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Reproduction and temporary disequilibrium: a Classical approach

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Abstract. We build a bisector reproduction model with classical features in which the capitalists aim at maximizing accumulation of their profits. At variance with gravitation models, it is assumed that they invest their profits in their own industry. Their plans are based on actual productions and expected prices. Effective prices and effective allocations of resources are determined by a market-clearing mechanism. A simple law on the formation of expectations allows us to define the dynamics of disequilibria, which let appear endogenous self-sustained fluctuations, around a long-run path. The long-run rate of growth and the amplitude of the fluctuations depend on the initial conditions.

Keywords: Classical Reproduction, Market prices, Disequilibrium, Growth, Cycle.

J.E.L. Classification: E11, E30, E32, O41

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I. Introduction

It is often considered that the study of equilibria is the only solid scope for economic theory. Our approach is different: we study the reproduction of a bisector economy conceived as a simplified representation of an economy in which individual decisions are irreversible and taken independently from each other. As a consequence the analysis is centered on temporary market disequilibrium. In our model, the agents are capitalists who aim at accumulating capital. Given their expectations at a given date, they make up plans which, in general, are globally inconsistent. The market is not a tâtonnement process ensuring the instantaneous adjustment of individual plans, but a device which reveals their incompatibility. Such private economic decisions are thus submitted to a social evaluation which shows itself in differences between expected and effective profits and allocation. The model proposes a formalization of this feature of a market economy by introducing a market mechanism which determines market prices, the actual rates of profit and the final allocations of resources among the sectors, as a result of a set of individual decisions. The independent producers must adapt their productions to circumstances they had not foreseen at the beginning of the period. Then a new period opens and the economy moves from a temporary disequilibrium state to another one. A simple hypothesis on the formation of expectations allows us to link a period to the next and gives its dynamic structure to the model. It is only through a sequence of temporary or market disequilibria that a full or long-period equilibrium may be reached.

Note that in our model profits are accumulated in their own sector instead of moving from a sector to another according to their relative profitability, as in the Classical tradition of gravitation. A historical root of that idea is found in some parts of Torrens's and Marx's works, such as Torrens's "Principles of Demand and Supply" (1821, chap.VI, section VI) and Marx's extended reproduction scheme (1885, Book II, chap. XXI). Torrens is a little-known but important author who emphasized the physical constraints on the reproduction of capital if the whole profits are accumulated in their original own sectors. His main ideas on value and distribution have been set out by De Vivo (1986). More recently, Benetti *et alii* (2007) studied the dynamics of a classical disequilibrium model when a part of profits is unproductively used. In the present paper, we propose a new approach centered on a market mechanism determining the relevant disequilibrium variables on which our notion of temporary disequilibrium is based.

In section II, we introduce the model, study its properties and define the equilibrium and disequilibrium concepts. Section III examines the dynamics by decomposing them into a long-run tendency and short-run cycles. In section IV, we discuss the market mechanism we have retained and compare our results with those obtained when the price rule is a Walrasian tâtonnement.

II. Temporary disequilibrium in a reproduction model

We consider an economy where capitalists are the only economic agents: workers do not appear explicitly. The economy is made of two sectors. In each sector, there is a unique method with constant returns, which makes use of both types of inputs (the inputs include an exogenously given wage basket). All capital is circulating. When the accumulation of goods is economically impossible, capitalists may get rid of them without cost, for instance by consumption. Since we assume that all capitalists have the same price expectations, the producers in the same sector can be aggregated and everything proceeds as if there were only one capitalist producing good 1 and another good 2.

The model has a sequential three-step structure. Production takes one period. Depending on price expectations at date *t* (end of period *t*-1 and beginning of period *t*), a part of the quantities produced during the previous period is brought to the market. Exchanges occur at *t* and determine market prices and the allocation of inputs. The effective production of period *t* is then determined.

II.1. Producers' plans

Let us start with the determination of the *i*th producer's plan (i = 1, 2), when he knows the quantity q_i^- obtained as a result of his previous investment and, before the opening of the market, expects a relative price $p_{ij}^e = p_i^e \,/\, p_j^e$ for his product at that date (for the moment, the price expectations are considered as exogenously given). His objective being maximum accumulation, the producer retains a part of his product in order to invest it and brings the remaining part to the market in order to exchange it for the complementary input *j*. He thus aims at converting his product into a basket of inputs which fits the proportions required by his method of production, the expected level of production q_i^e being determined by his technique and his budget constraint at the expected prices. Let a_{ii} and a_{ij} the quantities of inputs *i* and *j* necessary to produce one unit of good *i* (these production coefficients are assumed to be positive). The expected budget constraint being written $q_i^e(a_{ii}p_i^e + a_{ij}p_j^e) \le q_i^- p_i^e$, the production plan can be expressed as a function of the previous production and the expected relative price:

$$q_i^e = \frac{q_i^-}{a_{ii} + a_{ij}p_{ji}^e} \qquad (1)$$

The quantity s_i of commodity *i* brought to the market is the difference between the previous production and the quantity of input *i* required by the production plan: $s_i = q_i^- - a_{ii}q_i^e$. The producers of the other sector have a similar behaviour. Thanks to the aggregation of all producers belonging to the same sector, the economy can be represented in an Edgeworth box (Figure 1).

