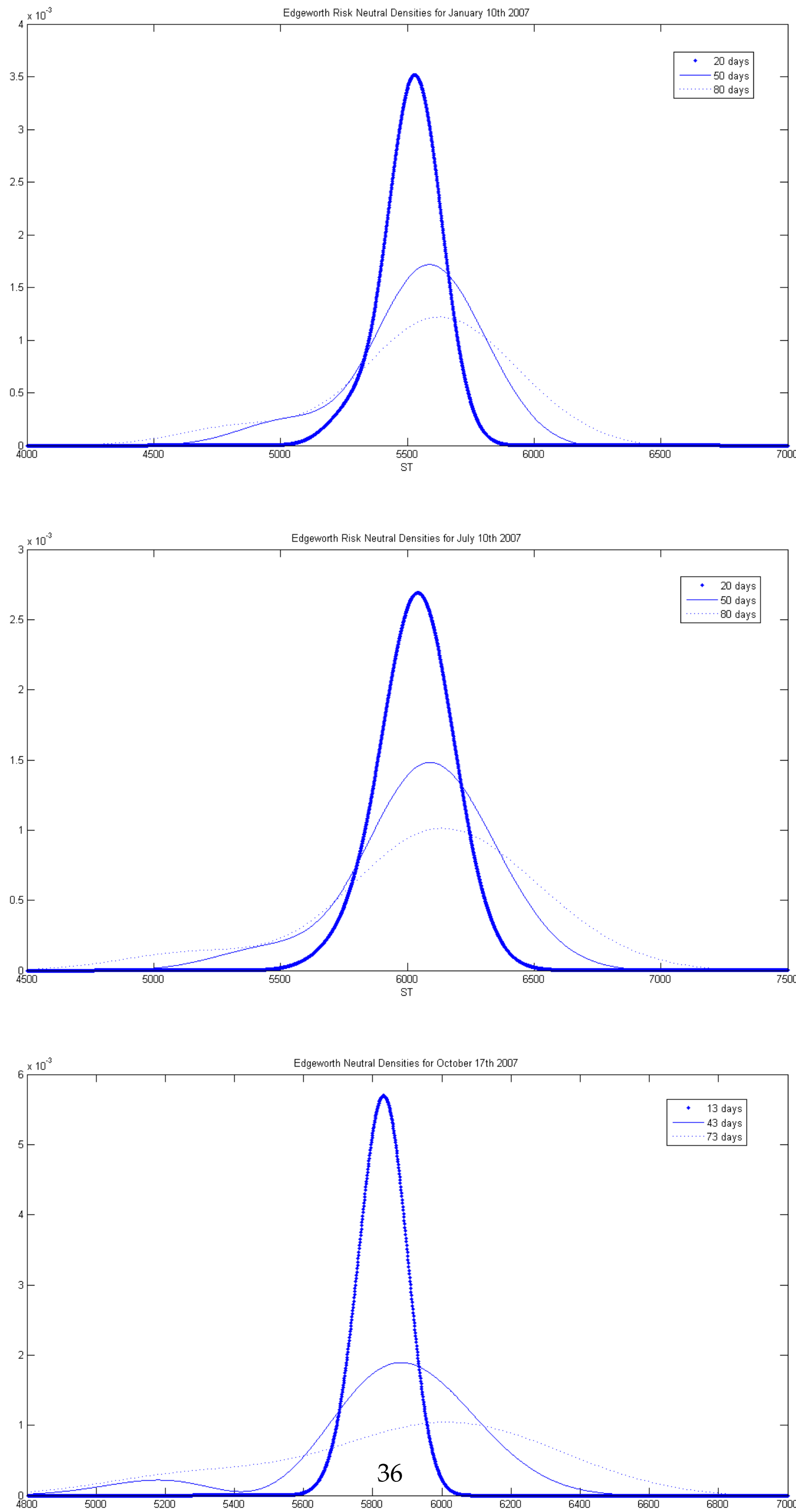

Figure 9: Edgeworth expansions Risk Neutral Densities for different dates and various maturities

