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Minimum Variance Portfolio Optimisation under Parameter Uncertainty: A Robust Control Approach

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The global minimum variance portfolio computed using the sample covariance matrix is known to be negatively affected by parameter uncertainty. Using a robust control approach, we introduce a portfolio rule for investors who wish to invest in the global minimum variance portfolio due to its strong historical track record but seek a rule that is robust to parameter uncertainty. Our robust portfolio theoretically corresponds to the global minimum variance portfolio in the worst-case scenario, with respect to a set of plausible alternative estimators of the covariance matrix, in the neighbourhood of the sample covariance matrix. Hence, it provides protection against errors in the reference sample covariance matrix. Monte Carlo simulations illustrate the dominance of the robust portfolio over its non-robust counterpart, in terms of portfolio stability, variance and risk-adjusted returns. Empirically, we compare the out-of-sample performance of the robust portfolio to various competing minimum variance portfolio rules in the literature. We observe that the robust portfolio often has lower turnover and variance and higher Sharpe ratios than the competing minimum variance portfolios.

Key words: Global minimum variance portfolio, Parameter uncertainty, Robust control approach, Robust portfolio.

1. Introduction

Modern portfolio theory suggests a positive relationship between risk and expected returns and that the market portfolio with non-diversifiable risk should generate the highest risk-adjusted returns. However, much empirical work reports underperformance of market capitalisation-weighted portfolios relative to certain popular minimum-risk investment strategies, such as the global minimum variance portfolio (Haugen and Baker, 1991; Chan et al., 1999; Jagannathan and Ma, 2003; Clarke et al., 2006; etc). This outcome can be explained by the historical long-term success of low-risk stock portfolios compared to high-risk stock portfolios. For instance, Baker et al. (2011) report that for the US investment universe restricted to the top 1000 stocks by market capitalisation, a dollar invested in the lowest-volatility portfolio in January 1968 had increased to \$10.12 in real terms by December 2008, while a dollar invested in the highest-volatility portfolio had declined to less than 10 cents in real terms over the same period. This striking result, which is well documented in the literature, is typically referred to as the "low-volatility anomaly." This anomaly appears to hold regardless of whether risk is defined as total volatility (Baker et al., 2011) or as idiosyncratic volatility (Ang et al., 2006, 2009) and is robust across markets and regions (Frazzini and Pederson, 2010; Blitz and Van Vliet, 2007). Some papers seek to provide rational explanations for the existence and the persistence of the low-volatility anomaly. The common theoretical approach has been to allow for the existence, independently of risk, of certain institutional and behavioural aspects of equity markets (e.g., benchmarking, preference for lotteries and representativeness) with dominant predictive power over returns, causing markets to be inefficient (see Baker et al., 2011 and reference therein).

Beyond these empirical considerations, the global minimum variance strategy is not feasible in practice as the optimal allocation depends on the covariance matrix of stock returns, which is not observable. The traditional plug-in approach consists of replacing the unknown covariance matrix with its empirical counterpart, *i.e.*, the sample covariance matrix. This estimator is the most efficient estimator, under the assumption that stock returns are independent and identically normally distributed. However, the plug-in approach is optimal only under the condition that the number n of available historical observations exceeds the size k of the stock universe. In relying on the sample covariance matrix, global minimum variance portfolio managers face parameter uncertainty when this condition is violated.

Because k is of the same order or even larger than n in real-world portfolio optimisation problems, efforts have been made in the literature to cope with the effect of parameter uncertainty in the computation of the global minimum variance portfolio. Common solutions include the use of factor models, the imposition of short-sale constraints (Jagannathan and Ma, 2003), the use of Bayesian shrinkage covariance matrices (Ledoit and Wolf, 2003, 2004), and the normed-constrained approach in DeMiguel et al. (2009b). These solutions have been proven to successfully reduce the amount of parameter uncertainty, thus improving portfolio stability and out-of-sample risk-adjusted returns.

The goal of this paper is to develop a method of mitigating the impact of parameter uncertainty on the performance of the global minimum variance portfolio. Formally, we introduce a robust strategy useful to an investor who (*i*) wants to invest in the global minimum variance portfolio due to its strong historical track record but (*ii*) who seeks a portfolio strategy that is robust to parameter uncertainty. Hence, our paper is related to the above-cited works, with the difference that we treat parameter uncertainty using an approach from the robust control literature in the field of optimisation under uncertainty (Ben-Tal and Nemirovski, 1998, 1999; El Ghaoui and Lebret, 1997; etc.). The idea behind robust control is to provide robust solutions to optimisation problems given uncertain data, where such solutions are guaranteed to be a good fit for all or most of the possible realisations of uncertainty in the data.

To reach our objective, we develop a methodology that can be divided into two steps. In the first step, we show that the optimal solution of the sample global minimum variance portfolio[1] can be reformulated as a least squares regression with uncertain data (with uncertainty in both the dependent and explanatory variables). The uncertainty in the least squares data is more significant as the sample covariance matrix is noisier. Hence, we obtain the robust global minimum variance portfolio in the second step by solving this least squares regression using the robust control approach introduced by El Ghaoui and Lebret (1997). This approach solves the least squares regression in the presence of uncertain data, providing the investor with protection against data uncertainty.

Our method of constructing the robust global minimum variance portfolio has several attractive features. First, by construction, this portfolio corresponds to the global minimum variance portfolio in the worst-case scenario with respect to the set of plausible alternative estimators of the covariance matrix in the neighbourhood of the sample covariance matrix. Hence, it provides protection against estimation errors in the sample covariance matrix. Second, the robust global minimum variance portfolio is a compromise between the sample global minimum variance portfolio that ignores uncertainty and the equally weighted portfolio that is free of parameter uncertainty. More precisely, the optimal portfolio corresponds to the sample global minimum variance portfolio when the level of parameter uncertainty is zero, and it converges to the equally weighted portfolio when the level of parameter uncertainty is high. We provide a data-dependent solution for the choice of the uncertainty level. Third, using simulated and real stock return data, we show that our robust portfolio dominates the sample global minimum variance portfolio as well as various competitive minimum variance portfolio strategies in the literature. This dominance is consistent across different data sets and evaluation criteria, including out-of-sample portfolio turnover, variance and Sharpe ratio.

From a theoretical point of view, our article is related to several papers in the literature on portfolio selection that are robust to parameter uncertainty or that incorporate aversion to parameter uncertainty (ambiguity). These papers, which draw on the multi-prior approach of Gilboa and Schmeidler (1989), include Goldfarb and Iyengar (2003), Tütüncu and Koenig (2004) and Garlappi et al. (2007), to cite a few. They adopt a max-min approach, maximising investors wealth in the worst case scenario with respect to the unknown parameters. Goldfarb and Iyengar (2003) and Tütüncu and Koenig (2004) introduce optimisation algorithms to solve max-min saddle-point problems and apply them to the classical mean-variance problem of Markowitz (1952). Garlappi et al. (2007) modify the classical mean-variance rule in such a way that the optimal portfolio is chosen so as to maximise its expected utility under a worst-case scenario, where the worst case is based

[1] By "sample global minimum variance portfolio", we mean the global minimum variance portfolio computed using the sample or empirical covariance matrix.

on uncertainty about the unknown expected returns.[2] Our paper differs from these in the sense that we seek to make robust the global minimum variance portfolio instead of the mean-variance portfolio. Our main reason for focusing on the global minimum variance portfolio is the existence and persistence of the low-volatility anomaly, as described above, which explains the historical performance of this particular portfolio on the efficient frontier. Moreover, as many empirical works (Jagannathan and Ma, 2003; DeMiguel et al., 2009a, 2009b; etc.) emphasise the outperformance of the global minimum variance portfolio over the mean-variance portfolio of Markowitz (1952), robust versions of the former are likely to outperform robust versions of the latter.[3]

The remainder of the paper is organised as follows. In section 2, we provide an overview of the global minimum variance portfolio selection problem under parameter uncertainty. In section 3, we introduce our robust control method to address the issue of parameter uncertainty. Section 4 analyses the properties of the robust strategy via Monte Carlo simulations. Empirical applications are conducted in sections 5. The final section concludes the paper.

