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Abstract

The parametric Lemke algorithm finds an odd number of solutions to the linear complementarity problem LCP (q, M), for a matrix M with zero blocks on the diagonal and vector q within a certain domain. A criterion for monotonicity and uniqueness is given. The algorithm applies to the determination of a long-run equilibrium in the presence of scarce resources, and its first description can be traced back to the nineteenth century economist David Ricardo.

Keywords. Oddity, parametric Lemke algorithm, Ricardo, uniqueness

JEL classification. B12, C61, C63

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1 Introduction

For a given matrix M, the parametric Lemke algorithm aims to solve the linear complementarity problem LCP(q_*, M) by transferring a known solution of LCP$(q(0), M)$ along a curve joining vector $q(0)$ to vector q_*. Most of the time, in a neighborhood of a point $q(t)$, the transfer involves a smooth adaptation of the solution obtained at $q(t)$; from time to time, some positive component of a solution vanishes and a more dramatic change is required to avoid it becomes negative. This change is mechanically determined by the algorithm, but it cannot be excluded that it leads to a U-turn on the path (antitone move), and then a second solution is obtained for some $q(t)$. Under which circumstances does the algorithm find a solution associated with q_*? Assume that the algorithm has the following properties: it works everywhere in a neighborhood of a point on the curve (U-turns being admitted), the solution for $q(0)$ is unique, there are finitely many solutions (if any) for any vector on the path and, finally, the algorithm never returns to a previously examined solution (that last property follows from a reversibility property of the algorithm). Then, starting from the unique solution attached to $q(0)$, transfers along the path define a unique sequence of transformed solutions, and the algorithm must eventually reach q_*: a solution of LCP(q_*, M) is obtained. If the curve is prolongated beyond q_* and goes to a point $q(1)$ for which the solution is also unique, it may be the case that other solutions are found at q_*, because U-turns may lead the algorithm to return to that point. However, since the trajectory starts from $q(0)$ and reaches $q(1)$, it goes an odd number of times to q_* (except if q_* is itself a point of U-turn), s times in the direct way from $q(0)$ to $q(1)$, $s-1$ in the opposite way. This suggests that the working of the parametric Lemke is intrinsically linked with an oddity property of the number of solutions. Uniqueness of the solution at any point of the path goes hand in hand with the absence of antitone move, and a global uniqueness result can be expected under some additional hypothesis.

These ideas are applied to a specific type of matrix M, with two blocks of zeroes on the diagonal. Section 2 describes the algorithm and Section 3 studies its properties when q_* belongs to some domain. Section 4 shows that the algorithm finds an odd number of solutions. Section 5 states a criterion for monotonicity, local and global uniqueness. The problem here examined stems from economic theory and can be traced back to the English economist David Ricardo (1772-1823), who studied how, in the presence of a scarce resource (land), a long-term equilibrium is affected by increasing demand. Ricardo may be considered as a precursor of the parametric Lemke algorithm (Section 6), which we therefore dub 'Ricardo-Lemke algorithm'.

Notations: For a real vector x, notation $x >> 0$ means that it is positive (all its components are positive), $x > 0$ that it is semipositive (x nonnegative and nonzero) and $x \geq 0$ that it is nonnegative. The notation $x \geq 0 \quad [y]$ means x and y are nonnegative and componentwise complementary. For a vector x, x^T denotes the transpose of x, and the same for matrices.

2 The Ricardo-Lemke algorithm

Let matrices A and B be given $m \times n$ real matrices, while $q_1 \in R^n$ and $q_2 \in R^m$ are given vectors. For $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and $M = \begin{bmatrix} 0 & A^T \\ B & 0 \end{bmatrix}$, the problem LCP$(q, M)$ consists in finding nonnegative vectors $z_1 \in R^m$, $z_2 \in R^n$ such that

$$A^T z_2 + q_1 \geq 0 \quad [z_1] \tag{1}$$
$$B z_1 + q_2 \geq 0 \quad [z_2] \tag{2}$$