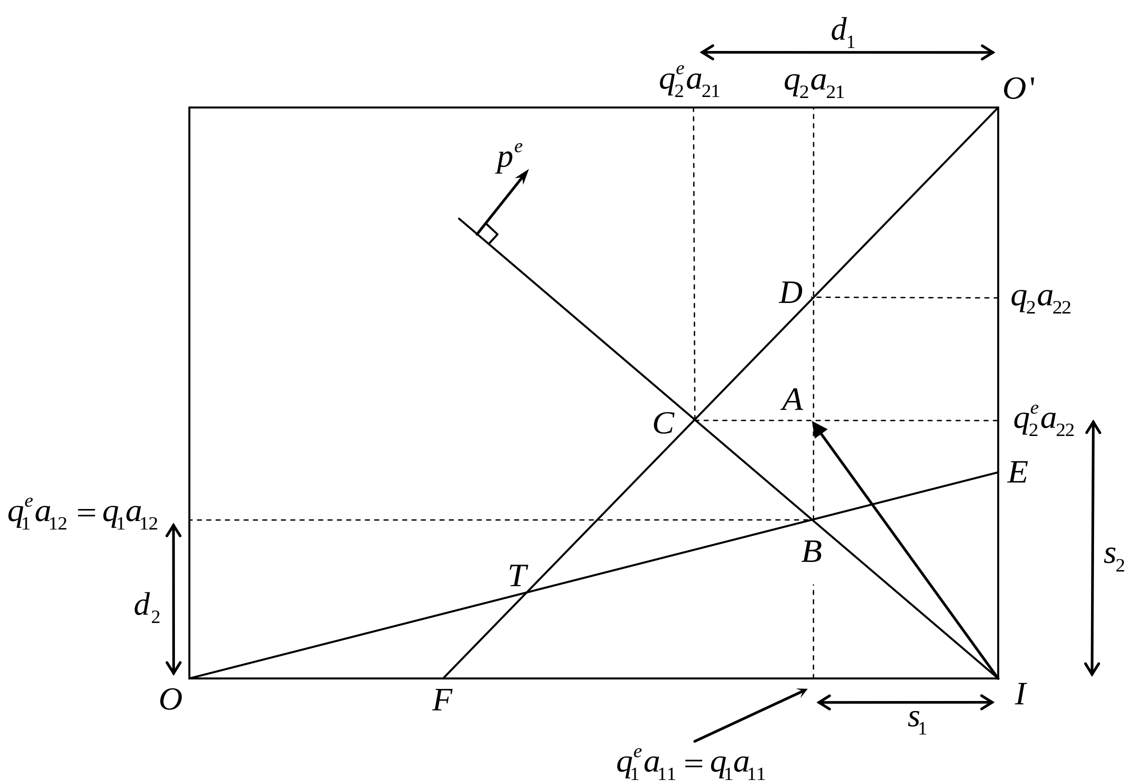

Figure 1

The origin of the axes is the south-west corner for sector 1 and the north-east corner for sector 2, and the sides of the box are the quantities q_1^- and q_2^-. The initial endowments are represented by point I at the south-east corner. The production methods are respectively represented by the line *OE* with slope a_{12} / a_{11} (for the production of good 1) and the line *O'F* with slope a_{22} / a_{21} (for that of good 2). For an expected price vector p^e, the expected budget constraint is orthogonal to p^e. The production plans are represented by point *B* for the first producer and point *C* for the second. Supplies s_1 and s_2, and demands d_1 and d_2, result.

II.2. Market mechanism and production

Flukes apart, the capitalists' plans are not compatible (the points *B* and *C* are distinct) and cannot be both implemented. According to the market clearing mechanism we introduce, inspired from Cantillon (1755; see Benetti and Cartelier, 2001), the market price is established through the exchange of the supplied quantities:

$$p_{ij} = \frac{s_j}{s_i} = \frac{q_j^- - a_{jj} q_j^e}{q_i^- - a_{ii} q_i^e} \tag{2}$$

In Figure 1, the relative price is the slope of segment *IA* and the effective exchange is represented by the move from point *I* to point *A*. A comparison of the expected and market prices leads to the property:

$$p_{ij} \gtrless p_{ij}^e \Leftrightarrow s_j \gtrless d_j \Leftrightarrow d_i \gtrless s_i \tag{3}$$

The first equivalence stems from the equality $p_{ij} = s_j / s_i$ and the expected budget constraint $d_j = s_i p_{ij}^e$, and the second from the Walras law $(d_i - s_i) p_{ij}^e + (d_j - s_j) = 0$.

Let us dub 'scarce' the commodity in excess demand (good 1 in the case of Figure 1) and 'superabundant' that in excess supply. The scarce commodity (denoted by *r* for ‘rare’) is therefore the one that the producers underevaluated. That commodity is entirely accumulated. The producer of the scarce good gets windfall profits, since its effective price is higher than expected: he obtains more of the other good than expected. Nevertheless he cannot accumulate it because of the lack of his own input. After having got rid of the quantity *AB* of the superabundant good (for instance by consuming it), the producer of the scarce commodity implements his investment plan at *B*, therefore the effective rate of growth (or accumulation) in that sector is equal to both the expected rate of growth and the expected rate of profit ($g_r = g_r^e = \pi_r^e < \pi_r$).[1] By contrast, the producer of the superabundant good (denoted by *a*) receives less of the other commodity than expected and therefore cannot invest the amount of his own commodity he had put aside for production. He must revise his plan and downsize production from *C* to *D* by eliminating the quantity *AD* of the superabundant good, therefore $g_a < \pi_a < \pi_a^e$. Note that both producers get rid of the same good, so that the definitions of scarcity and superabundancy are consistent with the effective uses of the goods. In this economy in temporary disequilibrium, each agent must withdraw a part of the same superabundant good from accumulation: were the market opened again, both producers would demand the same input and no exchange would take place. Nevertheless the outcome of the market is an efficient input allocation (*A* in Figure 1):[2] the production of a sector could only be improved by obtaining some quantity of the scarce commodity at the expense of the other sector.

The effectively produced quantities are determined by the inputs owned by the producers after the exchange: $q_i = \min\ \ q_i^e a_{ii} / a_{ii}\ ,\ s_j / a_{ij}$, or:

$$q_i = \min\left\{ q_i^e \ , \frac{q_j^{-} - a_{jj} q_j^e}{a_{ij}} \right\} \tag{4}$$

In that equality, let us replace the expected production by its expression as a function of past production and expected prices, as given by (1). The effective production is then defined as a function of initial quantities and expected prices:

$$q_i = \min\left\{ \frac{q_i^{-}}{a_{ii} + a_{ij} p_{ji}^e} \ , \frac{a_{ji}}{a_{ij}} \frac{q_j^{-}}{a_{ji} + a_{jj} p_{ji}^e} \right\} \tag{5}$$

Similarly, starting from given quantities and price expectations, the effective market prices are

[1] The rate of profit π is calculated as a value ratio between net production and inputs at current prices, i.e. at their replacement cost: $1 = (1 + \pi_i)(a_{ii} + a_{ij} p_{ji})$. This rate measures the capital reproduction capability.