2. Global minimum variance portfolio with parameter uncertainty

Suppose that at each date t, there are k stocks in the investment universe. Let r_t be the vector of length k of excess (over the risk-free rate) returns on the k stocks. We consider a risk-averse investor, with a one-period investment horizon, who allocates his wealth to the k stocks by minimising the overall risk of the optimal portfolio, that is

$$\omega^* \ = \ \arg\min_{\omega} \omega' \Sigma \omega \quad \text{s.t.} \quad \omega' \vartheta = 1, \tag{1}$$

where ω is the vector of length k of portfolio weights, ϑ is a vector of length k with all entries equal to one, and Σ is the $k \times k$ covariance matrix of stock returns. The solution to the global minimum variance portfolio problem in (1) is well known and given by

$$\omega^* = \frac{\Sigma^{-1}\vartheta}{\vartheta' \Sigma^{-1} \vartheta}. \tag{2}$$

[2] For further contributions on robust portfolio optimisation, refer to Fabozzi et al. (2007a, 2007b), Fabozzi et al. (2010) and Kim et al. (2012). These contributions include robust formulations for the markowitz mean-variance model, as well as works on deriving robust counterparts for value-at-risk and conditional value-at-risk problems.

[3] Other papers in finance (Chen and Epstein, 2002; Epstein and Miao, 2003; Uppal and Wang, 2004; Maenhout, 2004; etc.) solve the (dynamic) portfolio choice problem under uncertainty aversion, using the robust control approach of Hansen and Sargent (2001), Anderson, Hansen and Sargent (2003). As these papers focus on the mean-variance model of Markowitz (1952), they also differ from our contribution. Moreover, they treat the problem of allocating wealth between risky assets and a risk-free asset rather than that of large-scale portfolio optimisation, which is the focus of our paper. This last remark also holds for another recent class of papers that address parameter uncertainty through learning (Epstein and Schneider, 2007; Miao, 2009).

An estimator of the covariance matrix Σ is needed to make the above solution operational. Under the assumption that stock returns are independent, identically distributed and have a multivariate normal density with mean μ and covariance matrix Σ, the sample covariance matrix defined as

$$\widehat{\Sigma}_{\mathrm{S}} = \frac{1}{n-1}\sum_{t=1}^{n}\left(r_t - \widehat{\mu}\right)\left(r_t - \widehat{\mu}\right)', \tag{3}$$

is the most efficient estimator, where n is the available sample size and $\widehat{\mu}$ is the sample mean. Hence, in practice, the global minimum variance strategy may be implemented, approximating the true covariance matrix using its sampling estimator and yielding

$$\widehat{\omega}^{*} \;=\; \arg\min_{\omega}\, \omega'\widehat{\Sigma}_{\mathrm{S}}\omega \quad \text{s.t.} \quad \omega'\vartheta = 1, \tag{4}$$

with the following allocation:

$$\widehat{\omega}^{*} = \frac{\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}{\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}. \tag{5}$$

Nevertheless, as nicely summarised by Ledoit and Wolf (2004), no one should use the sample covariance matrix for the purpose of portfolio optimisation because it contains parameter uncertainty of a kind likely to perturb a mean-variance optimiser. Parameter uncertainty occurs when the size of the investment universe k is the same as or larger than the number n of observations available. Indeed, it is well known that when the ratio k/n is not close to zero, the eigenvalues of the sample covariance matrix will be more dispersed than the true unobservable eigenvalues (Marcenko and Pastur, 1967), and the eigenvectors will not be consistent (Johnstone and Lu, 2009).

To gain more insight into the impact of dimensionality on the statistical properties of the sample covariance matrix, let us consider a simple simulation experiment. We simulate a time series of length n of k monthly stock returns using a 1-factor model:

$$r_t = \beta f_t + \varepsilon_t, \;\; t = 1, ..., n, \tag{6}$$

where r_t is the vector of length k of excess returns, f_t is excess returns on the market factor, β is the vector of length k of the market factor loadings, and ε_t is the vector of length k of residuals. To make our simulation realistic in the context of portfolio selection, we follow MacKinlay and Pastor (2000), DeMiguel et al. (2009a) and Tu and Zhou (2011) in assuming an annual excess return of 8% and an annual standard deviation of 16% on the market factor. The factor loadings are evenly spread between 0.5 and 1.5. The residuals ε_t are drawn from a multivariate normal distribution with mean zero and a covariance matrix that is assumed to be diagonal, with the diagonal elements drawn from a uniform distribution with support $[0.10, 0.30]$. With the $n \times k$ matrix of simulated stock returns, we compute the sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$ and measure parameter uncertainty

(PU) using the Frobenius norm of the difference between $\widehat{\Sigma}_{\mathrm{S}}$ and the true covariance matrix Σ from the factor model in (6):

$$\mathrm{PU} = \left\| \widehat{\Sigma}_{\mathrm{S}} - \Sigma \right\|_{\mathrm{F}}, \tag{7}$$

where the Frobenius norm of a matrix A is defined as $\|A\|_{\mathrm{F}} = \{\mathrm{tr}\,(AA')\}^{1/2}$, with tr(.) the trace operator.

Table 1 in Appendix B displays the mean of PU across 1,000 simulations, for different values of $k \in \{10, 50, 100\}$ and $n \in \{30, 60, 120, 360, 6000\}$. The results confirm that the sample covariance matrix is indeed noisy, with high levels of parameter uncertainty in a large k and small n setting. This property is known to affect the sample global minimum variance portfolio, leading to unstable portfolio weights. An illustration is given in Figures 1 and 2 (see Appendix B), which display the boxplots of the weights of the sample global minimum variance portfolio across simulations. To save space, we only present the boxplots for $k = 25$ and $n = 30$, 6000. We observe that the estimated portfolio weights are highly unstable for the smallest sample size ($n = 30$) and appear to be more stable for the largest sample size ($n = 6000$). Indeed, while the weights in Figure 1 range from -3.17 to 3.31, the weights in Figure 2 range from -0.38 to 0.73. Note that portfolio weight instability is generally accompanied by substantial losses in risk-adjusted returns (Jagannathan and Ma, 2003; Ledoit and Wolf, 2003, 2004; DeMiguel et al., 2009b; etc.).

In the sequel, we introduce a robust global minimum variance portfolio, using the framework of robust control regression with uncertain data (El Ghaoui and Lebret 1997). The new optimal portfolio accounts for parameter uncertainty because it is computed in the worst-case scenario with respect to the set of alternative estimators of the covariance matrix in the neighbourhood of the reference noisy sample covariance matrix. We will show how to identify this set of alternative plausible estimators and how the investor can build a global minimum variance portfolio that is robust to parameter uncertainty in the reference sample covariance matrix as well as in the alternative plausible covariance matrices.

3. A robust control approach to the global minimum variance portfolio

3.1. Problem formulation and optimal solution

Consider the problem (4) of minimising portfolio variance using the noisy sample covariance matrix, with solution

$$\widehat{\omega}^* = \frac{\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}{\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}. \tag{8}$$

Let Q be the $k \times (k-1)$ matrix with the following properties:

$$Q'\vartheta = 0 \quad \text{and} \quad Q'Q = I_{k-1}, \tag{9}$$

where again ϑ is a vector of length k with all elements equal to one, and I_{k-1} the identity matrix of dimension $k-1$. Following Van Trees (2002) and Sélen et al. (2008), the columns of the matrix Q are easily identified as the eigenvectors corresponding to the $k-1$ non-zero eigenvalues of the matrix Θ, defined as

$$\Theta = I_k - \frac{\vartheta\vartheta'}{k}. \tag{10}$$

The following proposition, which is at the core of our methodology, shows that $\widehat{\omega}^*$ in (8) can also be obtained using the classical least squares regression.