Let us assume:
(H_1) $A + B \geq 0$

Lemma 1 *Under assumption (H_1), the unique solution to (1)-(2) for $q >> 0$ is $z_1 = z_2 = 0$.*

Proof. Consider a solution (z_1, z_2) of LCP(q, M) with $q >> 0$. By summing up the equalities $0 = z_1^T A^T z_2 + z_1^T q_1$ and $0 = z_2^T B z_1 + z_2^T q_2$, one obtains $0 = z_2^T (A + B) z_1 + z_1^T q_1 + z_2^T q_2$, and assumption ($H_1$) implies $z_1 = z_2 = 0$. ■

It is convenient to transform inequalities (1)-(2) into a more tractable system. Let vectors

$$\overline{q}_2 = \begin{pmatrix} q_2 \\ 0 \end{pmatrix} \in R_+^{m+n}, \overline{z}_2 = \begin{pmatrix} z_2 \\ \omega \end{pmatrix} \in R_+^{m+n}$$

be obtained by extending the previous vectors q_2 and z_2 with n additional components ($\omega \geq 0$). Similarly, let I_n be the identity matrix of dimension n and let

$$\overline{A} = \begin{bmatrix} A \\ -I_n \end{bmatrix}, \overline{B} = \begin{bmatrix} B \\ I_n \end{bmatrix}$$

be extended matrices of dimension $(m+n) \times n$. Clearly enough, any solution (z_1, z_2) gives birth to a solution $(z_1, \overline{z}_2)$ of the problem

$$\overline{A}^T \overline{z}_2 + q_1 = 0 \quad (3)$$

$$\overline{B} z_1 + \overline{q}_2 \geq 0 \quad [\overline{z}_2] \quad (4)$$

where the last n components of vector $\overline{z}_2$ are $\omega = A^T z_2 + q_1 \geq 0$, and vice-versa.

The assumptions on matrices A and B are better expressed on their extensions $\overline{A}$ and $\overline{B}$. Consider a continuous curve (C) $q_1 = q_1(t)$ in R^n ($t \in [0,1]$) with unchanged q_2. The curve is oriented according to increasing values of t and is assumed to have the following properties (H_2) and (H_3). For any q_1 on (C) and any solution to (3)-(4):

(H_2) The equality $(\overline{B} z_1 + \overline{q}_2)_i = 0$ holds for at most n components; then the corresponding n rows of $\overline{A}$ are independent, and the same for $\overline{B}$.

(H_3) The curve (C) cuts the cones positively generated by $n-1$ rows of $\overline{A}$ at finitely many points, and no point of (C) belongs a cone generated by $n-2$ rows.

Conditions (H_2) and (H_3) avoid degeneracies in the working of the Ricardo-Lemke algorithm (or parametric Lemke algorithm) we now describe.

