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# Robust Portfolio Protection: A Scenarios-Based Approach

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## **Abstract:**

This paper constructs a robust optimization framework of the uncertain worst-case return. The model defines an adjustable discrete uncertainty set which controls the conservatism of the optimal asset allocation. Without prior assumptions on the data generating process, the model also develops an a priori probabilistic guarantee of the robust solution. Unlike previous measures that depend solely on the uncertainty model, the new measure is also sensitive to asset allocation and investment horizon. We provide an application of international stock indexes portfolio protection during the 2008 financial crisis. Computational experiments and ex-post analysis provide evidence for the effectiveness of our model.

**Keywords:** Portfolio protection, Robust optimization, Multivariate tail dependence, Non-parametric predictive inference.

**JEL Classification:** C14, D81, G11, G15

## 1. Introduction

Investment protection against extreme stocks price movement is an important issue of the asset allocation problem (Harlow, 1991 and Brogan and Stidham, 2005). The optimization of the downside risk generally relies on predefined probability distributions of returns or on estimations of their moments (Roy, 1952; Telser, 1955 and Katoaka, 1963). It goes without saying that small changes in these input parameters lead to non-efficient portfolios giving rise to the well-known problem of estimation risk.<sup>1</sup> In this regard, Michaud (1989) and Chopra and Ziemba (1993) document that portfolios obtained from sample mean and covariance matrix estimations show poor out-of-sample performance. DeMiguel, Garlappi, and Uppal (2009) cast further doubt on the relevance of the estimated mean-variance portfolios in comparison to their naive diversification counterparts. These weaknesses justify the need for new models that take into account data uncertainty.

The existing literature dealing with this issue can be categorized into two main groups.<sup>2</sup> The first one is related to the ambiguity concept developed by Ellsberg (1963), where probability distributions associated to chance constraints are unknown, but assumed to belong to some ambiguity set. Gilboa and Schmeidler (1989) consider ambiguity for the asset allocation problem by extending the classical utility theory to allow for multiple priors among the set of subjective probabilities. Maccheroni et al. (2006) present an alternative model (variational preferences), which relaxes the independence condition assumed for the minimax framework. Ben Tal et al. (2010) make two critiques to this approach. First, how should one specify the set of distributions? Second, why does one has to treat distributions within the set equally while ignoring those outside?

A second group of models substitutes the concept of ambiguity by the uncertainty set using limited amount of information on parameters. It merges the steps of estimating unknown parameters and finding a solution that remains feasible for any realization of the uncertain coefficients within predefined sets. Soyster (1973) proposes the first robust model which specifies intervals' bounds of uncertain parameters. By ignoring information on correlations, this model provides an over-conservative solution in that it gives up too much profitability to ensure robustness (Quaranta and Zaffaroni, 2008). To reduce the price of robustness, new models take account of asymmetries and dependencies among uncertain parameters. For instance, Bertsimas and Sim (2004) construct a model that controls the level of conservatism. Chen et al. (2007) propose a generalized framework, which captures the distributional asymmetry and preserves the convexity and the tractability of the initial optimization problem. In the same vein, Bertsimas and Sim (2004<sup>b</sup>) develop an affine model of random factors to describe correlations between the parameters. Miao et al. (2007) construct an autoregressive mobile average model (*ARMA*) to estimate the coefficients associated with the perturbation factors.

For all these models, there is a lack of empirical justification motivating the choice of uncertainty sets. Conventional approaches construct these sets around point estimates of uncertain parameters or use past realizations. Uncertain parameters may hence vary in continuous intervals or in convex sets (Ben-Tal et al, 2000; Goldfarb and Iyengar, 2003; Bertsimas and Sim, 2004; Chen et al, 2007 and Averbakh and Zhao, 2008). Uncertainty may also be described by scenarios (Mulvey et al, 1995; Kouvelis and Yu, 1997; Bertsimas and Thiele, 2006; Bertsimas and Brown, 2009; Natarajan et al, 2009 and Bertsimas et al., 2010).

A second important issue of the robust optimization theory is to identify a priori probabilistic guarantees of the robust solution since uncertainty set does not necessarily cover the whole uncertain space. Chen et al. (2007), Bertsimas and Sim (2004<sup>a</sup>) and Ben-Tal and Nemirovski

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<sup>1</sup> Other portfolio risk measures such as  $VaR$  and  $CVaR$  are also affected by this shortcoming (Fabozzi et al., 2010).