Proposition 1. *With Q as defined in (9), we have*

$$\widehat{\omega}^* = \frac{\vartheta}{k} - Q\widehat{\eta}^*, \tag{11}$$

where $\widehat{\eta}^*$ is the solution of the least squares regression

$$\widehat{\eta}^* \ = \ \arg\min_{\eta} \|y - X\eta\|_2\,, \tag{12}$$

with y and X equal to

$$X = \widehat{\Sigma}_{\mathrm{S}}^{1/2} Q \in \mathbb{R}^{k\times(k-1)}; \qquad y = \widehat{\Sigma}_{\mathrm{S}}^{1/2}\frac{\vartheta}{k} \in \mathbb{R}^{k}. \tag{13}$$

See Appendix A for the proof.[4] The proposition indicates that the solution $\widehat{\omega}^*$ for the sample global minimum variance portfolio can be computed using (11), where $\widehat{\eta}^*$ is the solution of the least squares regression in (12). Note that (11) guarantees that the stock weights sum to one. Indeed, we have

$$\widehat{\omega}^{*\prime}\vartheta = \left(\frac{\vartheta}{k} - Q\widehat{\eta}^*\right)'\vartheta = \frac{\vartheta'\vartheta}{k} - \widehat{\eta}^{*\prime}Q'\vartheta = 1. \tag{14}$$

The main point we wish to emphasise here is that both the dependent variable y and the independent variables X in the least squares regression model (12) depend on the noisy sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$, via the expressions in equation (13). Therefore, parameter uncertainty is incorporated into the regression variables y and X via the uncertain noisy sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$. Our objective in this paper is to use the literature of robust control least squares regression with uncertain data (Golub and Van Loan, 1980; El Ghaoui and Lebret, 1997; Chandrasekaran et al., 1998; Sayed et al., 2002; Calafiore and Dabbene, 2005) to solve the regression equation (12) to enable the investor to invest in a global minimum variance portfolio that is robust to parameter uncertainty.

[4] Note that the seminal paper of Britten-Jones (1999) shows similar results, but for the tangency portfolio. Our results differ from those of Britten-Jones (1999), as we focus on the global minimum variance portfolio. However, the literature offers other reformulations of the global minimum variance portfolio using least squares regressions (Kempf and Memmel, 2006; Candelon et al., 2012).

Because the inputs X and y in the least squares regression equation (12) are uncertain, it is obvious that one way to account for this uncertainty is to assume that this equation is not defined by a single pair (X, y) but by a family of matrices $(X', y\prime) = (X + \Delta X, y + \Delta y)$, with $\Delta = [\Delta X \ \ \Delta y]$ an unknown-but-bounded matrix of perturbations

$$\|\Delta\|_F = \|[\Delta X \ \ \Delta y]\|_F \leq \rho, \tag{15}$$

where $\rho \geq 0$ and $\|.\|_F$ denotes the Frobenius norm of a matrix. In other words, the noisy sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$, which contains some degree of parameter uncertainty, defines the inputs X and y of the regression equation (12) via the relations in (13), and these inputs are uncertain. We seek to incorporate this uncertainty, allowing for the existence of other pairs of inputs $(X', y\prime) = (X + \Delta X, y + \Delta y)$ close enough to the original pair (X, y), where ΔX and Δy are perturbations. The degree of closeness is measured by the parameter ρ, which represents the level of parameter uncertainty in the reference sample covariance matrix.

A popular alternative to the least squares method with uncertain inputs is total least squares (TLS), introduced by Golub and Van Loan (1980). TLS solves the following problem:

$$\Delta X^*, \Delta y^*, \widehat{\eta}^* \ = \ \underset{\Delta X, \Delta y, \eta}{\arg\min} \|(y + \Delta y) - (X + \Delta X)\,\eta\|_2 \,. \tag{16}$$

Thus, compared to the traditional least squares method in (12), TLS allows for perturbations in both the dependent and the explanatory variables. Nevertheless, in practical situations, TLS has some drawbacks that degrade its performance. For instance, it may unnecessarily overemphasise the effect of uncertainties and it does not take into account the issue of robustness, which is the main focus of this paper. Hence, we do not rely on the TLS method, but we instead rely on the robust control least squares regression of El Ghaoui and Lebret (1997), which treats the problem of robustness by minimising the worst-case residual. For a fixed value of the regression parameter η, the worst-case residual is defined as

$$r\,(X, y, \rho, \eta) \ = \max_{\|\Delta X, \Delta y\|_F \leq \rho} \|(y + \Delta y) - (X + \Delta X)\,\eta\|_2 \,. \tag{17}$$

The robust global minimum variance portfolio we introduce in this paper is obtained by minimising the worst-case residual $r\,(X, y, \rho, \eta)$ with respect to the regression parameter vector η.

Definition 1. Let $\overline{\omega}\,(\rho)$ be the robust global minimum variance portfolio weights. Thus, we have

$$\overline{\omega}\,(\rho) = \frac{\vartheta}{k} - Q\overline{\eta}\,(\rho)\,, \tag{18}$$

with $\overline{\eta}\,(\rho)$ defined by

$$\overline{\eta}\,(\rho) \ = \ \underset{\eta}{\arg\min} \max_{\|\Delta X, \Delta y\|_F \leq \rho} \|(y + \Delta y) - (X + \Delta X)\,\eta\|_2 \,. \tag{19}$$

Therefore, our methodology implies robustness in the sense that we solve the regression problem (12) in proposition 1 for the least favourable outcome with respect to the set of plausible inputs $(X', y\prime) = (X + \Delta X, y + \Delta y)$.

Note that our robust global minimum variance portfolio encompasses the sample global minimum variance portfolio, which ignores parameter uncertainty. Indeed, when the uncertainty parameter ρ is set to zero, the worst-case residual in (17) is equal to the usual least squares residual, and the robust solution degenerates to the usual solution, yielding

$$\overline{\omega}(0) = \frac{\vartheta}{k} - Q\overline{\eta}(0) = \frac{\vartheta}{k} - Q\widehat{\eta}^* = \widehat{\omega}^*. \tag{20}$$

In the more general case, where $\rho > 0$, the worst-case residual $r(X, y, \rho, \eta)$ can be rewritten (see El Ghaoui and Lebret 1997) as follows:

$$r(X, y, \rho, \eta) = \|y - X\eta\|_2 + \rho \left\| \begin{bmatrix} \eta \\ 1 \end{bmatrix} \right\|_2. \tag{21}$$

Hence, for a given value of the uncertainty parameter $\rho > 0$, the robust solution

$$\overline{\omega}(\rho) = \frac{\vartheta}{k} - Q\overline{\eta}(\rho) \tag{22}$$

is found by solving the program

$$\overline{\eta}(\rho) = \min_{\eta} \ \|y - X\eta\|_2 + \rho \left\| \begin{bmatrix} \eta \\ 1 \end{bmatrix} \right\|_2. \tag{23}$$

The limiting behaviour of the robust portfolio is summarised in the following proposition.

Proposition 2. *The robust global minimum variance portfolio, with weights $\overline{\omega}(\rho)$, converges to the equally weighted portfolio at very high levels of uncertainty, i.e.,*

$$\lim_{\rho \to \infty} \overline{\omega}(\rho) = \frac{\vartheta}{k}. \tag{24}$$

The proof is straightforward. Indeed, when ρ increases indefinitely, $\overline{\eta}(\rho)$ in (23) converges to the null vector and $\overline{\omega}(\rho)$ in (22) converges to the vector of length k, with all elements equal to $1/k$. This is a nice feature of our robust approach, as it suggests that an investor with very high uncertainty in the reference noisy sample covariance matrix invests in the equally weighted portfolio, which is free of parameter uncertainty. Moreover, the results in DeMiguel et al. (2009a), obtained both from simulated and real market data, show that the uncertainty-free, equally weighted portfolio is a competitive investment strategy, as many other investment rules cannot beat it.

Note that when k is much larger than n, the sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$ is usually singular. Hence, our robust methodology is not directly applicable because, in this case, the least squares input data X and y, defined in equation (13), cannot be computed. As a solution, one can regularise the sample covariance matrix using the following formula:

$$\widehat{\Omega}_{\mathrm{S}} = \widehat{\Sigma}_{\mathrm{S}} + \xi I_k, \tag{25}$$

where ξ is a relatively small positive scalar (*e.g.*, $\xi = 10^{-4}$). This has the effect of rendering the sample covariance matrix non-singular without significantly affecting its statistical properties. The data, X and y in this case, are defined as

$$X = \widehat{\Omega}_{\mathrm{S}}^{1/2} Q \in \mathbb{R}^{k\times(k-1)}; \qquad y = \widehat{\Omega}_{\mathrm{S}}^{1/2} \frac{\vartheta}{k} \in \mathbb{R}^{k}, \tag{26}$$

and the robust control least squares methodology can now be employed.