At a given step of the algorithm, let the starting point be a value $\widehat{t} \in [0,1]$ which is not one of the intersection points mentioned in (H_3), a direction of change of t (say, increasing values), and a known solution $(z_1(\widehat{t}), \overline{z}_2(\widehat{t}))$ of (3)-(4) for $q_1(\widehat{t}) \in$(C) such that $\overline{z}_2(\widehat{t})$ has exactly n positive components $I \subset \{1, ..., m+n\}$. By the complementarity condition (4) and (H_2), equality $(\overline{B} z_1 + \overline{q}_2)_i = 0$ holds exactly for these components $i \in I$, and the knowledge of the set I suffices to identify the solution $(z_1(\widehat{t}), \overline{z}_2(\widehat{t}))$. With $q_1(t)$ moving on the curve, the problem is to find a solution such that $\overline{z}_2(t)$ varies continuously. The algorithm describes the unique answer, if any, to that problem. For small variations of t around $\widehat{t}$, the second condition in (H_2) implies the existence of $\overline{z}_2(t) \geq 0$ such that equality $\overline{A}^T \overline{z}_2(t) + q_1(t) = 0$ holds, with the same positive components I as for $\overline{z}_2(\widehat{t})$. Then conditions (3) and (4) are met for an unchanged vector z_1, therefore the solution is extended by continuity to a neighborhood of $\widehat{t}$. The process works in an interval $[t_0, t_1]$ containing $\widehat{t}$ until t reaches a value t_1 ($t_1 > \widehat{t}$, given the above assumption that t is increasing) such that some positive component $j \in I$ of $\overline{z}_2(t_1)$ vanishes and would become negative for a higher value of t. The point $q_1(t_1) \in$(C) is one of the points referred to in condition (H_3), and that condition ensures that the components of $\overline{z}_2(t_1)$ other than j remain positive. The question is to extend the solution beyond that limit, with a continuous change in $\overline{z}_2(t)$. Then the complementarity conditions (4) imply that equalities $(\overline{B} z_1 + \overline{q}_2)_i = 0$ hold in a neighborhood of t_1 for the $n-1$ components $i \in I\setminus\{j\}$. According to the third condition in (H_2), the general solution $z_1(t)$ of these $n-1$ affine equalities is of the type $z_1(t) = z_1(\widehat{t}) + \lambda z_1'$ where $z_1(\widehat{t})$ is the previous solution, z_1' is a nonzero solution of $(\overline{B}z_1')_i = 0$ for $i \in I\setminus\{j\}$ and λ is an arbitrary scalar. z_1' is unique up to a factor that we choose in order that $(\overline{B}z_1')_j > 0$. Then inequality (4) for $z_1 = z_1(t)$ holds as an equality for any $i \in I\setminus\{j\}$, and as a strict inequality for component j if and only λ is positive.

Assume that the following condition (D) holds (otherwise, there is no solution of LCP$(q(t), M)$ with positive components in $I\setminus\{j\}$, and *a fortiori* no continuous extension of the previous solution):

(D) $\overline{B}z_1' \ngeq 0$

Condition (D) means that $\overline{b}_k z_1' < 0$ for some row k of $\overline{B}$ (then $k \notin I$). For any row of that type, the initial inequality $(\overline{B}z_1(\overline{t}) + \overline{q}_2)_k \geq 0$ is transformed into an equality for $z_1(t) = z_1(\overline{t}) + \lambda z_1'$ and some positive λ. Let us pick up the component k corresponding to the minimum positive value of λ for which that equality occurs (that minimum rule might alternatively be described in geometric terms as the choice of a new facet). For that minimum value, the vector inequality (4) holds with exactly n equalities for the components in the n-set $J = \{k\} \cup I\setminus\{j\}$. According to ($H_2$), there exists a vector $\overline{z}_2 = \overline{z}_{J2}(t)$, with null components outside J, such that equality (3) holds. At $t = t_1$, $\overline{z}_{J2}(t_1)$ is a strictly positive combination of the $n-1$ rows of $\overline{B}$ belonging to $I\setminus\{j\} = J\setminus\{k\}$. By the second condition in assumption (H_2), a unique decomposition ($\overline{a}_i$ is the ith row of $\overline{A}$)

$$-q_1(t) = \sum_{i \in J\setminus\{k\}} [z_{J2}(t)]_i \, \overline{a}_i + [z_{J2}(t)]_k \, \overline{a}_k \tag{5}$$

is obtained in a neighborhood of t_1, with $[z_{J2}(t)]_i$ close to $[z_{J2}(t_1)]_i$ and therefore positive, while $[z_{J2}(t)]_k$ is close to $[z_{J2}(t_1)]_k = 0$. According to (H_3), $[z_{J2}(t)]_k$ is nonzero for t close but different from t_1, therefore $[z_{I2}(t)]_k$ is positive either on an interval $]t_1, t_1 + \alpha[$ (monotone move) or on $]t_1, t_1 - \alpha[$ (antitone move). A solution of (3)-(4) is thus found on that half-interval. A new starting point $t_1 \pm \alpha$, a new orientation for t and a new set J of positive components for $\overline{z}_2(t_1 \pm \alpha)$ are thus defined, so that the algorithm works locally.