<sup>2</sup> Further robust methods such as robust estimators, resampling techniques, and Bayesian approaches were also developed to improve portfolio stability.

(2000) develop specific probability measures for interval-based model. To the best of our knowledge, this issue has not been addressed for discrete uncertainty model.

Our focus in this paper is on developing a deterministic robust optimization model to account for data uncertainty. We contribute to the literature of portfolio protection in the presence of estimation risk by proposing a scenario-based model related to the branch of literature originating from Soyster (1973). This approach controls the price of robustness by varying lower tail dependencies between returns. Our first goal is to study the interactions between performance and robustness of optimal portfolios for various uncertainty sets. As a second goal, we measure the a priori probabilistic guarantee of the robust portfolio using a nonparametric predictive inference technique. Unlike previous measures that depend fully on the uncertainty model, our measure is also sensitive to the optimal solution and the investment horizon. Such property is well suited when new deterministic constraints are added to the initial problem.

The remainder of the paper is organized as follows. Section 2 introduces the portfolio protection problem and discusses its robust counterpart for some common uncertainty models. Section 3 presents the new uncertainty model and formulates the problem. Section 4 discusses the probabilistic guarantee of the robust portfolio. Section 5 introduces the dataset and provides summary statistics. The results of computational experiments and *ex post* analysis are discussed in section 6. Finally, section 7 concludes.

## 2. Robust portfolio protection

Let us consider an asset allocation problem of  $J$  risky assets taking place at  $t=0$  and kept unchanged until the end of the investment horizon. The vectors of uncertain return and assets weights are denoted by  $\mathbf{r}=(\tilde{r}_1, \dots, \tilde{r}_J)^T$  and  $\mathbf{x}=(x_1, \dots, x_J)^T$ , respectively. The set  $\mathbf{X}=\{\mathbf{x} \in [0,1]^J / \mathbf{x}^T \mathbf{e} \leq 1\}$  refers to the impossibility of short selling and to the budget constraint. Let us denote by  $r_{wc}$  the worst-case return reached by the portfolio over the investment period. Here, the investor's first goal is to choose an asset allocation having the less conservative  $r_{wc}$  and with sufficient probabilistic guarantees not to fall under this threshold. Clearly, the more extreme is  $r_{wc}$ , the more likely the last condition will be fulfilled. The essence of the problem is thus the arbitrage between these two contradictory objectives. This setting can be seen as a special case of the Kataoka (1963) model, where the target return is endogenously determined and the probability to achieve this target is tending to one.

In practice, investors also attach importance to profitability. Ding and Zhang (2009) add a minimum expected return constraint to the Katoaka model, whereas Bienstock (2007) introduces a parameter( $\lambda$ ) that balances the two objectives. When the joint distribution of returns is known with certainty, one can express the problem using a chance constraint as:

$$\begin{aligned} \max_{\mathbf{x} \in \mathbf{X}, r_{\min}} \quad & \bar{\mathbf{r}}^T \mathbf{x} + \lambda r_{\min} \\ \text{s.t} \quad & \mathbf{P}(\tilde{\mathbf{r}}^T \mathbf{x} \leq r_{\min}) \leq \alpha \\ & \mathbf{A} \mathbf{x} \leq \mathbf{b} \end{aligned} \tag{1}$$

where  $\alpha$  is a predetermined critical level of not achieving the target,  $\bar{\mathbf{r}}$  is a statistical estimate of the expected returns and  $\lambda \in \mathbb{R}$  stands for the tradeoff parameter.<sup>3</sup> The deterministic constraints  $\mathbf{A} \mathbf{x} \leq \mathbf{b}$  may describe, for example, a maximum allocation per asset or per group of assets. However, it is still computationally challenging to the problem because of its NP-hardness. When

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<sup>3</sup> When  $\lambda = 0$ , the investor focuses only on the maximization of the expected return. On the other hand, when  $\lambda$  goes to infinity, the investor seeks to maximize the minimum guaranteed return.

the joint distribution is not exactly known, it is unlikely to protect the portfolio because of the unidentified extent of adverse variation of parameters. Robust programming also known as “min-max” or “worst-case” approach aims to determine a solution which is feasible for all possible instances of uncertain return. The robust problem has a simple deterministic formulation:

$$\begin{aligned} \max_{\mathbf{x} \in X} \quad & \bar{\mathbf{r}}^T \mathbf{x} + \left[ \min_{\mathbf{r}} \lambda(\mathbf{r}^T \mathbf{x}) \right] \quad \forall \mathbf{r} \in U \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \quad (2)$$

where  $U$  is the uncertainty set of returns.