3.2. Choice of the uncertainty parameter ρ

A data-driven estimation of the uncertainty parameter ρ is required to make our robust global minimum variance portfolio operational. To estimate ρ we employ bootstrapping techniques. Our method consists in randomly resampling stock returns from the original data set, generating a sample S of B (for example, $B = 1,000$) different values of the sample covariance matrix

$$S = \left\{ \widehat{\Sigma}_{\mathrm{S,b}},\ b = 1, ..., B \right\}. \tag{27}$$

With the sample S, alternative values of the uncertain least squares regression inputs X and y in proposition 1 can be generated, yielding

$$S_{X,y} = \left\{ (X_b, y_b) : X_b = \widehat{\Sigma}_{\mathrm{S,b}}^{1/2} Q; \quad y_b = \widehat{\Sigma}_{\mathrm{S,b}}^{1/2} \frac{\vartheta}{k},\ b = 1, ..., B \right\}. \tag{28}$$

It follows that the set S_ρ defined as

$$S_\rho = \left\{ \rho_b : \rho_b = \left\| \Delta X_b\ \Delta y_b \right\|_F = \left\| X - X_b\ y - y_b \right\|_F,\ b = 1, ..., B \right\}, \tag{29}$$

contains reasonable values of ρ. Let $q_\xi(S_\rho)$ be an upper quantile of this set, with, for example, $\xi = 99\%$. The calibrated value of ρ is thus

$$\widehat{\rho} = q_\xi(S_\rho). \tag{30}$$

This method of calibrating the parameter ρ is fully data-adaptive. Indeed, when the original sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$ is not affected by parameter uncertainty, the resampled covariance matrices $\widehat{\Sigma}_{\mathrm{S,b}}$, $b = 1, ..., B$, should be close to $\widehat{\Sigma}_{\mathrm{S}}$. Therefore, the generated inputs X_b and y_b should also be close to X and y, respectively, with an estimated value of $\widehat{\rho}$ close to zero. In this configuration, our robust global minimum variance portfolio $\overline{\omega}(\widehat{\rho})$ should not diverge too much from the sample global minimum variance portfolio $\widehat{\omega}^*$. The converse case will arise for a very noisy sample covariance matrix ($\widehat{\rho} \to \infty$), with our robust global minimum variance portfolio converging to the equally weighted portfolio.

3.3. Relation to the 2-norm-constrained portfolio

Our robust control formulation of the global minimum variance portfolio can be interpreted as the 2-norm constrained portfolio, introduced by DeMiguel et al. (2009b). More precisely, the optimal solution in (23) can be alternatively written as

$$\begin{cases} \overline{\eta}(\delta) = & \arg\min_{\eta} \|y - X\eta\|_2 \\ \text{s.t.} \left\| \begin{bmatrix} \eta \\ 1 \end{bmatrix} \right\|_2 \leq \delta \end{cases} \tag{31}$$

with $\delta \in \mathbb{R}_+$ a constant inversely related to ρ. Because the constraint in (31) is equivalent to

$$\sqrt{\eta'\eta + 1} \leq \delta,$$

the robust portfolio can be obtained by solving the following optimisation problem

$$\begin{cases} \overline{\eta}(\delta) = & \arg\min_{\eta} \|y - X\eta\|_2 \\ \text{s.t.} \sqrt{\eta'\eta + 1} \leq \delta, \end{cases} \tag{32}$$

or equivalently

$$\begin{cases} \overline{\eta}(\delta) = & \arg\min_{\eta} \|y - X\eta\|_2 \\ \text{s.t.} \ \eta'\eta \leq \pi, \end{cases} \tag{33}$$

where the parameter π is equal to

$$\pi = \delta^2 - 1.$$

The optimisation problem in (33) shows that our robust global minimum variance portfolio is equivalent to the 2-norm constrained portfolio in DeMiguel et al. (2009b). These authors propose a general framework for finding minimum variance portfolios that perform well out-of-sample in the presence of parameter uncertainty. Their framework, as in (33), solves the global minimum-variance problem, subject to the additional constraint that the 2-norm of the portfolio-weight vector is smaller than a given threshold. Nevertheless, although our robust control approach leads to a solution equivalent to the 2-norm constraint portfolio, its advantage over the latter lies in the calibration of the tuning parameter ρ (or equivalently δ or π). Indeed, whereas DeMiguel et al. (2009b) suggest using cross-validation to calibrate the level of the tuning parameter, our robust control formulation, as shown in the above subsection, offers a very simple and intuitive way to calibrate this parameter endogenously, via bootstrapping techniques. By the term "endogenously", we mean that in our framework the tuning parameter ρ is estimated such that it matches the level of parameter uncertainty in the sample covariance matrix. Empirical applications conducted in Section 5 indeed demonstrate that our robust portfolio outperforms the 2-norm constrained portfolio in DeMiguel et al. (2009b).

4. Properties of the robust portfolio strategy

In this section, Monte Carlo simulations are conducted to measure to what extent the new robust global minimum variance strategy mitigates the impact of parameter uncertainty. In the evaluation of portfolio strategies, Monte Carlo simulations are useful, as they help to draw conclusions which are not affected by the existence of market anomalies like momentum, mean-reversion, calendar effects, small-firm effect, etc. Moreover, with Monte Carlo simulations, the vector of the true optimal portfolio weights is known exactly, and the error that arises from the implementation of a given portfolio strategy can be easily computed.

4.1. Design of the Monte Carlo simulations

Our Monte Carlo simulations design is similar to that in DeMiguel et al. (2009a) and Tu and Zhou (2011). We generate a data set of k monthly stock returns $r_t = (r_{1t}, r_{2t}, ..., r_{kt})'$, assuming a multivariate normal distribution and relying on a three-factor model

$$r_t = \beta f_t + \varepsilon_t, \tag{34}$$

where f_t is the 3×1 vector of excess returns on the three factors, β is the $k \times 3$ matrix of factor loadings, and ε_t is the $k \times 1$ vector of residuals. The first two moments (means and covariance matrix) of the factors are calibrated based on the monthly data from July 1963 to August 2007 for the market factor and for Fama-French's (1993) size and book-to-market portfolios. The stock factor loadings are randomly paired and evenly spread between 0.9 and 1.2 for the market β's, -0.3 and 1.4 for the size portfolio β's, and -0.5 and 0.9 for the book-to-market portfolio β's. The residual variance-covariance matrix is assumed to be diagonal, with the diagonal elements drawn from a uniform distribution with support $[0.10, 0.30]$, so that the cross-sectional average annual idiosyncratic volatility is 20%.

Using the three-factor model in (34), we simulate a sample of stock returns of dimension $N \times k$ where $N = 12,000$ $(1,000$ years) is the number of months and k is the number of stocks. With the simulated data, we use a rolling-window procedure to compare the sample global minimum variance strategy (with weights $\widehat{\omega}^*$) with its robust counterpart introduced in this paper (with weights $\overline{\omega}(\widehat{\rho})$). More precisely, the simulated returns for the first n months (where n is the estimation sample length) are used to compute $\widehat{\omega}^*$ and $\overline{\omega}(\widehat{\rho})$, which are considered to be competitive portfolios for the month $n+1$, and the two corresponding out-of-sample portfolio returns. This process is iterated by repeatedly moving the estimation window forward one month (including the data for a new month and dropping the data for the earliest month) until the last observation is reached. Note that at the end of this procedure, for each of the two global minimum variance strategies, we have computed $N-n$ portfolio weights $\widehat{\omega}_t^*$ and $\overline{\omega}_t(\widehat{\rho})$, $t = n, ..., N-1$, with corresponding out-of-sample returns $R_t = \widehat{\omega}_t^{*\prime} r_{t+1}$ and $\overline{R}_t(\widehat{\rho}) = \overline{\omega}_t'(\widehat{\rho})\, r_{t+1}$, where r_{t+1} is the $k \times 1$ vector of stock returns at time $t+1$.

4.2. Evaluation of the bootstrap procedure

Before comparing the two global minimum variance strategies, it is important to examine the above rolling-window procedure to determine whether the bootstrap method (see subsection 3.2) used to calibrate the level of ρ in our robust framework is relevant. Indeed, it is worth checking whether $\widehat{\rho}$, the estimator of the uncertainty coefficient ρ, increases as parameter uncertainty in the sample covariance matrix increases. To this end, Table 2 in Appendix B displays the mean of $\widehat{\rho}$ across the $N-n$ optimisations. We consider different configurations of $k \in \{10, 50, 100\}$, the number of stocks, and $n \in \{120, 240, 360, 6000\}$, the estimation sample length. The results reveal that for a given n, the calibrated level of parameter uncertainty $\widehat{\rho}$ increases, on average, with the number of stocks k. For example, with $n = 120$, the mean of $\widehat{\rho}$ is 0.0323, for $k = 10$, and 0.1466 and 0.2885, for $k = 50$ and $k = 100$ respectively. Moreover, when the size of the stock universe k is kept fixed, the mean of $\widehat{\rho}$ decreases, on average, with n and converges to zero for the largest estimation sample size ($n = 6000$). These results are as expected because, as stressed above, when the ratio k/n diverges from zero, parameter uncertainty in the reference sample covariance matrix is high. Hence, our bootstrap method used to estimate the uncertainty coefficient ρ is indeed fully data-adaptive.