3 Properties of the algorithm

The oriented curve (C) from $q_1(0)$ to $q_1(1)$ being definitely given, consider a pair $(I, [t_0, t_1])$, where I is an n-subset of $\{1, ..., m+n\}$ sustaining a solution of LCP$(q(t), M)$ with positive components of $\overline{z}_2(t)$ in I for t in $[t_0, t_1]$, and $[t_0, t_1]$ is an oriented interval (t moves towards t_1). The knowledge of I suffices to identify the interval $[t_0, t_1]$ or $[t_1, t_0]$, not its orientation, as well as the associated solution of LCP$(q(t), M)$ for any t in that interval. The set S made of all pairs of that type is finite. If the algorithm works locally, the knowledge of $(I, [t_0, t_1])$ also suffices to identify the pair $(J, [t_1, t_2])$ which succeeds it, which is denoted $suc(I, [t_0, t_1])$ (however, the function $suc : S \to S$ is not defined if $t_1 = 0$ or 1, as the algorithm stops if either $q_1(0)$ or $q_1(1)$ is reached). The algorithm can therefore be represented as a finite directed graph whose nodes are the elements of S. The algorithm, which starts from a given solution $(I_0, [0, \tau])$ at $q_1(0)$, finds a solution at $q_1(1)$ if the successor of each node is well defined and if it admits no loops. Lemma 2 deals with the first condition, while the second condition results from the reversibility property (Lemma 3).

Let $\mathcal{D} \subset R^n$ be the open and convex set, which contains the strictly positive orthant, defined as

$$\mathcal{D} = \left\{ q_1; \exists y > 0 \quad q_1 >> B^T y \right\} \subset R^n \tag{6}$$

Lemma 2 means that the algorithm transfers a solution from interval $[t_0, t_1]$ to the next $[t_1, t_2]$.

Lemma 2 *Let the curve (C) lie in $\mathcal{D}$. Each element $(I, [t_0, t_1])$ of S admits a unique successor, except if $t_1 = 0$ or 1.*

Proof. From Section 2, we know that the successor of $(I, [t_0, t_1])$ is well defined if condition (D) holds, where $z_1^{'}$ is defined by the conditions $\overline{b}_i^T z_1^{'} = 0$ for $i \in I \backslash \{j\}$ and $\overline{b}_j^T z_1^{'} > 0$. If $z_1^{'}$ has a negative component, then the condition is met by the submatrix I_n of $\overline{B}$. Assume that $z_1^{'}$ is semipositive. It follows from equality (3) and hypothesis (H$_1$) that $z_1^{'T} q_1(t_1) = -z_1^{'T} \overline{A}^T \overline{z}_2(t_1) \leq z_1^{'T} \overline{B}^T \overline{z}_2(t_1)$, and the last term is zero because $\overline{B}^T \overline{z}_2(t_1)$ is a combination of the columns of matrix $\overline{B}$ belonging to the set $I \backslash \{j\}$, which are all orthogonal to $z_1^{'}$. According to definition (6), there exists $y > 0$ such that $z_1^{'T} B^T y < z_1^{'T} q_1(t_1) \leq 0$, therefore condition (D) is met. ∎

The reversibility property considers the effects of a reverse move of t:

Lemma 3 *For $t_1 \neq 0, 1$, let $suc(I, [t_0, t_1]) = (J, [t_1, t_2])$. Then $suc(J, [t_2, t_1]) = (I, [t_1, t_0])$.*