[2] In Figure 1 the set of the efficient allocations is made of the two triangles *OFT* and *O'ET*.

obtained by eliminating q_i^e and q_j^e between relationships (1) and (2), namely:

$$p_{ij} = \frac{a_{ji}q_j^-}{a_{ij}q_i^-}\frac{a_{ii}p_{ij}^e + a_{ij}}{a_{jj}p_{ji}^e + a_{ji}} \tag{6}$$

II.3. A remarkable property

We know that the producer of the scarce commodity implements his production plan and that this commodity is entirely accumulated. The first condition implies that the producer of the scarce commodity brings to the market a quantity s_r which has the same expected value as the quantity of input *a* he plans to invest and does invest: $s_r p_r^e = q_r a_{ra} p_a^e$; the second condition implies that the quantity s_r is actually invested by the producer of the superabundant good: $s_r = q_a a_{ar}$. The equality $q_a a_{ar} p_r^e = q_r a_{ra} p_a^e$ follows. In Figure 1, the segment *IB* makes visible that the (partial) use of the superabundant commodity available in sector r (i.e., the quantity $q_1 a_{12} = d_2$) has the same expected value as the (full) use of the scarce commodity in sector a (i.e., the quantity $q_2 a_{21} = s_1$). As the scarce and the superabundant goods play symmetric roles in this equality, it can also be written:

$$\frac{q_i}{q_j} = \frac{a_{ji}}{a_{ij}} p_{ij}^e \tag{7}$$

This equality might also have been found by direct computation from (5). It shows that the proportion between the effective productions only depend on expected prices and not on the past activity levels: the economic intuition is that the share of the existing quantity of the scarce commodity between the sectors is determined by its producer only on the basis of the expected price, and determines the ongoing productions.[3] The relationship between the relative activity level and the expected price is linear and increasing, therefore a rise of the expected price leads to a proportional rise of the relative production of the corresponding commodity. The study of the dynamics will shed light on the strength and the limits of this market regulation.

[3] The same idea can be found in Marx's schemas of enlarged reproduction in which the total accumulation of means of production determines the growth path of the economy (1885, Book II, chap. XXI).

II.4. Equilibrium

Let a full (or long-period) equilibrium of the economy be defined by the implementation of two conditions: (i) *market temporary equilibrium*, that is the equality between demand and supply at the expected price before the opening of the current market; and (ii) *reproduction equilibrium*, that is the equality between the two rates of accumulation.

Market equilibrium presumes the equality $p_{ij}^{e} = p_{ij}$, as shown by relations (3). No commodity is then under- or over-evaluated, there is no scarce or superabundant commodity, and all plans for exchange and production are implemented. Both arguments of the *min* functions are equal. In Figure 1, this condition occurs if p^{e} is orthogonal to segment *IT*. Note that market equilibrium holds only by a sheer fluke. At variance with the Hicksian temporary equilibrium (Hicks, 1938), it is not the result of an underlying stable adjustment process, and it is not maintained from a period to the next. The dynamics of our model are those of a sequence of temporary disequilibria.

Reproduction equilibrium is defined by $q_i / q_j = q_i^{-} / q_j^{-}$. Let us set $p = p_1 / p_2 = p_{12}$ and $q = q_1 / q_2$. At full equilibrium, both equalities $p = p^{e}$ and $q = q^{-}$ prevail. Then, by eliminating p between (6) and (7), it turns out that the full equilibrium proportion $\bar{q}$ is defined by:

$$\bar{q} = \frac{a_{11}\bar{q} + a_{21}}{a_{12}\bar{q} + a_{22}} \qquad (8)$$

This formula shows that the full equilibrium row vector $(\bar{q},1)$, or $(\bar{q}_1, \bar{q}_2)$ as well, is the positive Perron-Frobenius eigenvector of the matrix $\mathbf{A} = \ a_{ij}$. Let α_1 be the dominant root of **A**, which is smaller than 1 if the economy is productive. In equilibrium both goods are entirely accumulated and the uniform rate of growth and accumulation is $\alpha_1^{-1} - 1$: the economy is on a von Neumann maximum growth path. In that case the rate of profit is uniform and equal to the maximum growth rate of the economy, and the market prices – the column positive Perron-Frobenius eigenvector of **A** – coincide with the classical prices of production.

Point *T* in Figure 1 is named after Torrens: the historical reference to our notion of equilibrium is found in Torrens's numerical model, in which all profits are accumulated and enduring reproduction goes on only when Torrens's "good proportions", which are those defined by equality (8), are met.

III. Dynamics of disequilibria

A hypothesis on the making of the expectations will allow us to study the dynamics of disequilibria. For simplicity, we assume static expectations: the expected price at some date is the last known market price

$$p^e = p^- \qquad (9)$$

Then the market prices depend on the produced quantities and the past prices (formulas (6) and (9)) and the same for the quantities produced (formulas (5) and (9)). Equality (7), which is written as:

$$q = (a_{21} / a_{12)}) p^- \qquad (10)$$

shows that the dynamics of the relative price and the relative quantity are basically identical up to a one-period lag. It therefore suffices to study the evolution of quantities. We shall examine in the first place the long-term tendency of the quantities, and then the fluctuations of their proportion.

III.1. Long-run growth

Out of equilibrium the evolution of quantities is irregular. We proceed to computer simulations to study their long-run dynamics. These experiments show that the average rate of growth is roughly the same for both sectors in the medium run (let us say over 10 periods) and is approximately constant afterwards. When the initial distance to the equilibrium proportion increases, the long-run rate of growth decreases (in the long-run, no sector can grow faster than the other because its expansion would ultimately require to absorb more than the gross product obtained by the other). Figure 2 plots the long-run rate – computed over 1000 periods – against the initial value of relative quantity $q(0)$ (for simplicity, we have chosen $p(0) = \overline{p}$ and $\overline{q} = 1$).

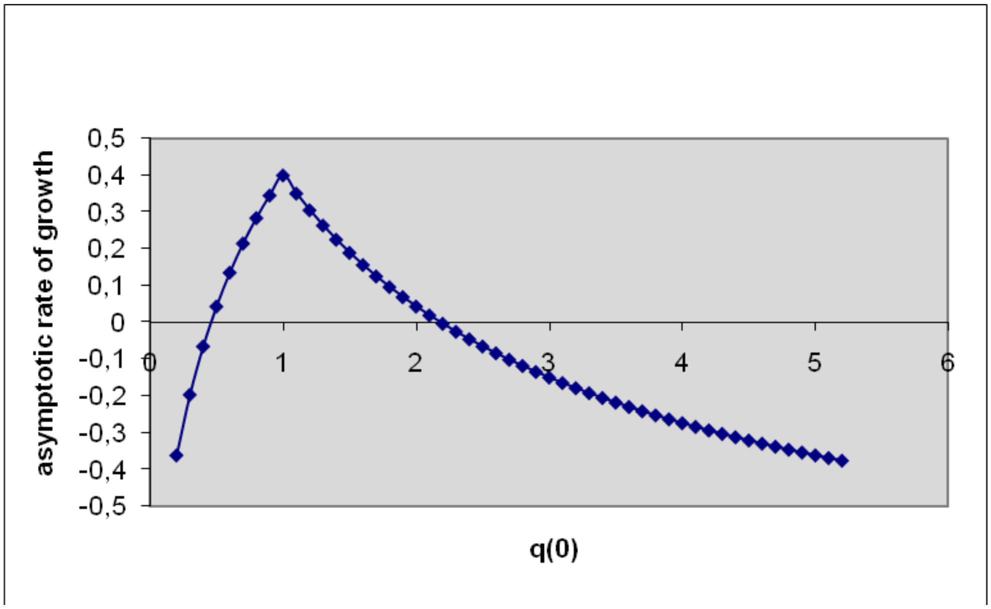

Figure 2

The rate of growth reaches its (von Neumann) maximum value for $q(0) = \bar{q}$ and decreases with the distance between $q(0)$ and $\bar{q}$. It may become negative, though the economy is productive, because the superabundant commodity is partly 'spoilt' at each date: the dynamics of disequilibrium imply the exclusion of some commodities from accumulation.