4.3. Mean square errors of the estimated portfolio weights

The comparison of the new robust global minimum variance portfolio with its non-robust sample counterpart is now conducted from a statistical point of view using the mean square errors (MSE) of the estimated portfolio weights. For the two strategies, the MSEs are computed as follows:

$$\mathrm{MSE}\left(\omega, \widehat{\omega}^*\right) = \frac{1}{N-n} \sum_{t=n}^{N-1} \sum_{j=1}^{k} \left(\widehat{\omega}_t^{*(j)} - \omega_j \right)^2, \tag{35}$$

$$\mathrm{MSE}\left(\omega, \overline{\omega}\left(\widehat{\rho}\right)\right) = \frac{1}{N-n} \sum_{t=n}^{N-1} \sum_{j=1}^{k} \left(\overline{\omega}_t^{(j)}\left(\widehat{\rho}\right) - \omega_j \right)^2, \tag{36}$$

where ω is the vector of length k of the true global minimum variance portfolio from the specified three-factor model in (34). Table 3 in Appendix B displays, for different values of n and k, the MSE of the sample global minimum variance portfolio weights followed in brackets by the MSE of its robust analogue based on the robust control least squares approach. The results show that when the number k of stocks under consideration is not low relative to the number n of historical return observations available, such that the ratio k/n is not close to zero, the sample estimator $\widehat{\omega}^*$ of the global minimum variance portfolio weights ω is very noisy, leading to large mean square errors. The results also suggest that in these configurations, the robust global minimum variance portfolio strategy developed in this paper decreases the MSE of the estimated weights. For instance, with $(n, k) = (120, 100)$, the MSE of the sample estimator $\widehat{\omega}^*$ is 2.4865, whereas it is only 0.1839 for the robust estimator $\overline{\omega}\left(\widehat{\rho}\right)$. Note that for the largest sample size ($n = 6000$), the sample estimator $\widehat{\omega}^*$ becomes more precise and slightly outperforms its robust competitor. However, this configuration is not empirically relevant.

4.4. Portfolio turnover, variance and risk-adjusted returns

Economically, we compare the properties of the two global minimum variance portfolio strategies using three criteria: the out-of-sample portfolio turnover that provides insight into the temporal stability of each strategy, the out-of-sample variances and the Sharpe ratios. The out-of-sample Sharpe ratio $\widehat{SR}_{\widehat{\omega}^*}$ of the sample global minimum variance portfolio $\widehat{\omega}^*$ is defined as

$$\widehat{SR}_{\widehat{\omega}^*} = \frac{\widehat{\mu}_{\widehat{\omega}^*}}{\widehat{\sigma}_{\widehat{\omega}^*}}, \tag{37}$$

where $\widehat{\mu}_{\widehat{\omega}^*}$ and $\widehat{\sigma}^2_{\widehat{\omega}^*}$ estimate, respectively, the mean and the variance of R_t

$$\widehat{\mu}_{\widehat{\omega}^*} = \frac{1}{N-n} \sum_{t=n}^{N-1} R_t, \tag{38}$$

$$\widehat{\sigma}^2_{\widehat{\omega}^*} = \frac{1}{N-n-1} \sum_{t=n}^{N-1} \left(R_t - \widehat{\mu}_{\widehat{\omega}^*}\right)^2, \tag{39}$$

and the same statistic $\widehat{SR}_{\overline{\omega}(\widehat{\rho})}$ for the robust global minimum variance portfolio $\overline{\omega}\left(\widehat{\rho}\right)$ can be computed by replacing R_t by $\overline{R}_t\left(\widehat{\rho}\right)$ in equations (38-39). The portfolio turnover is defined as

$$\text{Turnover}_{\widehat{\omega}^*} = \frac{1}{N-n-1} \sum_{t=n}^{N-1} \sum_{j=1}^{k} \left(\left| \widehat{\omega}^{*(j)}_{t+1} - \widehat{\omega}^{*(j)}_{t+} \right| \right), \tag{40}$$

where $\widehat{\omega}^{*(j)}_t$ is the weight of stock j in the optimal portfolio based on $\widehat{\omega}^*_t$, $\widehat{\omega}^{*(j)}_{t+}$ is the weight of stock j before rebalancing at $t+1$, and $\widehat{\omega}^{*(j)}_{t+1}$ the desired weight of stock j at time $t+1$ (after rebalancing). The same expression in equation (40) is used to define the turnover of the robust portfolio, with weights $\overline{\omega}\left(\widehat{\rho}\right)$.

Table 4 in Appendix B displays the out-of-sample portfolio turnover for different values of n and k. For each pair (n, k), we first report the turnover of the sample global minimum variance portfolio, followed by the turnover of our robust global minimum variance portfolio in brackets. The results convincingly demonstrate that our robust estimation of the optimal global minimum variance portfolio succeeds in reducing the time instability of the estimated portfolio weights. For example, with $(n, k) = (120, 50)$, the turnover of the sample portfolio is 0.5903, while that of the robust portfolio is only 0.1442. However, we see that for a given value of k, the turnover of the sample global minimum variance portfolio decreases as the estimation sample length n increases, and the relevance of our robust strategy decreases asymptotically.

The out-of-sample variances of the two competitive global minimum variance portfolios are reported in Table 5. For each (n, k) pair, the out-of-sample variance of the sample global minimum variance portfolio is first reported, followed in brackets by the out-of-sample variance of the robust portfolio. The third reported value in parentheses is the p-value used to test for equality between the variances. We compute p-values using the robust test of variances comparison in Ledoit and Wolf (2011). Based on the results, it appears that our robust methodology does not cause the out-of-sample portfolio variances to increase significantly when the ratio k/n is not close to zero. In other words, the robust strategy is successful in reducing portfolio variances relative to the sample non-robust strategy, for medium and large portfolio selection problems ($k = 50, 100$) and for realistic small sample sizes ($n = 120, 240, 360$). Indeed, in these configurations, the reported p-values are less than 5%. Obviously, if large sample sizes are available ($n = 6000$), such that the sample covariance matrix is less noisy, our robust approach is not beneficial, because the portfolio variances would increase significantly. To illustrate, for $(n, k) = (100, 6000)$, the out-of-sample variances for the sample and robust portfolios are, respectively, 0.00121 and 0.00127, with the difference between the two values statistically significant at the 5% level.

Table 6 displays the out-of-sample Sharpe ratios of the two global minimum variance portfolios. The presentation is similar to Table 5, except that the differences in the Sharpe ratios are tested using the studentised circular block bootstrapping methodology of Ledoit and Wolf (2008). The overall picture from Table 6 is that our robust approach significantly improves the performance of the sample global minimum variance portfolio. For example, with $(n, k) = (100, 120)$, the monthly Sharpe ratio of the robust portfolio is 0.1597, while that of the sample portfolio is just 0.0450, less than a third of the value of the Sharpe ratio of the robust portfolio. Note that the superior performance of the robust portfolio holds asymptotically ($n \to \infty$). This result means that although the variance of the robust portfolio exceeds that of the sample portfolio for large sample size, this is not at the cost of risk-adjusted performance (Sharpe ratio) because of the positive effect of our robust approach on the out-of-sample portfolio returns.

To summarise, the Monte Carlo simulations show that for realistic values of the sample size n and number of stocks k, the robust global minimum variance portfolio strategy performs well in (i) stabilising portfolio turnover, (ii) reducing portfolio variances, and (iii) improving portfolio risk-adjusted returns, when parameter uncertainty is a concern.

5. Out-of-sample evaluation with real data

In this section, we stress the empirical relevance of the robust global minimum variance portfolio strategy using three different data sets. The first (respectively, second) data set includes monthly excess returns on the ten (respectively, forty-nine)-Industry portfolios representing the U.S. stock markets over the period from July 1963 to December 2010. The third data set contains monthly excess returns on the one hundred Fama and French portfolios of firms sorted by size and book-to-market. All data sets are extracted from Kenneth French's website[5] and are frequently used in

[5] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

empirical applications to compare alternative portfolio strategies (DeMiguel et al., 2009a, 2009b; Tu and Zhou, 2011; etc).