Proof. $(J, [t_1, t_2])$ being constructed as the successor of $(I, [t_0, t_1])$, let us now start from $(J, [t_2, t_1])$. The limit of the solution sustained by J is reached at value t_1 when the kth component of $\overline{z}_{J2}(t)$ vanishes. Let T be the n-set which succeeds J. According to the construction examined in Section 2, the new vector z_{T1} differs from z_{J1} by $\mu y_1^{'}$, where $y_1^{'}$ is a solution of the equations $(\overline{B}y_1^{'})_i = 0$ for $i \in J\backslash\{k\}$. As $J\backslash\{k\} = I\backslash\{j\}$, these equations are the same as those determining $z_1^{'}$ in the passage from $(I, [t_0, t_1])$ to $(J, [t_1, t_2])$, so that $y_1^{'}$ is proportional to $z_1^{'}$. We have already noticed that $(\overline{B}z_1^{'})_k = \overline{b}_k z_1^{'} < 0$, whereas $y_1^{'}$ satisfies inequality $(\overline{B}y_1^{'})_k > 0$ by construction. Therefore $y_1^{'} = -z_1^{'}$ up to a positive factor, and $z_{T1} = z_{J1} + \mu(-z_1^{'})$. The scalars λ and μ are both defined as the minimum value for which equality $(\overline{B}z_{J1} + \overline{q}_2)_i = 0$, which holds for the $n-1$ components in $J\backslash\{k\}$, also holds for one more component. Therefore $\lambda = \mu$, $T = I$, and the successor of $(J, [t_2, t_1])$ is $(I, [t_1, t_0])$. ■

Let *rev* be the function which associates with each pair $(I, [t_0, t_1])$ of S its reverse $(I, [t_1, t_0])$. Lemma 3 applied to $(I, [t_1, t_0])$ states that the function $suc \circ rev \circ suc \circ rev$ is the identity on S.

Lemma 4 *For $t_0 \neq 0$, each element $(I, [t_0, t_1])$ of S admits a unique predecessor.*

Proof. The function $pre = rev \circ suc \circ rev$ defines the unique predecessor of an element of S. ■

Lemma 5 *From a given starting point, the algorithm does not reach a node and its reverse.*

Proof. It the algorithm reaches $(I, [t_0, t_1])$ at step μ and $(I, [t_1, t_0])$ at step ν, with $\nu > \mu$, it reaches $suc(I, [t_0, t_1])$ at step $\mu + 1$ and $pre(I, [t_1, t_0])$ at step $\nu - 1$. By Lemma 3 these nodes are also reverse from each other but the gap $\nu - \mu$ is reduced by two. But $\nu - \mu$ cannot be reduced either to zero (this would mean $t_0 = t_1$) or to one (this would mean that $(I, [t_1, t_0])$ succeeds $(I, [t_0, t_1])$, when two consecutive sets differ by one element). ■

Lemma 6 *If the solution of LCP($q(0), M$) is unique, the algorithm never returns either in a neighborhood of $t = 0$ or on a previously examined solution at $q(t)$.*

Proof. Let $(I_0, [0, \tau])$ be the starting point in S. If the solution at $q(0)$ is unique and the algorithm returns at step σ in a neighborhood of $t = 0$, the set I_σ of positive components of $z_2(t)$ must be the same by the uniqueness of the solution, and the values of t are decreasing, therefore $(I_\sigma, [t_\sigma, t_{\sigma+1}]) =$

$rev(I_0, [0, \tau])$, which is excluded by Lemma 5. The same Lemma also implies that if the algorithm returns at steps μ and ν to a previously examined solution (then $I_\mu = I_\nu$), the direction must be the same in both cases, therefore $(I_\mu, [t_\mu, t_{\mu+1}]) = (I_\nu, [t_\nu, t_{\nu+1}])$. That identity also holds for their predecessors at steps $\mu - 1$ and $\nu - 1$, and a contradiction is obtained by considering the first pair (μ, ν) for which a return occurs. ■

4 Oddity property

The first part of Theorem 1 considers an oriented curve from $q_1(0) >> 0$ to q_{1*} (in the literature on the parametric LCP, it is generally the segment joining these points), the second part prolongates that curve, which returns into the positive orthant.