A noteworthy property of the model is that the hysteresis effects concern the long-run rates and not only the levels of the growth paths: the transitory shocks of productivity, what the Classicals named "a good or bad crop", have permanent effects on the long-run rates of growth.

III.2. Cycles

The study of cycles is based on the observation of the relative quantity. The elimination of the relative price from the one-step induction formulas (6), (9) and (10) allows us to obtain the two-step induction formula governing the evolution of the relative quantity:

$$q_t = \frac{q_{t-1}}{q_{t-2}} \frac{a_{11}q_{t-1} + a_{21}}{a_{12}q_{t-1} + a_{22}} \qquad (11)$$

In case of uniform composition of capital ($k = a_{11} / a_{12} = a_{21} / a_{22}$), a simplification occurs in this formula ($q_t = k\dfrac{q_{t-1}}{q_{t-2}}$) and the sequence $x_t = \ln(q_t / k)$ satisfies the simple induction formula $x_t = x_{t-1} - x_{t-2}$, from which it follows that $x_t = (x_{t-2} - x_{t-3}) - x_{t-2} = -x_{t-3}$ and $x_t = x_{t-6}$: the sequence of relative quantities is periodic of period six. So is the sequence of relative prices, and the long-run rate of growth is then equal to the average rate of growth over six consecutive periods. Outside this simple case, we study first the fluctuations in a neighborhood of the equilibrium, then in the large.

III.2.1. Local fluctuations

The fixed point of the dynamics (11) is the unique positive solution $\bar{q}$ of (8). The standard method to identify the nature of the dynamics in a neighbourhood of the fixed point consists in linearizing the dynamical equation $q_t = h(q_{t-1}, q_{t-2})$ and writing it as $X_t = h_1 X_{t-1} + h_2 X_{t-2}$, where X_t stands for $q_t - \bar{q}$ and h_1 and h_2 are the partial derivatives of h at point $(\bar{q}, \bar{q})$. Calculation shows that $h_2 = -1$ and $h_1 = 1 + (\det \mathbf{A})/(a_{12}\bar{q} + a_{22})^2$. Let α_1 and α_2 be the dominant and second eigenvalue of matrix **A**. Since $(\bar{q}, 1)$ is the positive row-Perron-Frobenius eigenvector, we have $\alpha_1 = a_{12}\bar{q} + a_{22}$, hence $h_1 = 1 + \alpha_1\alpha_2 / \alpha_1^2$. As $|\alpha_2| \leq \alpha_1$, it is convenient to introduce the scalar θ defined by:

$$2\cos\theta = 1 + \frac{\alpha_2}{\alpha_1} \quad \left(\theta \in \left]0, \frac{\pi}{2}\right[\right) \tag{12}$$

The linearized dynamics obey the law:

$$X_t - 2X_{t-1}\cos\theta + X_{t-2} = 0 \tag{13}$$

The characteristic polynomial $X^2 - 2X\cos\theta + 1$ has two complex roots $e^{\pm i\theta}$ of modulus one, which is unfortunately the case in which linearization cannot conclude to the local convergence or divergence.

Let $M_t = (q_t, q_{t-1})$ and $\overline{M} = (\bar{q}, \bar{q})$. A more precise study of the local dynamics makes us introduce the quadratic form δ^2 defined by**:**

$$\delta^2(M_t, \overline{M}) = X_t^2 - 2X_t X_{t-1} \cos\theta + X_{t-1}^2 \tag{14}$$

Since $\delta(M_t, \overline{M})$ is zero if and only if M_t coincide with $\overline{M}$, δ is a distance. Moreover, when X_t is replaced by its value, given by (13), as a function of X_{t-1} and X_{t-2}, $\delta^2(M_t, \overline{M})$ is expressed as a function of X_{t-1} and X_{t-2}, and it turns out that the expression then obtained is exactly $\delta^2(M_{t-1}, \overline{M})$. This means that the distance to $\overline{M}$ remains constant from one point to the next, therefore, up to first order approximations, the trajectory of the point M_t is such that $\delta(M_t, \overline{M}) = \delta_0$, where δ_0 is the initial distance. The curve defined by this equation is an ellipse: its centre is the fixed point $\overline{M}$ and its main axis the 45-degree line. The economic interpretation is that the relative quantities q_t fluctuate around a central value $\bar{q}$. Appendix I shows that the period only depends on the technical coefficients (of course, the amplitude depends also on initial conditions). In general, the period is not an integer. The case of uniform composition of capital serves as a benchmark and a divide between quicker and smaller fluctuations on the one hand, or slower and larger fluctuations on the other hand.

Because these calculations rely on a first order approximation, they do not prove the cyclical nature of the relative quantities: the points M_t might actually move on a spiral with a slight inwards or outwards deviation at each loop. Then the approximation by an ellipse holds for a certain time in a neighbourhood of equilibrium but is not admissible globally.

III.2.2. Global fluctuations

Apart the case of uniform composition of capital, the study of the global dynamics is complex because the dynamics are not linear. Let us plot the points $M_t = (q_{t+1}, q_t)$ and proceed to computer simulations for an arbitrary starting point M_0. It turns out that these points move on a closed curve. Figure 3 represents three such curves, which depend on the initial condition (500 points plotted on each curve). These curves are fitted into each other and gradually lose their initial ellipse shape as they wander away from equilibrium. In Figure 4, we have drawn the segments $M_{t-1}M_t$, and the fact their envelope is smooth shows that the movement of M_t on its curve is regular. The observed regularities are the expression of cyclical movements in the relative activity level, and similarly for the relative price.