For each data set, we apply the rolling-window procedure described in the last section to compute the out-of-sample turnover of the sample global minimum variance portfolio and that of its robust version introduced in this paper.[6] The results are displayed in Panel A of Table 7. To provide practitioners with sufficient incentive to implement our robust portfolio strategy, in Panel B of Table 7, we report the out-of-sample turnover of various performing minimum variance portfolio strategies in the literature: the minimum variance portfolio with shortsale constraints (Jagannathan and Ma, 2003), the minimum variance portfolios based on shrinkage estimators of the covariance matrix (Ledoit and Wolf, 2003, 2004), and the 2-norm constrained portfolio in DeMiguel et al. (2009b). We also display the out-of-sample turnover of the equally weighted portfolio. We consider this last portfolio because it constitutes a relevant benchmark strategy that outperforms many other strategies in the presence of parameter uncertainty (DeMiguel et al., 2009a). Moreover, we have shown that our robust minimum variance portfolio degenerates to the equally weighted portfolio asymptotically ($\rho \rightarrow \infty$). Therefore, it is useful to compare our portfolio strategy to this target portfolio. Beyond the Monte Carlo simulation results in the last section, Table 7 provides additional proofs that our robust strategy dramatically improves the stability of optimised minimum variance portfolios. For example, with the 100FF data set, the sample global minimum variance portfolio has a turnover of 7.97, while the turnover of our robust global minimum variance portfolio using the same data set is only 0.15. This observation is consistent across the three data sets. Except for the equally weighted portfolio and the shortsale constrained strategy of Jagannathan and Ma (2003), the turnover of the robust portfolio is less than for the minimum variance portfolios in the literature. This result favours our robust methodology because high turnover is accompanied by high transaction costs, which can alter the overall return of a given portfolio strategy.

Table 8 reports the out-of-sample variances for the different global minimum variance portfolio strategies and the corresponding p-values indicating whether the portfolio variance for a given strategy is significantly different from that for the robust strategy. The p-values are computed using the robust test of variances comparison in Ledoit and Wolf (2011). In panel A of Table 8, we observe that the out-of sample variances of the robust portfolio are always lower than those of the sample portfolio, with the differences statistically significant for the 49-Industry and the 100FF data sets. Comparing the variances of the robust portfolio to those of various global minimum variance portfolios in the literature, Panel B of Table 8 shows that our robust strategy is a competitive

[6] In applying the rolling-window procedure, we set the estimation sample length to $n = 120$. The results for other values of n are quite similar and are available from the authors upon request.

alternative. For instance, with the 100FF data set, the robust portfolio always achieves out-of-sample variances lower than those of the minimum variance portfolios considered. The differences are statistically significant in the cases of the shortsale constraint minimum variance portfolio of Jagannathan and Ma (2003), the shrinkage minimum variance portfolio in Ledoit and Wolf (2004), and the equally weighted portfolio.

Lastly, we compare the out-of-sample Sharpe ratios of the competitive global minimum variance portfolios. To measure the impact of transactions costs on out-of-sample performance, we compute the Sharpe ratio net of transaction costs. For a given portfolio strategy with estimated weights $\widehat{\omega}$, the latter is defined as

$$\widehat{SR}_{\widehat{\omega}}^{net} = \frac{\widehat{\mu}_{\widehat{\omega}}^{net}}{\widehat{\sigma}_{\widehat{\omega}}^{net}}, \tag{41}$$

where $\widehat{\mu}_{\widehat{\omega}}^{net}$ and $\widehat{\sigma}_{\widehat{\omega}}^{net}$ measure the mean and the standard error, respectively, of the returns net of transaction costs (R_t^{net}) over the out-of-sample period:

$$\widehat{\mu}_{\widehat{\omega}}^{net} = \frac{1}{N-n} \sum_{t=n}^{N-1} R_t^{net}, \tag{42}$$

$$\left(\widehat{\sigma}_{\widehat{\omega}}^{net}\right)^2 = \frac{1}{N-n-1} \sum_{t=n}^{N-1} \left(R_t^{net} - \widehat{\mu}_{\widehat{\omega}}^{net}\right)^2. \tag{43}$$

The returns net of transaction costs are expressed as follows:

$$R_t^{net} = (1 + R_t) \left(1 - c \times \sum_{j=1}^{k} \left(\left|\widehat{\omega}_{t+1}^{(j)} - \widehat{\omega}_{t^+}^{(j)}\right|\right)\right) - 1, \tag{44}$$

where R_t is raw out-of-sample portfolio returns, and c is proportional transaction costs, which we set to 50 basis points per transaction, as in DeMiguel et al. (2009a).

Table 9 in Appendix B displays, for the three data sets, the out-of-sample Sharpe ratios net of transaction costs and in parentheses the corresponding p-values indicating whether the portfolio net Sharpe ratio for a given strategy is significantly different from that for the robust strategy. We obtain the p-values using the studentised circular block bootstrapping methodology in Ledoit and Wolf (2008), where the number of bootstrap replications and the size of each block are set, respectively, to $B = 1{,}000$ and $b = 5$. The results in Panel A show, once again, the relevance of our robust approach. For instance, with the 100FF data set, the sample global minimum variance portfolio has a negative net Sharpe ratio of -36%, which suggests that, by ignoring parameter uncertainty, investors lose money on a risk-adjusted basis. By contrast, the robust strategy leads to a positive net Sharpe ratio of 27%. In addition, Panel B of Table 9 shows that the robust portfolio often has statistically higher net Sharpe ratios than competing strategies in the literature. This picture is more pronounced for the two largest data sets, where the computed p-values are in almost all cases below 5%.

6. Conclusion

Because of the troublesome but persistent historical track records of low-volatility stocks, global minimum variance investing has gained repute in both the academic and financial spheres. Nevertheless, applying this investment strategy in practice requires estimating the covariance matrix of stock returns. The plug-in approach that consists in approximating the covariance matrix by its sampling counterpart is known to not be economically valuable. Indeed, parameter uncertainty, which affects the sample covariance matrix, leads to unstable portfolio weights and low risk-adjusted returns.

For an investor optimising a portfolio using the global minimum variance portfolio strategy, this paper provides a robust control approach to mitigate the impact of parameter uncertainty. Our robust portfolio provides protection against parameter uncertainty because it corresponds to the global minimum variance portfolio in the worst-case scenario with respect to the set of plausible alternative estimators of the covariance matrix, in the neighbourhood of the sample covariance matrix. The robust portfolio has the nice property that it is equivalent to the sample global minimum variance portfolio when the level of parameter uncertainty is zero, and it converges to the equally weighted portfolio under very high levels of uncertainty. For intermediate values of the uncertainty coefficient, the optimal portfolio shrinks the sample global minimum variance portfolio towards that of the equally weighted portfolio. We provide a data-adaptive method to calibrate the uncertainty coefficient.

Using Monte Carlo simulations, we show that, in the presence of parameter uncertainty, the robust strategy dominates the plug-in approach in term of portfolio weight stability, portfolio variance and risk-adjusted returns. We also empirically compare, across three data sets, the out-of-sample performance of our robust strategy to that of various competing minimum variance strategies in the literature. We find that the robust portfolio often has lower turnover and variances and higher Sharpe ratios than the competing minimum variance strategies.