Theorem 1 *Let (C) $q_1 = q_1(t)$ be a curve in $\mathcal{D}$, joining from $q_1(0) >> 0$ to q_{1*}. Under assumptions (H_1),(H_2) and (H_3), the Ricardo-Lemke algorithm finds a solution of the linear complementarity problem (1)-(2) at (q_{1*}, q_2). If the curve (C) returns to $q_1(1) >> 0$, the algorithm finds an odd number of solutions (finitely many points on the curve apart), the solutions corresponding to a monotone move exceeding by one those corresponding to an antitone move.*

Proof. The algorithm admits a representation in terms of a finite directed graph, with the $(I_\sigma, [t_\sigma, t_{\sigma+1}])$'s as nodes. The starting point is unique (Lemma 1), each node has a unique successor (except if $t_{\sigma+1} = 1$) and a unique predecessor (except if $t_\sigma = 0$), and loops are excluded. Therefore, the trajectory starting from $q_1(0)$ must reach q_{1*} and defines a solution at that point. Assume moreover that the curve (C) continues and goes to $q_1(1) >> 0$. The unique solution at $q_1(0)$ is transferred along the path and transformed into one or more solutions at q_{1*}, because of possible antitone moves, then is transferred to $q_1(1)$. Among the solutions thus generated at q_{1*}, those corresponding to monotone moves exceed by one those corresponding to antitone moves, except if q_{1*} is a point of U-turn. ■

Theorem 2 *For q_{1*} in $\mathcal{D}$, the number of solutions of LCP(q_*, M) is generically odd.*

Proof. Take an arbitrary solution at q_{1*} as starting point, with an initial direction towards $q_1(1)$. If the trajectory (T) generated by the algorithm reaches either $q_1(0)$ or $q_1(1)$, it is a sub-trajectory of the unique trajectory joining $q_1(0)$ to $q_1(1)$. These solutions are those examined in Theorem 1 and their number is odd. It may also be the case that (T) never reaches $q_1(0)$ or $q_1(1)$ because the graph admits a loop. Then, as the successor and the predecessor of a node are unique, that disconnected part of the graph is itself is a loop. The trajectory oscillates around q_{1*}, with alternate monotone and antitone moves, and a new solution is found everytime it reaches q_{1*}, until it comes back to the solution one started with. The number of solutions thus generated at q_{1*} is even, with an equal number of monotone and antitone moves. If the starting point is anyone of these new solutions, the same set of solutions is found. A partition of those additional equilibria is thus obtained, with an even number of solutions in each subset. ■

5 Monotonicity and Uniqueness

Let $(J, [t_1, t_2])$ succeed $(I, [t_0, t_1])$.By assumption (H$_2$), the four $n \times n$ submatrices $\overline{A}_I$, $\overline{A}_J$, $\overline{B}_I$ and $\overline{B}_J$ made of the rows I and J of $\overline{A}$ and $\overline{B}$ are regular. The following criterion determines if the move of t is the same for the consecutive nodes (t is either increasing or decreasing in both cases) or if it changes (monotone and antitone moves, therefore the values of t overlap).

Theorem 3 *When J succeeds I, the direction of the move of t is unchanged if and only if $\det \overline{A}_I / \det \overline{B}_I$ and $\det \overline{A}_J / \det \overline{B}_J$ are the same sign. Otherwise, a U-turn on the curve (C) occurs.*

Proof. We have $J \backslash \{k\} = I \backslash \{j\}$. By (2), inequality $\overline{B} z_{I1} + \overline{q}_2 \geq 0$ holds for the vector z_{I1} associated with I, and since it holds as an equality for the I-components, assumption (H$_2$) implies that the inequality is strict for component k: $\beta_k = \overline{b}_k^T z_{I1} + \overline{q}_{2k} > 0$. As $\overline{b}_k^T z_{J1} + \overline{q}_{2k} = 0$, we have $\beta_k = \overline{b}_k^T (z_{I1} - z_{J1}) > 0$. Similarly, $\beta_j = \overline{b}_j^T (z_{J1} - z_{I1}) > 0$. Vector $z_{J1} - z_{I1}$ is orthogonal to the row-vector $\beta_k \overline{b}_j^T + \beta_j \overline{b}_k^T$. By construction, $z_{J1} - z_{I1} = \lambda z_1'$ is also orthogonal to the $n-1$ rows of $\overline{B}$ belonging to $I \backslash \{j\} = J \backslash \{k\}$. Thus matrix $\beta_k \overline{B}_J + \beta_j \overline{B}_K$ transforms $z_{J1} - z_{I1}$ into the null vector, therefore $\beta_k \det \overline{B}_I + \beta_j \det \overline{B}_J = 0$ and $\det \overline{B}_I$ and $\det \overline{B}_J$ have opposite signs.