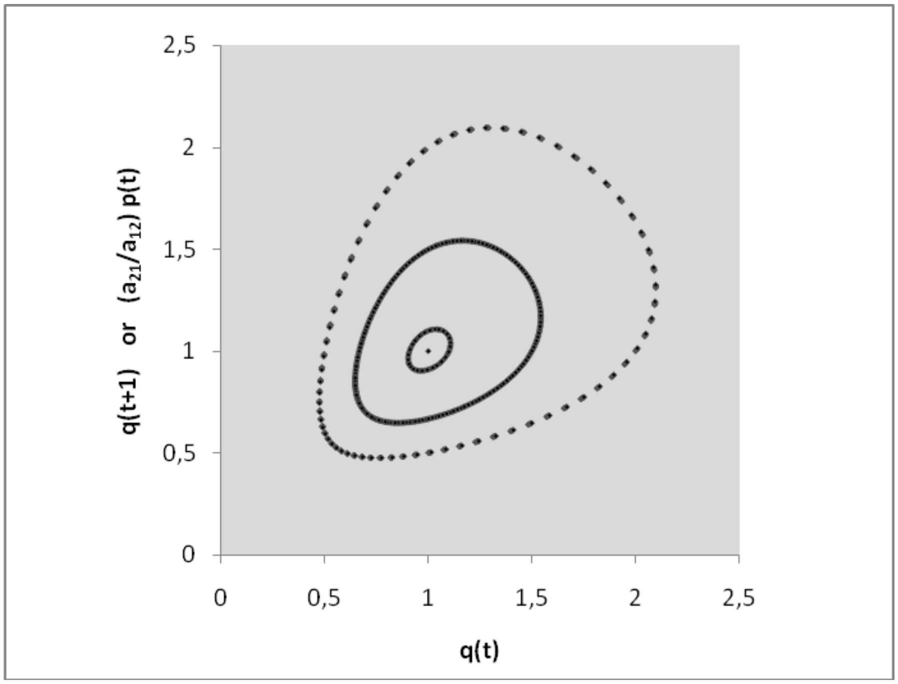

Figure 3

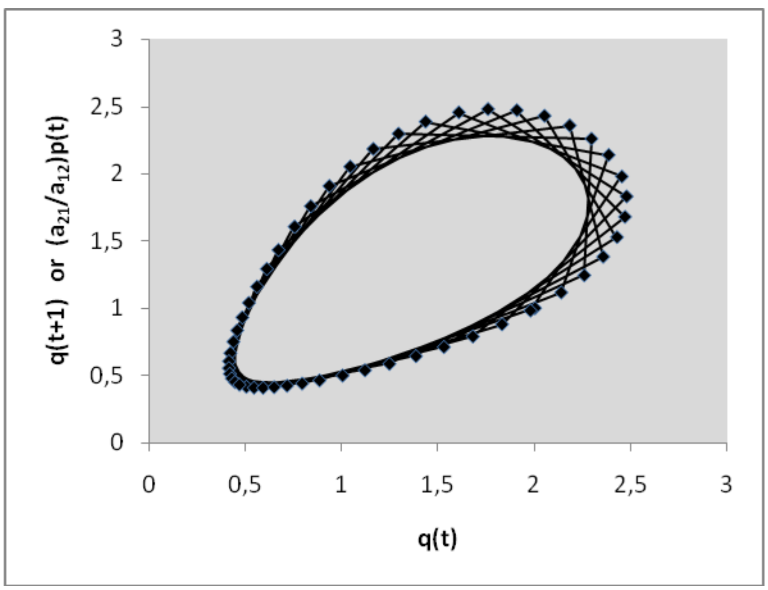

Figure 4

Figure 5, which represents a simulation of q_t over 1000 periods, is another significant aesthetical contribution to the study of the reproduction model, in that it suggests the existence of hidden regularities underlying complex moves.

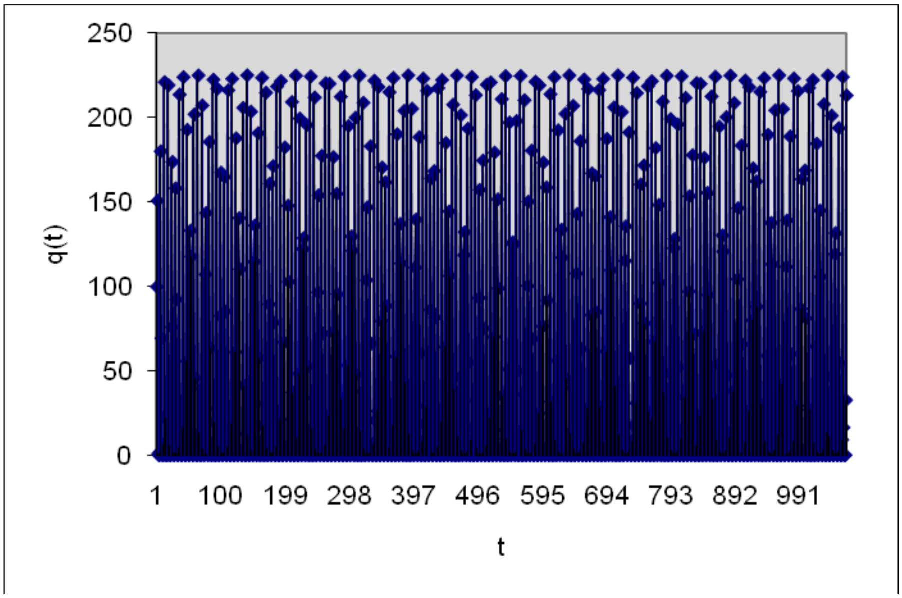

Figure 5

An important property of the formula (11) is its temporal symmetry, as it is written $q_t q_{t-2} = f(q_{t-1})$, or $q_{t+1}q_{t-1} = f(q_t)$: q_{t-1} and q_{t+1} play symmetric roles for a given value of q_t. Time can be reversed in a strong sense as the future and the past are defined by the *same* formulas. This temporal symmetry implies the geometrical symmetry of the curves with respect to the 45-degree line, a property illustrated by Figure 3. That property excludes the slightly inwards and outwards spirals as possible outcomes, the existence of which has been suggested in the previous subsection. Therefore there are strong reasons to think that the dynamics are those of a self-sustained cycle. Appendix II goes into some working tracks concerning the property that the points M_t move on a closed curve.