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Appendix A: Proof of proposition 1

With Q as defined in (9), *i.e.*,

$$Q'\vartheta = 0 \quad \text{and} \quad Q'Q = I_{k-1}, \tag{45}$$

consider the reparameterization of the weights vector ω, *i.e.*,

$$\omega = \frac{\vartheta}{k} - Q\eta, \tag{46}$$

with η a vector of length $k-1$. This reparameterization does not impose any restrictions on ω other than $\omega'\vartheta = 1$. Given the reparameterization, the program of minimizing the portfolio variance with the sample covariance matrix $\widehat{\Sigma}_{\mathrm{S}}$

$$\widehat{\omega}^* \;=\; \underset{\omega}{\arg\min}\; \omega'\widehat{\Sigma}_{\mathrm{S}}\omega \quad \text{s.t.} \quad \omega'\vartheta = 1, \tag{47}$$

can be equivalently rewritten as

$$\begin{cases} \widehat{\omega}^* = \frac{\vartheta}{k} - Q\widehat{\eta}^* \\ \widehat{\eta}^* = \underset{\eta}{\arg\min}\; \left(\frac{\vartheta}{k} - Q\eta\right)' \widehat{\Sigma}_{\mathrm{S}} \left(\frac{\vartheta}{k} - Q\eta\right), \end{cases} \tag{48}$$

which gives after some simple algebraic calculus

$$\begin{cases} \widehat{\omega}^* = \frac{\vartheta}{k} - Q\widehat{\eta}^* \\ \widehat{\eta}^* = \underset{\eta}{\arg\min}\; \|y - X\eta\|_2, \end{cases} \tag{49}$$

with

$$X = \widehat{\Sigma}_{\mathrm{S}}^{1/2} Q; \quad y = \widehat{\Sigma}_{\mathrm{S}}^{1/2} \frac{\vartheta}{k}. \tag{50}$$

Note that $\widehat{\eta}^*$ in (49) is nothing but the solution of a least squares regression problem. Therefore, we have

$$\widehat{\eta}^* = (X'X)^{-1} X'y \tag{51}$$

$$= \left(Q'\widehat{\Sigma}_{\mathrm{S}}Q\right)^{-1} Q'\widehat{\Sigma}_{\mathrm{S}} \frac{\vartheta}{k}, \tag{52}$$

and $\widehat{\omega}^*$ in (49) takes the following expression

$$\widehat{\omega}^* = \frac{\vartheta}{k} - Q\left(Q'\widehat{\Sigma}_{\mathrm{S}}Q\right)^{-1} Q'\widehat{\Sigma}_{\mathrm{S}} \frac{\vartheta}{k}, \tag{53}$$

which can be simplified as follows:

$$\begin{aligned} \widehat{\omega}^* &= \left[I_k - Q\left(Q'\widehat{\Sigma}_{\mathrm{S}}Q\right)^{-1} Q'\widehat{\Sigma}_{\mathrm{S}}\right] \frac{\vartheta}{k} \\ &= \widehat{\Sigma}_{\mathrm{S}}^{-1/2} \underbrace{\left[I_k - \widehat{\Sigma}_{\mathrm{S}}^{1/2} Q \left(Q'\widehat{\Sigma}_{\mathrm{S}}Q\right)^{-1} Q'\widehat{\Sigma}_{\mathrm{S}}^{1/2}\right]}_{A} \widehat{\Sigma}_{\mathrm{S}}^{1/2} \frac{\vartheta}{k}. \end{aligned} \tag{54}$$

Remarks that the matrix A is the orthogonal projection matrix onto the complement of the column space of $\widehat{\Sigma}_{\mathrm{S}}^{1/2}Q$. Moreover, with the definition of the matrix Q, we have

$$Q'\vartheta = 0 \tag{55}$$

$$Q'\widehat{\Sigma}_{\mathrm{S}}^{1/2}\widehat{\Sigma}_{\mathrm{S}}^{-1/2}\vartheta = 0 \tag{56}$$

$$\left(\widehat{\Sigma}_{\mathrm{S}}^{1/2}Q\right)' \left(\widehat{\Sigma}_{\mathrm{S}}^{-1/2}\vartheta\right) = 0. \tag{57}$$

The last expression implies that $\widehat{\Sigma}_{\mathrm{S}}^{-1/2}\vartheta$ spans the complement of the column space of $\widehat{\Sigma}_{\mathrm{S}}^{1/2}Q$, and equation (54) becomes

$$\begin{aligned}
\widehat{\omega}^* &= \widehat{\Sigma}_{\mathrm{S}}^{-1/2}\underbrace{\left[\widehat{\Sigma}_{\mathrm{S}}^{-1/2}\vartheta\left(\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta\right)^{-1}\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1/2}\right]}_{A}\widehat{\Sigma}_{\mathrm{S}}^{1/2}\frac{\vartheta}{k} \\
&= \widehat{\Sigma}_{\mathrm{S}}^{-1/2}\frac{\widehat{\Sigma}_{\mathrm{S}}^{-1/2}\vartheta\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1/2}}{\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}\widehat{\Sigma}_{\mathrm{S}}^{1/2}\frac{\vartheta}{k} \\
&= \frac{\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}{\vartheta'\widehat{\Sigma}_{\mathrm{S}}^{-1}\vartheta}. \qquad (58)
\end{aligned}$$

The last equation corresponds to the familiar expression of the sample global minimum variance portfolio.

Appendix B: Tables and Figures

Figure 1 Boxplots of the weights of the sample global minimum variance portfolio: sample size $n = 30$

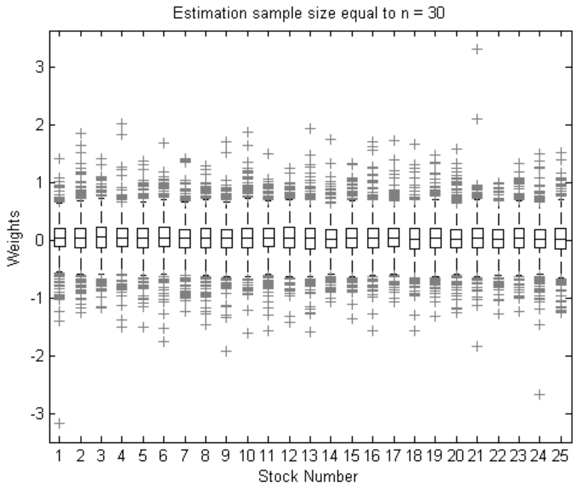

Figure 2 Boxplots of the weights of the sample global minimum variance portfolio: sample size $n = 6000$

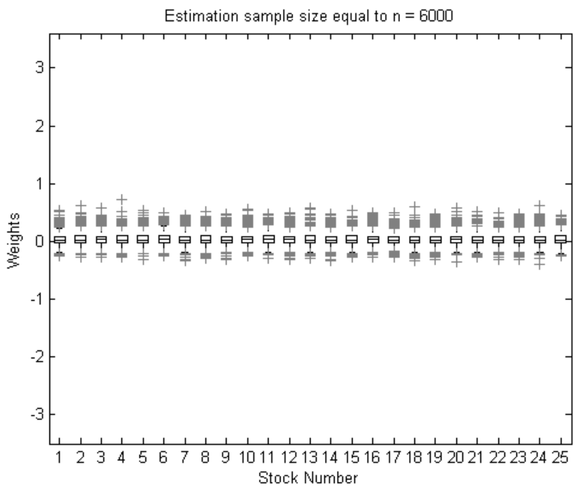

Table 1 Sample covariance matrix and parameter uncertainty

	$k=10$	$k=50$	$k=100$
$n=30$	0.0116	0.0570	0.1122
$n=60$	0.0084	0.0404	0.0807
$n=120$	0.0058	0.0286	0.0572
$n=360$	0.0034	0.0166	0.0333
$n=6000$	0.0008	0.0040	0.0081

Notes: For different values of k the number of stocks and n the estimation sample length, this table reports the mean across $1,000$ simulations of the estimated level of parameter uncertainty for the sample covariance matrix. The stock returns are simulated using a Gaussian one-factor model.

Table 2 Means for the estimated values of the level of parameter uncertainty $\widehat{\rho}$

	$k=10$	$k=50$	$k=100$
$n=120$	0.0323	0.1466	0.2885
$n=240$	0.0227	0.1024	0.1945
$n=360$	0.0185	0.0833	0.1580
$n=6000$	0.0045	0.0201	0.0382

Notes: For different values of k the number of stocks and n the estimation sample length, this table reports the means of the estimated (via bootstrap) values of the level of parameter uncertainty $\hat{\rho}$. The stock returns are simulated using a Gaussian three-factor model.

Table 3 Mean square errors of estimated portfolio weights

	$k=10$	$k=50$	$k=100$
$n=120$	0.0676	0.3976	2.4865
	[0.0378]	[0.1169]	[0.1839]
$n=240$	0.0319	0.1409	0.3469
	[0.0227]	[0.0930]	[0.1547]
$n=360$	0.0207	0.0855	0.1844
	[0.0165]	[0.0801]	[0.1376]
$n=6000$	0.0010	0.0048	0.0080
	[0.0011]	[0.0186]	[0.0335]

Notes: For different values of k the number of stocks and n the estimation sample length, this table reports the mean square errors of estimated portfolio weights for two different global minimum variance portfolios. For each couple (n, k), we first report the mean square error for the sample global minimum variance portfolio, followed in bracket by the mean square error for its robust counterpart obtained via the robust least squares approach. The stock returns are simulated using a Gaussian three-factor model.