For t in the interval associated with I and close to t_1, equality $\overline{A}_I^T \overline{z}_{I2}(t) + q_1(t) = 0$ holds with $\overline{z}_{I2}(t)$ positive, but the jth component of $\overline{z}_{I2}(t)$ vanishes at $t = t_1$ and would become negative beyond that limit. When row j is replaced by another row k, a second algebraic decomposition of vector $-q_1(t)$

is obtained:

$$-q_1(t) = \sum_{i \in I\setminus\{j\}} [z_{I2}(t)]_i \, \overline{a}_i + [z_{I2}(t)]_j \, \overline{a}_j = \sum_{i \in J\setminus\{k\}} [z_{J2}(t)]_i \, \overline{a}_i + [z_{J2}(t)]_k \, \overline{a}_k \quad (7)$$

At $t = t_1$ both decompositions coincide, therefore $[z_{J2}(t)]_i$, which is equal to $[z_{I2}(t_1)]_i > 0$ at $t = t_1$, remains positive in a neighborhood. $[z_{J2}(t)]_k$ is zero at $t = t_1$. The set J which succeeds I makes no U-turn if and only if $[z_{J2}(t)]_k$ is positive whereas $[z_{I2}(t)]_j$ has become negative. Equality (7) shows that the vectors $\overline{a}_i$ for $i \in I\setminus\{j\} = J\setminus\{k\}$ and vector $[z_{I2}(t)]_j \, \overline{a}_j - [z_{J2}(t)]_k \, \overline{a}_k$ are linearly dependent, therefore $\det([z_{I2}(t)]_j \, \overline{A}_I - [z_{I2}(t)]_k \, \overline{A}_J) = 0$. $[z_{I2}(t)]_j$ and $[z_{I2}(t)]_k$ have opposite signs if and only if it is the case for $\det \overline{A}_I$ and $\det \overline{A}_J$. This amounts to saying that $\det \overline{A}_I / \det \overline{B}_I$ and $\det \overline{A}_J / \det \overline{B}_J$ have the same sign. ■

Corollary 1 *Let I be the set of positive components of $\overline{z}_2(t)$. The move is monotone or antitone according as* $\det \overline{A}_I / \det(-\overline{B}_I)$ *is positive or negative.*

Proof. In a neighborhood of the starting point $q_1(0) >> 0$ the move is monotone, $\overline{A}_I$ and $-\overline{B}_I$ are the identity matrix and the ratio of their determinants is positive. In the next steps, a change of orientation coincides with a change in the sign of $\det \overline{A}_I / \det(-\overline{B}_I)$. ■

Corollary 2 *For q_{1*} in $\mathcal{D}$, and flukes appart, the number of solutions of LCP(q_*, M) for which* $\det \overline{A}_I / \det(-\overline{B}_I)$ *is positive exceeds by one that for which it is negative.*

Proof. By Theorem 1 and Corollary 1, the solutions which are reached by the algorithm have the property mentioned in the Corollary. The additional solutions studied in the proof of Theorem 2 have as many monotone as antitone moves. ■

Corollary 3 *For q_{1*} in $\mathcal{D}$, and flukes apart, if* $\det \overline{A}_I / \det(-\overline{B}_I)$ *is positive at any associated equilibrium, that equilibrium is unique.*

Proof. This follows from Corollary 2. ■

If $\det \overline{A}_I / \det(-\overline{B}_I)$ is everywhere positive in $\mathcal{D}$, the transformation along a path is monotone, and the unique solution in D is reached by the algorithm.