The economic interpretation of the clockwise move of the point M_t on its curve in Figure 3 is as follows. By relation (10), q_{t+1} is equal to p_t up to a constant factor so that, up to a linear deformation, Figure 3 can be seen as plotted in the plane (q_t, p_t). To draw the phase diagram in this plane, we consider the curves $\Delta q_t = 0$ and $\Delta p_t = 0$. Since $\Delta q_t = q_{t+1} - q_t = (a_{21}/a_{12})p_t - q_t$, the locus $\Delta q_t = 0$ is a straight line from the origin with slope (a_{12}/a_{21}) (from an economic point of view, $\Delta q_t = 0$ means that market equilibrium prevails at *t* (because $\Delta q_t = 0 \Rightarrow p_t = p_{t-1} = p_t^e$) and reproduction equilibrium at $t+1$ (because $q_{t+1} = q_t$). The equation of the curve $\Delta p_t = 0$ is obtained from (6) and (9) and is written:

$$\Delta p_t = p_{t+1} - p_t = \tfrac{a_{21}}{a_{12}} \tfrac{p_t}{q_t} \tfrac{a_{11}p_t + a_{12}}{a_{21}p_t + a_{22}} - p_t = 0 \quad \Rightarrow \quad p_t = \frac{a_{12}}{a_{21}} \frac{q_t a_{22} - a_{21}}{- q_t a_{12} + a_{11}} \qquad (15)$$

Its slope depends on the sign of the determinant. The arrows in Figure 6 result from the signs of Δq_t and Δp_t.

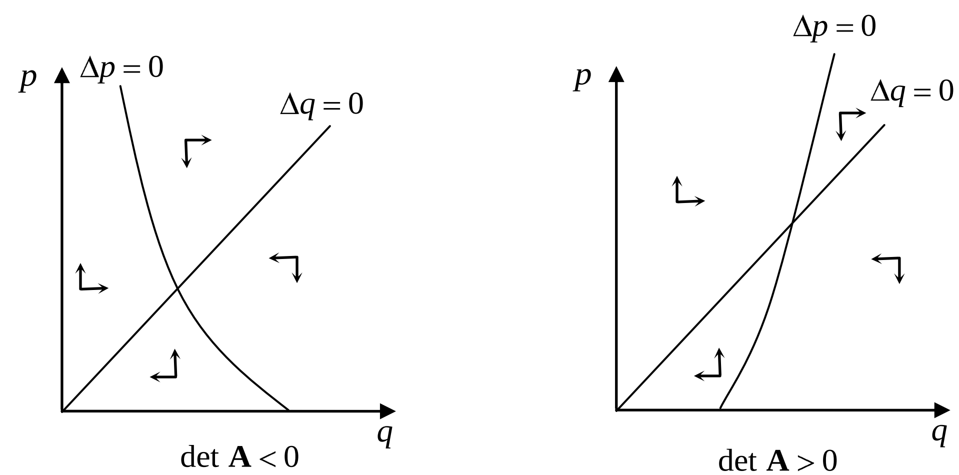

Figure 6

Above the line $\Delta q_t = 0$, q is increasing because commodity 1 is scarce: it had been under-evaluated before the opening of the market at date t ($p_t > p_{t-1} = p_t^e$) and is completely invested, whereas good 2 is partly excluded from accumulation. The market regulation, expressed by the increasing relationship (6) between the proportion q and the expected price, is at work. The price may keep rising for a while, but sooner or later it will fall because high price at t means high expectation at $t+1$, high activity plan in sector 1, high demand for good 2 and low supply of good 1 (because the producer 1 keeps a great quantity of his own product as input); and *mutatis mutandis* for the low activity plan in sector 2. As soon as the price has declined once, commodity 2 becomes the scarce good at the next period and is in excess demand. Then the same story holds the other way around, with commodity 2 being the scarce good during the other half of the cycle.

IV. Comparisons

IV.1. Disequilibrium prices

In our model, the agents are in disequilibrium and the baskets they buy differ from those they expected. The reason is that the mechanism retained for the market price formation presumes that the agents take irrevocable decisions before knowing the market prices. An obvious objection is that the customer knows the price of meat before he pays it to the butcher, and that is why he does not get more of it than the quantity he decided. At first sight, the opposite situation, which occurs in our model for the producer of the scarce commodity, might seem in contradiction with the idea of voluntary exchange. But the demand for meat is not meaningful: going out of the butcher's, the consumer may discover that chocolates are sold off and that he brings home more meat than he would have bought, had he been aware of that sale. Walras (1874) taught us that the demand for meat depends as much on its price as on those of the other goods and, therefore, assumed that the decisions are taken on the basis of a perfect knowledge of the entire price vector. In modelling the working of the market, some aspects of reality are ignored while others are stressed: our basic hypothesis that the expenditure decisions are taken on the basis of (generally erroneous) expectations of prices is less strong than the one retained by the general equilibrium theory. It allows us to deal with disequilibrium, which is not the case for the Walrasian theory. Suppose indeed that the price p^e is announced by an auctioneer and that, as in a Hahn process (Hahn and Negishi, 1962), transactions are effective at these prices. The producers of the scarce commodity would be in equilibrium and the quantity of the superabundant commodity not used in production (the segment *BD* in Figure 1) would remain in the hands of his producers. Then the produced quantities coincide with those we have determined but the prices at which the next transactions will take place are unknown. That is why the Walrasian theory must call on for an auctioneer who decides the price changes whereas, in our decentralized and perfect competition model, the price is determined (by equation (2)) as a function of the agents' decisions and independently of their wills. Other price formation rules might be taken into account, but the one we retain is simple and its origin can be traced back to Cantillon's (1734) analysis of market prices.

IV.2. A comparison with Walrasian tâtonnement

The differences between the dynamics generated by our price formation rule and the Walrasian tâtonnement can be put into light by comparing their consequences in the same situation. Since the treatment of complementarity is at stake here, let us first recall how the Walrasian theory deals with it and consider a two-good exchange economy with 2000 price-taking agents, all of them desiring one lump of sugar for one cup of coffee. The initial endowments are: 1000 agents hold 2 cups of coffee, 999 hold 2 sugar lumps, whereas the last agent has 3 sugar lumps. At the Walrasian equilibrium sugar is a free good, the 1000 agents who hold it do not get anything while the 1000 other agents can drink 2 cups of coffee.