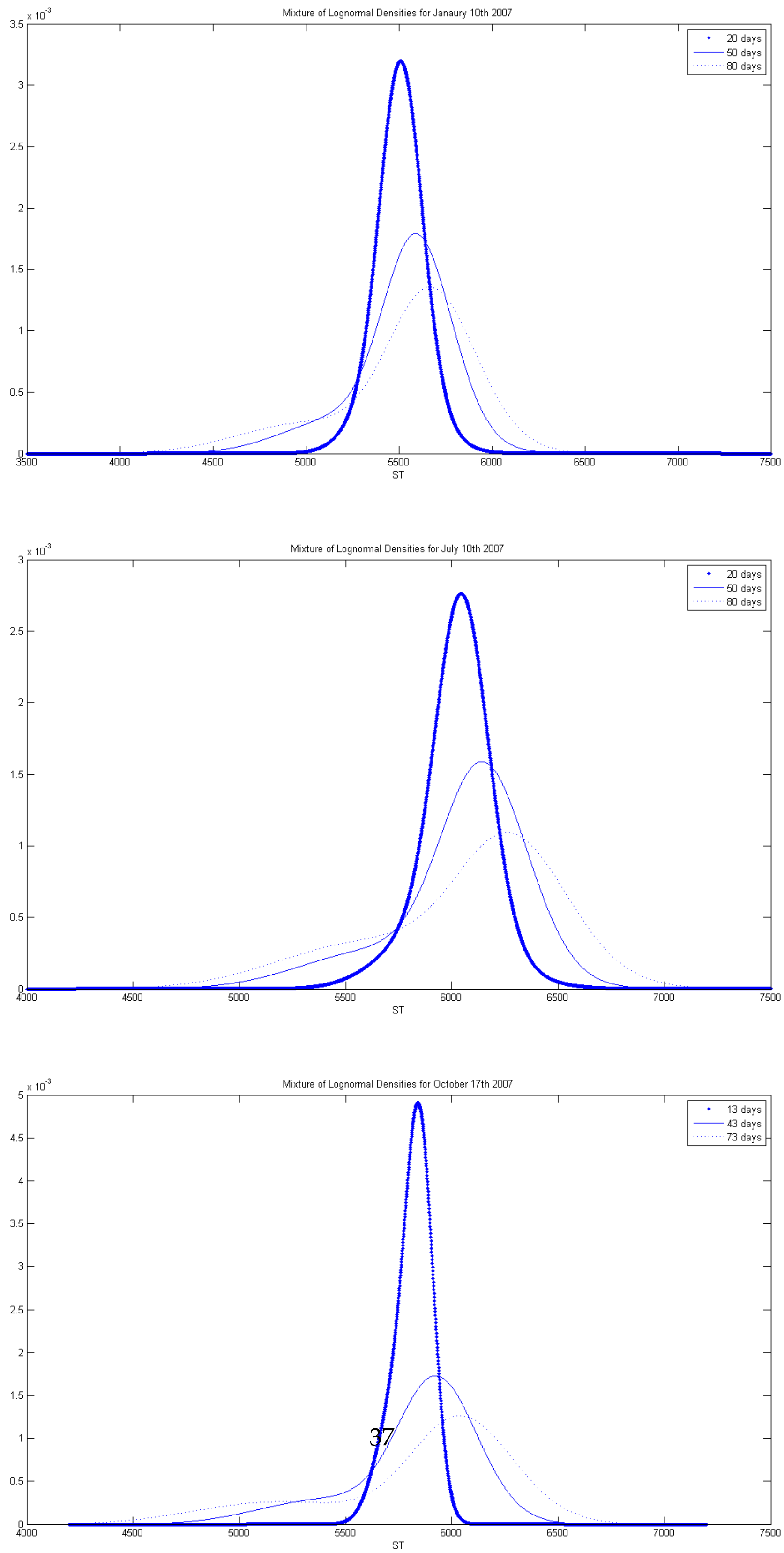

Figure 10: Mixture of log-normals Risk Neutral Densities for different dates and various maturities