6 Ricardo as a precursor

David Ricardo (1817) studied the long-run dynamics of a capitalist economy and was especially interested in the evolution of the distribution of national income, which he thought to be unfavourable to capitalists because, when the demand for corn increases, landowners reap higher rents. Sraffa (1960) proposed a formalization of the notion of long-run equilibrium that post-Sraffian authors completed. Let there be n goods and lands, while labor is homogenous. A method of production i is described by a vector $e_i \in R_+^n$ of material inputs and lands, a quantity of labor input $q_{2i} \in R_+$ and a vector $f_i \in R_+^n$ of outputs. The net product of that method is $a_i = f_i - e_i$. When each method i operates at activity level z_{2i},vector $A^T z_2$ represents the net product of goods and the negative of the quantity of lands used in the economy. With vector q_1 representing the negative of an exogenously given final demand vector and the available quantities of lands, inequality (1) means that final demand and the scarcity constraints on lands are met. The complementary vector z_1 represents the prices of goods and the rents on lands: the price of a good in excess supply or the rent on a partially cultivated land is zero. Let r be a given, nonnegative and uniform rate of profit. With $b_i = (1+r)e_i - f_i$, inequality (2) means that no method yields more than the ruling rate of profits at the price-and-rent vector z_1 when the nominal wage is equal to unity by convention. The complementarity relationships (2) with the vector of activity levels z_2 mean that any operated method yields the normal rate of profit. Conditions (1) and (2) define respectively the quantity side and the value side of a long-term equilibrium. Condition (H_1) holds. If one introduces free disposal and fallowing as methods of production (these methods dispose at no cost of any surplus of goods and lands), the quantity side of an equilibrium is equivalently written as equality (3). In Chapter 2 of the *Principles* (1817), Ricardo studied the effects of an increase of demand, therefore of a change $q_1 = q_1(t)$ on a given equilibrium. He stressed that, most of the time, only activity levels $z_2(t)$ need to be adjusted, with no influence on the price-and-rent vector z_1. But, spasmodically, the adjustment is no longer possible, for instance because some quality of land becomes fully cultivated. Then the price of corn must rise, and the rents too. Even if Ricardo did not mention in Chapter 2 the effects on the other prices, he was fully aware that the rise in the price of corn triggers changes in those of all commodities whose production requires corn. The process suggested by Ricardo thus corresponds to the choice of a new price-and-rent vector as described in Section 2, the new operated method k being the first that yields the normal rate of profit r when the price of rice increases (minimum rule for λ). On the whole, that process may be identified with what is known today as the parametric Lemke algorithm, a variant of Lemke's (1965) initial algorithm. What Ricardo and his followers did not see, however, is that the method k selected on the basis of a profitability criterion may not be productive enough, i.e. they did not see the possibility of an antitone move.

For the type of matrix M here considered, an existence result for q_1 in $\mathcal{D}$ was established by Dantzig and Manne (1974). Their proof makes reference to another variant of the Lemke algorithm. Salvadori (1986) applied that result to the existence of a long-term equilibrium. Starting from geometrical considerations, Erreygers (1990, 1995) stated the local uniqueness condition on the signs of the determinants and assumed that the global uniqueness problem can be reduced to that of local uniqueness. None of these studies refers to the parametric Lemke algorithm. Corollary 2 generalizes Bidard and Erreygers's (1998) result on the oddity of the number of equilibria in the absence of lands (see also Lemke and Howson (1964) on oddity). Bidard (2012) criticized Ricardo's attempt to get rid of rent in his analysis.

From a formal point of view, the initial formalization (1)-(2) is symmetric in the indices 1 and 2 but the equivalent formulation (3)-(4) introduces an asymmetry. A dual treatment of the same problem is therefore possible, with no clear economic interpretation.

7 Conclusion

The algorithmic Lemke algorithm is usually used to find a solution of an LCP by transferring a given solution along a segment. The transfer along a curve which starts from and comes back into a domain in which the uniqueness of the solution is ensured suggests that the working of the algorithm is intrinsically connected with the possibility to assign an orientation to each of these equilibria, with one more solution with a direct orientation. An open problem connected with the possibility to find all solutions concerns the distinction between the equilibria which are reached by the parametric method along a curve from those which are not.

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