Let us return to the economy examined in section II but replace our market mechanism by a Walrasian tâtonnement (future markets are not included in the model). p^e is now interpreted as a price called by the auctioneer and taken by producers as a parameter for their calculations. In order to reach maximum accumulation, each producer maximizes his future production under the given constraints. Let q_i be the available endowments (previously noted as q_i^-). The *i*th (i = 1,2) producer's program is written:

$$\max_{s_i, d_j} \ \min \ \frac{q_i - s_i}{a_{ii}}, \frac{d_j}{a_{ij}} \qquad \text{s.t.} \qquad d_j = s_i p_{ij}$$

hence the aggregate demand and supply functions:

$$\begin{aligned} s_1 &= \frac{q_1}{1+k_1 p} \qquad & d_2 &= \frac{q_1 p}{1+k_1 p} \\ s_2 &= \frac{q_2 k_2 p}{1+k_2 p} \qquad & d_1 &= \frac{q_2 k_2}{1+k_2 p} \end{aligned} \tag{16}$$

where, by definition, $p = p_{12}$ and $k_1 = a_{11} / a_{12}$ and $k_2 = a_{21} / a_{22}$. Disregarding the case of equal composition of capital, i.e. assuming $k_1 \neq k_2$, suppose for a moment that:

$$\min \ k_1, k_2 \ < q < \max \ k_1, k_2 \tag{17}$$

where $q = q_1 / q_2$. Then:

$$p_T = \frac{1}{k_2}\frac{q - k_2}{k_1 - q} \tag{18}$$

is an equilibrium price, at which the resources are fully employed (in Figure 1, this price supports the equilibrium point *T*). For any other price p, the excess demand $z_1(p) = d_1 - o_1$ for good 1 amounts to:

$$z_1(p) = \frac{k_2 q_2}{1 + k_2 p \quad 1 + k_1 p} \;\; k_1 - q \;\; (p - p_T) \tag{19}$$

From condition (17), the sign of $k_1 - q$ is that of $k_1 - k_2$. It results from (19) that the sign of $z_1(p)$ is that of $p - p_T$ if $k_1 > k_2$, opposite otherwise. The equilibrium at p_T price is therefore unstable if $k_1 > k_2$ and locally stable if $k_1 < k_2$. Let us distinguish these two cases:

a) Weakly interdependent industries ($k_1 > k_2$)

The economic interpretation of the condition $k_1 > k_2$ is that the inputs of each industry come more from the product of that industry itself than from the other: production is 'introverted'. The equilibrium at price p_T is unstable, but the excess demand formula (19) also shows the existence of two 'hidden' equilibria with a free good. One of them corresponds to $p = 0$, i.e. $p_1 = 0,\ p_2 > 0$: then $z_1(p) < 0$ and $d_2 = s_2 = 0$. Since, for $p_1 = \varepsilon > 0,\ p_2 = 1$ we have $z_1(p) < 0$, the equilibrium is stable. The second equilibrium corresponds to $p = +\infty$, i.e. $p_1 > 0,\ p_2 = 0$ (then $d_2 < s_2$ and $d_1 = s_1 = 0$), and is also stable.

This case is represented in the Figure 1: equilibrium *T* is unstable but *E* and *F* are two more equilibria, with a zero price for commodity 2 at *E* and for commodity 1 at *F* (for instance at *E*, there is an excess supply for good 2 whereas good 1 is neither demanded nor supplied). As can be seen on the Figure, the tâtonnement process during the current period converges towards an equilibrium with one free commodity, which is either 1 or 2 according as the first called price is lower or higher than p_T. At the next date, the sector producing the zero-price good cannot buy any amount of the other input and its production will be nil. During the following period, the other sector will be in the same situation, therefore the economy collapses in two periods.

b) Strongly interdependent industries ($k_1 < k_2$)

If $k_1 < q < k_2$ there is no equilibrium with a zero price, because the excess demand for such a commodity would be positive. The equilibrium at T is unique and globally stable. The tâtonnement within the period leads to the full utilization of the resources.

Let us study the dynamics of quantities across the sequence of successive equilibria. Let $\mathbf{A}$ be the matrix of technical coefficients and $\mathbf{q}$ the row-vector of the produced quantities. The activity levels $\mathbf{q}^+$ of the next period are such that $\mathbf{q}^+\mathbf{A} = \mathbf{q}$, i.e. $\mathbf{q}^+ = \mathbf{q}\mathbf{A}^{-1}$ and $\mathbf{q}(t) = \mathbf{q}(0)\mathbf{A}^{-t}$. Let $q(0) = \lambda_1 y_1 + \lambda_2 y_2$, where y_1 and y_2 are row eigenvectors of **A**. Since the root α_2 with maximum modulus of A^{-1} is the inverse of the non dominant root of **A**, the vector $q(t) = \lambda_1(\alpha_1)^t y_1 + \lambda_2(\alpha_2)^t y_2$ takes the direction of y_2 in the long run, and therefore has a negative component. There exists a minimum integer τ such that $q(\tau)$ admits a negative component. In economic terms, the full utilization of resources at this date is impossible. At this date, the tâtonnement converges towards the only equilibrium, with a zero price for either good 1 (if $q_\tau > k_2$) or good 2 (if $q_\tau < k_1$). The production of the free good vanishes at date $\tau + 1$ and the big crunch of the economy occurs at date $\tau + 2$.

Summing up, the technical complementarities imply that, sooner or later, some commodities are excluded from accumulation: a planner who would aim at the full utilization of resources and would therefore select the allocation T at any date would meet the same difficulty. The Walrasian tâtonnement allows for the disposal of some good only if its price is zero: this will occur either immediately ($k_1 > k_2$) or after some time ($k_1 < k_2$), but the economy will then disappear. By contrast, the price formation rule we have considered allows for the partial disposal of the superabundant commodity even if its price is positive and the reproduction of the economy is possible at any date.

V. Conclusion

In the temporary disequilibrium model with Classical features we have dealt with, the conditions of production, the present quantities and the price expectations lead to a determination of market prices, quantities, rates of accumulation and profit. In disequilibrium no agent can accumulate the whole of his effective profit and some agents cannot implement their initial plans. The dynamics of our model exhibit endogenous self-sustained cycles around a long-term path. The level of the long-term rate of growth depends on the gap between the initial proportions and the equilibrium (or von Neumann) proportions. If the gap is significant, the amount of commodities discarded from accumulation is great, a phenomenon which generates lower long-term rates of growth and cycles with larger amplitudes. Since, in our formalization, no substitution is possible between inputs, the survival of the economy in disequilibrium requires a not-too-tough price mechanism. By contrast with the Walrasian tâtonnement, the one we have retained, even if its formulation is too simple and other market mechanisms are conceivable, allows for the determination of effective prices and quantities outside equilibrium. A natural extension of the model would consist in examining if the introduction of money into the market mechanism would help to improve the economic performances.