Table 4 Portfolio turnovers with simulated data

	$k=10$	$k=50$	$k=100$
$n=120$	0.0985	0.5903	3.5900
	[0.0677]	[0.1442]	[0.1744]
$n=240$	0.0636	0.2648	0.6270
	[0.0518]	[0.1116]	[0.1441]
$n=360$	0.0540	0.1956	0.4057
	[0.0468]	[0.1006]	[0.1329]
$n=6000$	0.0417	0.1123	0.2030
	[0.0404]	[0.0937]	[0.1487]

Notes: For different values of k the number of stocks and n the estimation sample length, this table reports the monthly out-of-sample turnovers for two different estimators of the global minimum variance portfolio. For each couple (n, k), we first report the turnover of the sample global minimum variance portfolio, followed in bracket by the turnover of its robust counterpart obtained via the robust least squares approach. The stock returns are simulated using a Gaussian three-factor model.

Table 5 Portfolio variances with simulated data

	$k=10$	$k=50$	$k=100$
$n=120$	0.00224	0.00255	0.00741
	[0.00219]	[0.00180]	[0.00178]
	(0.00)	(0.00)	(0.00)
$n=240$	0.00214	0.00187	0.00206
	[0.00214]	[0.00171]	[0.00165]
	(0.13)	(0.00)	(0.00)
$n=360$	0.00211	0.00172	0.00167
	[0.00211]	[0.00167]	[0.00158]
	(0.96)	(0.00)	(0.00)
$n=6000$	0.00207	0.00149	0.00121
	[0.00207]	[0.00150]	[0.00127]
	(0.61)	(0.00)	(0.00)

Notes: For different values of k the number of stocks and n the estimation sample length, this table reports the monthly out-of-sample variances for two different estimators of the global minimum variance portfolio. For each couple (n, k), we first report the variance of the sample global minimum variance portfolio, followed in bracket by the variance of its robust counterpart obtained via the robust least squares approach. The third reported value (in parenthese) is the p-value of the difference between the two reported values. The difference is tested using the robust test of variances comparison in Ledoit and Wolf (2011). The stock returns are simulated using a Gaussian three-factor model.

Table 6 Portfolio Sharpe ratios with simulated data

	$k = 10$	$k = 50$	$k = 100$
$n = 120$	0.1163	0.1040	0.0450
	[0.1283]	[0.1472]	[0.1597]
	(0.00)	(0.00)	(0.00)
$n = 240$	0.1171	0.1217	0.1146
	[0.1269]	[0.1495]	[0.1648]
	(0.00)	(0.00)	(0.00)
$n = 360$	0.1169	0.1235	0.1243
	[0.1251]	[0.1476]	[0.1642]
	(0.00)	(0.00)	(0.00)
$n = 6000$	0.1277	0.1377	0.1425
	[0.1296]	[0.1480]	[0.1604]
	(0.00)	(0.00)	(0.00)

Notes: For different values of k the number of stocks and n the estimation sample length, this table reports the monthly out-of-sample Sharpe ratios for two different estimators of the global minimum variance portfolio. For each couple (n, k), we first report the Sharpe ratio of the sample global minimum variance portfolio, followed in bracket by the Sharpe ratio of its robust counterpart obtained via the robust least squares approach. The third reported value (in parenthese) is the p-value of the difference between the two reported values. The difference in Sharpe ratios is tested using the studentized circular block bootstrapping methodology in Ledoit and Wolf (2008). The stock returns are simulated using a Gaussian three-factor model.

Table 7 Portfolio turnovers with real data

Rules	10-Industry	49-Industry	100FF
	Panel A		
Sample	0.1619	0.8267	7.9729
Robust	0.0695	0.1427	0.1587
	Panel B		
Shortsale	0.0521	0.0756	0.1168
LW-CC	0.1025	0.3849	1.4466
LW-SI	0.1427	0.3856	1.2683
2-norm constrained	0.2366	0.4324	1.4925
Equally-weighted	0.0235	0.0326	0.0257

Notes: For each of the data set considered, this table reports the monthly out-of-sample turnovers for different estimators of the global minimum variance portfolio. The first panel displays the turnovers of the sample global minimum variance portfolio, followed by the turnovers of its robust counterpart obtained via the robust least squares approach. The second panel displays the turnovers of various estimators of the global minimum variance portfolio from the existing literature. The estimation sample length is set to 120. The data set 10-Industry (resp. 49-Industry) includes monthly returns for the ten (resp. forty-nine)-Industry portfolios representing the U.S. stock markets, over the period July 1963-December 2010. The data set 100-FF includes monthly returns (over the same period) for the one hundred Fama and French portfolios of firms sorted by size and book-to-market.

Table 8 Portfolio variances with real data

Rules	10-Industry	49-Industry	100FF
	Panel A		
Sample	0.00134 (0.56)	0.00188 (0.00)	0.00708 (0.00)
Robust	0.00132 (1.00)	0.00140 (1.00)	0.00165 (1.00)
	Panel B		
Shortsale	0.00137 (0.05)	0.00141 (0.84)	0.00209 (0.00)
LW-CC	0.00132 (0.87)	0.00145 (0.56)	0.00233 (0.00)
LW-SI	0.00132 (0.98)	0.00138 (0.81)	0.00186 (0.21)
2-norm constrained	0.00130 (0.34)	0.00136 (0.29)	0.00171 (0.70)
Equally-weighted	0.00195 (0.00)	0.00255 (0.00)	0.00280 (0.00)

Notes: For each of the data set considered, this table reports the monthly out-of-sample variances for different estimators of the global minimum variance portfolio. The first panel displays the variances of the sample global minimum variance portfolio, followed by the variances of its robust counterpart obtained via the robust least squares approach. The second panel displays the variances of various estimators of the global minimum variance portfolio from the existing literature. For a given estimator of the global minimum variance portfolio, the value in parenthese is the p-value that the portfolio variance is different from that of the robust portfolio. The p-values are computed using the robust test of variances comparison in Ledoit and Wolf (2011). The estimation sample length is set to 120. The data set 10-Industry (resp. 49-Industry) includes monthly returns for the ten (resp. forty-nine)-Industry portfolios representing the U.S. stock markets, over the period July 1963-December 2010. The data set 100-FF includes monthly returns (over the same period) for the one hundred Fama and French portfolios of firms sorted by size and book-to-market.

Table 9 Portfolio Sharpe ratios (net of transactions costs) with real data

Rules	10-Industry	49-Industry	100FF
	Panel A		
Sample	0.2578 (0.49)	0.0673 (0.00)	−0.3636 (0.00)
Robust	0.2686 (1.00)	0.2291 (1.00)	0.2759 (1.00)
	Panel B		
Shortsale	0.2591 (0.29)	0.2359 (0.75)	0.1939 (0.00)
LW-CC	0.2598 (0.51)	0.1468 (0.00)	0.1114 (0.00)
LW-SI	0.2625 (0.69)	0.1511 (0.00)	0.1165 (0.00)
2-norm constrained	0.2449 (0.01)	0.1753 (0.00)	0.1086 (0.00)
Equally-weighted	0.2276 (0.06)	0.2100 (0.48)	0.1935 (0.00)

Notes: For each of the data set considered, this table reports the monthly out-of-sample Sharpe ratios (net of transactions costs) for different estimators of the global minimum variance portfolio. The first panel displays the Sharpe ratios (net of transactions costs) of the sample global minimum variance portfolio, followed by the Sharpe ratios (net of transactions costs) of its robust counterpart obtained via the robust least squares approach. The second panel displays the Sharpe ratios (net of transactions costs) of various estimators of the global minimum variance portfolio from the existing literature. For a given estimator of the global minimum variance portfolio, the value in parenthese is the p-value that the portfolio Sharpe ratio (net of transactions costs) is different from that of the robust portfolio. The difference in Sharpe ratios is tested using the studentized circular block bootstrapping methodology in Ledoit and Wolf (2008). The estimation sample length is set to 120. The data set 10-Industry (resp. 49-Industry) includes monthly returns for the ten (resp. forty-nine)-Industry portfolios representing the U.S. stock markets, over the period July 1963-December 2010. The data set 100-FF includes monthly returns (over the same period) for the one hundred Fama and French portfolios of firms sorted by size and book-to-market.