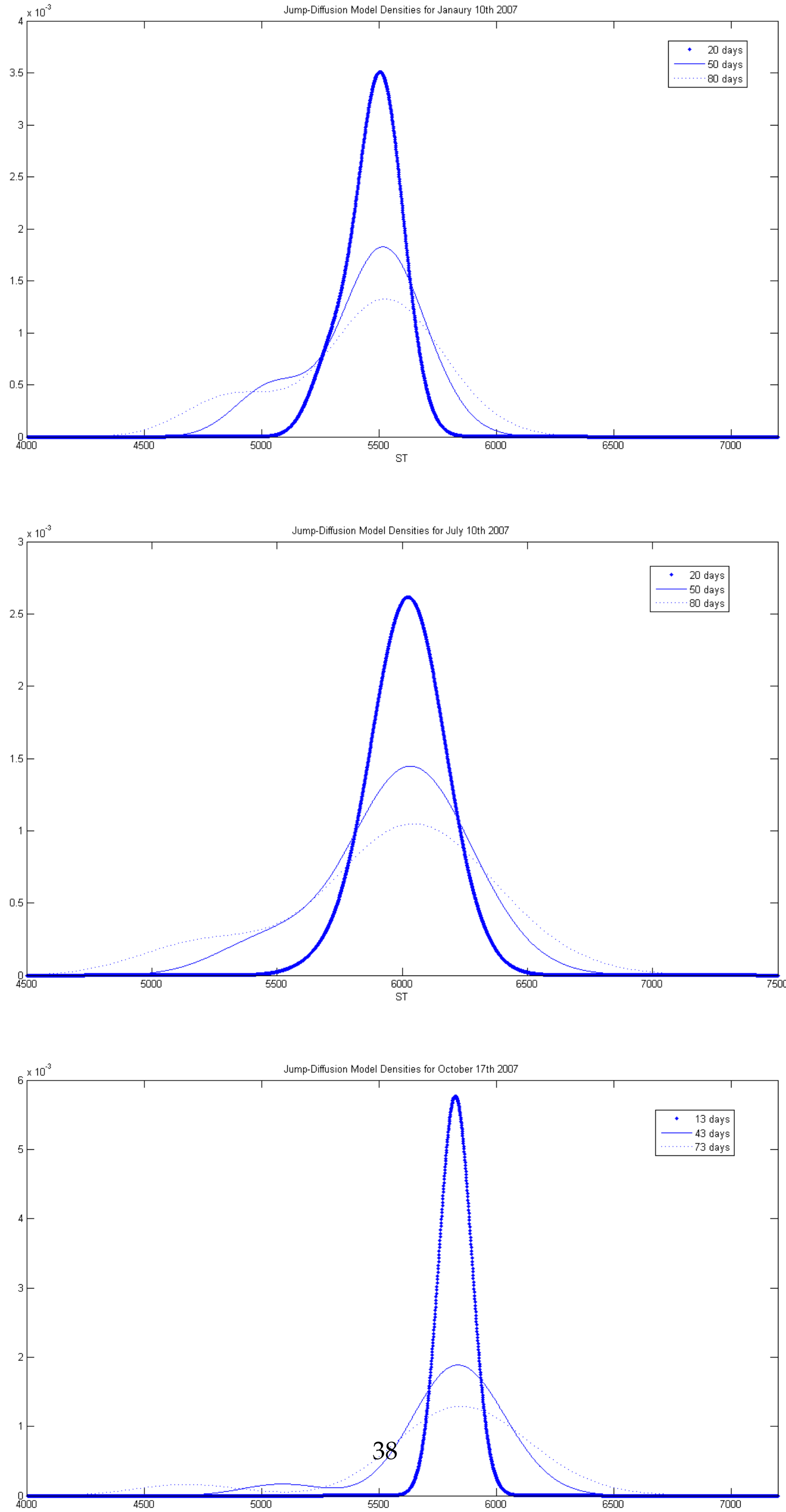

Figure 11: Jump diffusion Risk Neutral Densities for different dates and various maturities

B.Tables

Table 3 : Estimation of Black & Scholes parameters

Dates	τ	m	s
01-10-2007	20	8.6153	0.0211
	50	8.6185	0.0442
	80	8.6224	0.0600
07-10-2007	20	8.7050	0.0253
	50	8.7086	0.0485
	80	8.7127	0.0678
10-17-2007	13	8.6701	0.0122
	43	8.6739	0.0423
	73	8.6839	0.0590