Appendix I: Local fluctuations

In a neighbourhood of the equilibrium, the local fluctuations are defined by the dynamic equation (13) where the scalar θ depends on the ratio between the first and the second eigenvalues of matrix A by formula (12). The solution of the equation (13) being a linear combination of $e^{i\theta}$ and $e^{-i\theta}$, the movement is periodic of period

$$T = \frac{2\pi}{\theta} \tag{20}$$

The amplitude A is obtained as the difference between the extreme abscissas of the points of the ellipse (these points are those for which the tangent is vertical). Calculation gives:

$$A = \frac{2\delta_0}{\sin\theta} \tag{21}$$

The case of uniform composition of capital serves as a benchmark. If the inputs of an industry are mainly provided by the other industry ($a_{11}a_{22} < a_{12}a_{21}$), then $\det \mathbf{A}$ is negative,

$\alpha_2 = \det \mathbf{A} / \alpha_1$ is negative and θ defined by (12) is greater than $\pi / 3$: then the period is smaller than 6 and the amplitude is small. In intuitive terms, when the interindustrial relationships are strong, the transmission of fluctuations is rapid ($T < 6$) and the overall fluctuations small. On the contrary, when the interdependencies are weak ($\det \mathbf{A} > 0, \alpha_2 > 0, \theta < \pi / 3$), the period is long and the amplitude large: the economy drifts far from its regular state, comes back slowly and drifts again in the opposite direction.

Appendix II: Global fluctuations

As an application of the temporal symmetry established in section III.2.2, consider an invariant expression $I(X_t, X_{t-1})$, if any. For $t > 0$, the points (X_{t-1}, X_t) move on the curve $I(X_{t-1}, X_t) = I_0$ but, since they also represent the past of the sequence which starts at (X_1, X_0), they also move on a curve of equation $I(X_t, X_{t-1}) = C$, where C is a constant. This suggests that the invariant is symmetric: $I(X_{t-1}, X_t) = I(X_t, X_{t-1})$.

An idea to prove the existence of a cycle would be to generalize the local calculation and look for an invariant of order n, for any n. That is, we set $X_t = q_t - \bar{q}$ and consider a polynomial of the type $P_n(X,Y) = \sum b_{ij} X^i Y^j$ ($i + j \leq n$). The polynomial is an invariant of order n if, when one replaces the value X_t in $P_n(X_t, X_{t-1})$ by its value as a function of X_{t-1} and X_{t-2}, as derived from the equality (11) with $X_t = q_t - \bar{q}$, the result obtained coincides with $P_n(X_{t-2}, X_{t-1})$ up to the order n. The identification of the two polynomials leads to write down n^2 linear equations with the n^2 unknowns b_{ij}. These coefficients may be determined by induction on *n*. As a consequence of the temporal symmetry of the dynamical equation, one can set $b_{ij} = b_{ji}$. There are two difficulties with that method: first, the linear system defining the b_{ij} coefficients is not always of the Cramer type; second, once a sequence $P_n(X,Y)$ of invariants has been identified, its convergence towards some $P_\infty(X,Y) = I(X,Y)$ must be established to conclude that the point $M_t = (X_{t-1}, X_t)$ moves on the curve $I(X,Y) = I_0$.

Imagine that an invariant function $I(.,.)$ has been found, from which the equation of the curve $I(X_{t-1}, X_t) = I_0$ follows. Let us associate with a starting point (X_0, X_1) any magnitude such as the average value of the whole sequence X_t (or its minimum value, its period, etc). Such a magnitude $\overline{X}$ remains the same if the starting point was (X_{n-1}, X_n), and therefore $\overline{X}$ is itself invariant on the curve. This means that the average value $\overline{X}$ depends on the (X_0, X_1) only through the value $I(X_0, X_1)$. In that sense, the expression I can be viewed as a 'universal formula'. Expressed the other way round, the property becomes: the knowledge of the function f which gives the centre of the curve as a function of its starting point ($\overline{X} = f(X_0, X_1)$) suffices to know the equation of the curve itself, which is $f(X_{t-1}, X_t) = f(X_0, X_1)$.

A case in which a cyclical movement can be proved occurs when the starting point belongs to the 45° line $(X_0 = X_1)$ and the level of X_0 is chosen in order that two consecutive values X_t and X_{t+1} are also equal for some *t*. Then the reversibility property implies that *t* periods later the initial point will be reached again, therefore the dynamics admit period 2*t*.

In continuous time, let us write $x_{t+dt} = x_t + \dot{x}_t dt + \ddot{x}_t (dt)^2 / 2$, then $x_{t+dt} + x_{t-dt} = 2x_t + \ddot{x}_t (dt)^2$. The dynamic equation (11) can be transformed into an equation of the type $x(t+dt) + x(t-dt) = f(x(t)) + 2x(t)$, where $x(t) = \ln q(t-1)$, $dt = 1$ and $f(x) = \ln\left(e^x \dfrac{a_{11}e^x + a_{21}}{a_{12}e^x + a_{22}} \right) - 2x$. That equation is written $\ddot{x} = f(x)$. Function *f* admits a unique root $\overline{x} = \ln \overline{q}$, is negative for $x > \overline{x}$, positive for $x < \overline{x}$, and $|f(x)|$ is not close to zero when *x* is great enough. Let us consider a solution with $x(t_0) < \overline{x}$ and let $t > t_0$. As long as condition $x(t) < \overline{x}$ holds, the function $x(t)$ is convex. If $\dot{x}(t_0) < 0$, the value of $\dot{x}(t)$ increases and, since $\ddot{x}$ is positive and not close to zero, $\dot{x}(t)$ vanishes at some date t_1 and then is positive at some $t_2 > t_1$. If $\dot{x}(t_0) > 0$ (otherwise, the conditions $x(t) < \overline{x}$ and $\dot{x}(t) > 0$ are met at $t = t_2$) the first derivative is positive and increasing as long as $x(t) < \overline{x}$, therefore the function $x(t)$ reaches value $\overline{x}$ and goes beyond it. A point such that $x(t_3) > \overline{x}$ and $\dot{x}(t_3) > 0$ is reached. That position is qualitatively symmetric from the one considered in the first case we have examined and a similar argument applies: $x(t)$ increases, goes up to a local maximum, then decreases to $\bar{x}$ and below. The trajectory $x(t)$ thus admits incessant fluctuations around $\bar{x}$. Moreover, let us take the origin of time a date when a local extremum is reached. Then, if $x(t)$ is a solution of the differential equation $\ddot{x} = f(x)$ with the initial conditions $x(0)$ and $\dot{x}(0) = 0$, the same for $x(-t)$, therefore $x(t) = x(-t)$. The graph of $x(t)$ is therefore symmetric with respect to any vertical axis corresponding to a local extremum. Since there are several local extrema and that two consecutive symmetries amount to a translation, the graph is also invariant by some translation. That translation defines a period of the solution $x(t)$. The overall conclusion is that $q(t)$ oscillates and admits a periodic movement around the long-run level $\bar{q}$.

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