Table 4 : Estimation of Mixture of log-normal parameters

Dates	τ	m_1	m_2	s_1	s_2	α
01-10-2007	20	8.6092	8.6152	0.0332	0.0200	0.2876
	50	8.5705	8.6319	0.0671	0.0325	0.2713
	80	8.5361	8.6465	0.0835	0.0406	0.2729
07-10-2007	20	8.6928	8.7079	0.0407	0.0202	0.2894
	50	8.6496	8.7257	0.0681	0.0328	0.2672
	80	8.6487	8.7478	0.0835	0.0405	0.4042
10-17-2007	13	8.6541	8.6741	0.0154	0.0111	0.3019
	43	8.6043	8.6892	0.0623	0.0311	0.2515
	73	8.5620	8.7089	0.0796	0.0393	0.2729

Table 5 : Estimation of Hermite polynomials parameters

Dates	τ	σ	b_3	b_4
01-10-2007	20	0.0967	-0.4280	0.4845
	50	0.1298	-0.4302	0.5250
	80	0.1435	-0.4283	0.4960
07-10-2007	20	0.1113	-0.2324	0.1289
	50	0.1405	-0.4295	0.5314
	80	0.1612	-0.4320	0.4823
10-17-2007	13	0.0635	-0.0699	-0.0284
	43	0.1284	-0.3848	0.3246
	73	0.1582	-0.4263	0.4633

Table 6 : Estimation of Edgeworth expansion parameters

Dates	τ	σ	γ_1	γ_1
01-10-2007	20	0.0960	-0.4480	0.5247
	50	0.1288	-0.8317	0.3271
	80	0.1413	-0.7991	0.0590
07-10-2007	20	0.1092	-0.1656	0.2550
	50	0.1344	-0.5386	0.5747
	80	0.1572	-0.6779	0.395
10-17-2007	13	0.0634	-0.1087	-0.0683
	43	0.1389	-1.0724	1.9653
	73	0.1505	-0.6665	-0.9913

Table 7 : Estimation of Jump-Diffusion parameters

Dates	τ	σ	λ	κ
01-10-2007	20	0.0738	3.3166	-0.0366
	50	0.0858	1.4745	-0.0845
	80	0.0918	0.9700	-0.1168
07-10-2007	20	0.1032	1.0401	-0.0424
	50	0.1069	1.0299	-0.0921
	80	0.1125	0.7551	-0.1350
10-17-2007	13	0.0629	0.1108	-0.0010
	43	0.0979	0.6082	-0.1283
	73	0.1073	0.4525	-0.1999

Table 8: Estimation of Heston's Stochastic Volatility model parameters

Dates	τ	κ^*	θ^*	σ	ρ	V_t
01-10-2007	20	45.4325	0.0296	0.1481	-0.9738	0.0003
	50	9.9352	0.0308	0.4431	-0.7180	0.0000
	80	23.4851	0.0220	0.5867	-0.8887	0.0000
07-10-2007	20	7.1459	0.0713	0.2017	-0.6110	0.0021
	50	5.4654	0.0628	0.5107	-0.6415	0.0000
	80	11.1566	0.0434	0.7884	-0.7215	0.003
10-17-2007	13	44.5449	0.0113	0.0938	-0.9564	0.0001
	43	53.8001	0.0191	1.0621	-0.4742	0.0029
	73	21.5488	0.0382	2.1490	-0.4972	0.0000

Table 9 : Implicit standard deviation

Dates	τ	*B&S*	*Hermite*	*Edgeworth*	*ML*	*Jump*
01-10-2007	20	116.41	124.31	123.41	135.689	123.32
	50	244.92	263.06	261.03	281.65	262.55
	80	334.18	367.03	361.39	394.91	367.84
07-10-2007	20	152.70	156.50	153.54	171.0177	157.42
	50	294.1744	311.35	297.82	329.78	309.93
	80	413.62	450.76	439.54	460.74	447.08
10-17-2007	13	71.086	72.46	69.51	90.05	69.16
	43	247.71	255.24	276.14	312.04	277.18
	73	349.42	408.43	388.44	470.10	444.30

Note that τ is the maturity, *B&S* is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, and Jump is the jump-diffusion model.

Table 10 : Implicit skewness

Dates	τ	*B&S*	*Hermite*	*Edgeworth*	*ML*	*Jump*
01-10-2007	20	0.0633	-1.05	-0.44	-0.069	-0.367
	50	0.1328	-1.049	-0.83	-0.8106	-0.57
	80	0.1804	-10.49	-0.79	-0.9106	-0.57
07-10-2007	20	0.0759	-0.57	-0.16	-0.3668	-0.13
	50	0.1457	-1.52	-0.54	-0.9157	-0.4
	80	0.2039	-1.58	-0.68	-0.66	-0.45
10-17-2007	13	0.0366	-0.172	-0.11	-0.5126	0.0356
	43	0.127	-0.94	-0.07	-0.98	-1.05
	73	0.177	-1.04	-0.66	-0.9653	-1.18

Note that τ is the maturity, *B&S* is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, and Jump is the jump-diffusion model.

Table 11 : Implicit kurtosis

Dates	τ	*B&S*	*Hermite*	*Edgeworth*	*ML*	*Jump*
01-10-2007	20	3.007	5.37	3.52	3.8104	3.056
	50	3.0313	5.57	3.32	4.298	3.22
	80	3.05	5.42	3.06	3.9052	3.16
07-10-2007	20	3.01	3.63	3.25	4.4932	3.21
	50	3.037	5.6	3.57	4.2168	3.24
	80	3.07	5.36	3.39	3.239	3.07
10-17-2007	13	3.0024	2.86	2.97	3.3044	3
	43	3.0287	4.59	3.32	4.1335	4.93
	73	3.056	5.26	3.46	3.5049	4.77

Note that τ is the maturity, *B&S* is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, and Jump is the jump-diffusion model.

Table 12 : MSE

Dates	τ	*B&S*	*Hermite*	*Edgeworth*	*ML*	*Jump*	*Heston*
01-10-2007	20	167.902	44.33	47.1335	1419.4	**17.32**	2664.2
	50	2392.39	1196.2	190.317	190.65	**131.2532**	1342.9
	80	4866.7	1143.8	445.97	**352.0697**	463.5055	1018.1
07-10-2007	20	123.1794	62.724	**40.252**	203.95	85.52	2854.2
	50	1694.4	691.8055	**77.47**	2898.4	121.99	4641.8
	80	3888.4	1953.01	**205.27**	276.66	327.52	5561.3
10-17-2007	13	**44.48**	69.0462	70.5022	259.2161	64.2039	5362
	43	1869.8	1133	564.69	478.0566	**273.79**	2629.4
	73	852.52	8199.8	**151.83**	2236.9	777.85	7400

Note that τ is the maturity, *B&S* is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, Jump is the jump-diffusion model and Heston is the Heston's stochastic volatility model.

Table 13: ARE (10^4)

Dates	τ	*B&S*	*Hermite*	*Edgeworth*	*ML*	*Jump*	*Heston*
01-10-2007	20	**0.1781**	0.8607	5.5447	0.2046	7.12	54.61
	50	0.0452	0.0356	0.1832	0.0116	**0.000138**	3.09
	80	0.0408	0.0406	1.9181	0.0013	**0.000615**	1.61
07-10-2007	20	0.0358	**0.0194**	10.67	0.0227	0.1544	44.07
	50	0.0017	0.001	**4.9** 10^{-5}	0.0043	1.3 10^{-4}	73.1375
	80	8.6 10^{-4}	8.4 10^{-4}	**2.2** 10^{-5}	5.2 10^{-4}	2.5 10^{-5}	67.02
10-17-2007	13	781.1711	**31.286**	46.385	2416.6	3317.2	75.52
	43	0.0259	0.0122	0.0219	**0.0014**	0.0091	162.4
	73	2.05 10^{-4}	0.0012	**4.52** 10^{-5}	3.37 10^{-4}	1.7 10^{-4}	62.31

Note that τ is the maturity, *B&S* is the Black & Scholes model, Hermite is the Hermite model, Edgeworth is the Edgeworth model, ML is the mixture of log-normals model, Jump is the jump-diffusion model and Heston is the Heston's stochastic volatility model.