The optimal solution of program (2) is called a robust solution obtained through the construction of the robust counterpart of (2). The complexity of such a step depends on the structure of the original program under consideration and the uncertainty model. In this study, we essentially focus on linear programs. Next, we analyze the robust counterpart of (2) for some common uncertainty models.

### 2.1 Hypercube uncertainty model

The hypercube uncertainty set proposed by Soyster (1973) is by far the simplest model to formulate. It assumes that individual return varies in a bounded interval independently from other returns. Let us assume that the true value  $r_j$  of an uncertain return is then given by:

$$r_j = \bar{r}_j + z_j \hat{r}_j \quad \forall j=1, \dots, n \quad (3)$$

where  $\bar{r}_j$  is a statistical estimate of the expected value of  $r_j$ ,  $\hat{r}_j$  is a statistical estimate of the maximum distance that  $r_j$  is expected to deviate from  $\bar{r}_j$  and  $z_j$  is a deviation factor which varies in the interval  $[-1, 1]$ . Program (2) can be written as:

$$\begin{aligned} \max_{\mathbf{x} \in X} \quad & \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left[ \min_{\bar{\mathbf{r}} - \hat{\mathbf{r}} \leq \mathbf{r} \leq \bar{\mathbf{r}} + \hat{\mathbf{r}}} (\mathbf{x}^T \mathbf{r}) \right] \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \quad (4)$$

The dual formulation of the inner program leads to the robust form of the program (2):<sup>4</sup>

$$\begin{aligned} \max_{\mathbf{x} \in X, \mathbf{u}, \mathbf{v}} \quad & \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left[ \mathbf{u}^T (\bar{\mathbf{r}} + \hat{\mathbf{r}}) - \mathbf{v}^T (\bar{\mathbf{r}} - \hat{\mathbf{r}}) \right] \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{u} - \mathbf{v} = \mathbf{x} \\ & \mathbf{u} \geq 0, \mathbf{v} \geq 0 \end{aligned} \quad (5)$$

by replacing  $\mathbf{u}$  with  $\mathbf{x}$  and  $\mathbf{l}$ , we obtain:

$$\begin{aligned} \max_{\mathbf{x} \in X, \mathbf{v}} \quad & \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left[ \mathbf{x}^T (\bar{\mathbf{r}} + \hat{\mathbf{r}}) + 2\mathbf{v}^T \hat{\mathbf{r}} \right] \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \delta + \mu_j \geq \hat{r}_j x_j, \forall j=1, \dots, J \\ & \delta \geq 0, \mu \geq 0 \end{aligned} \quad (6)$$

Although the Soyster's model admits the highest protection level, it is also the most conservative in the sense that the robust solution has the lowest function value. Applying this model is equivalent to choose an optimal portfolio by considering extreme realizations of returns.

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<sup>4</sup> To simplify the presentation, we limit ourselves to the most important attributes of the optimization programs. Dual transformation can be found in Gabrel and Murat (2010).

## 2.2 Budget of uncertainty model

Bertsimas and Sim (2004) propose a model that highlights the tradeoff between the robustness and performance of optimal solution. They assume the uncertainty set has a polyhedral shape. By introducing the parameter  $(\Gamma_0)$ , called the budget of uncertainty, they control the maximum number of uncertain parameters taking their worst realization at the same time. By using the deviation factors to describe uncertainty, one has to solve this program:

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left( \mathbf{x}^T \bar{\mathbf{r}} + \min_{\sum_{j=1}^J |z_j| \leq \Gamma_0, -1 \leq z_j \leq 1} \mathbf{z}^T (\hat{\mathbf{r}}\mathbf{x}) \right) \quad (7)$$

Since,  $\mathbf{x} \geq 0$  (i.e. absence of short selling), the deviation factors that deteriorate the objective function are such that  $\mathbf{z} \leq 0$ . Therefore, (7) can be rewritten as follows:

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left( \mathbf{x}^T \bar{\mathbf{r}} - \max_{\sum_{j=1}^J z_j \leq \Gamma_0, 0 \leq z_j \leq 1} \mathbf{z}^T (\hat{\mathbf{r}}\mathbf{x}) \right) \quad (8)$$

Using the dual transformation for the inner program, we obtain the robust version of (8):<sup>5</sup>

$$\begin{aligned} \max_{\mathbf{x} \in X, \delta, \boldsymbol{\mu}} \quad & \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left( \mathbf{x}^T \bar{\mathbf{r}} + \Gamma_0 \delta + \sum_{j=1}^J \mu_j \right) \\ \text{s.t} \quad & A\mathbf{x} \leq b \\ & \delta + \boldsymbol{\mu} \geq \hat{\mathbf{r}}\mathbf{x} \\ & \delta \geq 0, \boldsymbol{\mu} \geq 0 \end{aligned} \quad (9)$$

A key issue addressed by Bertsimas and Sim (2004) is the probability of non-violation of the protection level. They demonstrate that for a given uncertainty budget  $(\Gamma_0)$  and for independently and uniformly distributed deviation factors in the interval  $[-1,1]$ , this probability is at least equal to:

$$\Phi \left( \frac{\Gamma_0 - 1}{\sqrt{J}} \right) \quad (10)$$

where  $\Phi$  is the normal cumulative distribution function and  $J$  is the number of uncertain parameters.

## 2.3 Polytope uncertainty model

Polytope uncertainty model can be seen as a generalization of the model Bertsimas and Sim (2004) that defines a set of affine restrictions between deviation factors. The robust counterpart of (2) has the following form:

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left( \mathbf{x}^T \bar{\mathbf{r}} + \min_{\mathbf{F}|\mathbf{z}| \leq \mathbf{g}, -1 \leq z_j \leq 1} \mathbf{z}^T (\hat{\mathbf{r}}\mathbf{x}) \right) \quad (11)$$

where  $F$  is an  $(p \times n)$  matrix and  $g$  is an  $(p \times 1)$  vector. Since  $\mathbf{x} \geq 0$ , deviation factors that deteriorate the objective function are such that  $\mathbf{z} \leq 0$ . Therefore, one can rewrite (11) as:

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<sup>5</sup> Details on the construction of the robust version using the duality can be found in Gregory et al. (2011).

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left( \mathbf{x}^T \bar{\mathbf{r}} + \min_{\mathbf{F}\mathbf{z} \leq \mathbf{g}, 0 \leq z_j \leq 1} \mathbf{z}^T (\hat{\mathbf{r}}\mathbf{x}) \right) \quad (12)$$

The robust counterpart problem is then equivalent to:

$$\begin{aligned} \max_{\mathbf{x} \in X, \boldsymbol{\mu}, \boldsymbol{\delta}} \quad & \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left( \mathbf{x}^T \bar{\mathbf{r}} + \sum_{i=1}^p \delta_i g_i + \sum_{j=1}^J \mu_j \right) \\ \text{s.t.} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{F}'\boldsymbol{\delta} + \boldsymbol{\mu} \geq \hat{\mathbf{r}}\mathbf{x} \\ & \boldsymbol{\delta} \geq 0, \boldsymbol{\mu} \geq 0 \end{aligned} \quad (13)$$

To control the conservatism of the robust solution, one may add new linear restrictions to reduce the size of the uncertainty set and thus to limit the occurrence of worst-case returns. Conversely, reducing the existing restrictions may enlarge the range of extreme returns and leads to conservative portfolios.

#### 2.4 Discrete uncertainty model

Unlike continuous models, discrete uncertainty models assume that the available information is a finite set of scenarios (Kouvelis and Yu, 1997). Formally, let us define a set of scenarios  $\Omega = \{s_1, \dots, s_T\}$  associated with a multivariate realization of returns belonging to  $U = \{\mathbf{r}^{s_1}, \dots, \mathbf{r}^{s_T}\}$ . In this case, program (2) can be written as:

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left[ \min_{s_i \in \Omega} (\mathbf{x}^T \mathbf{r}^{s_i}) \right] \quad (14)$$

Let  $U' = \{\mathbf{z}^1, \dots, \mathbf{z}^T\}$  the uncertainty set of deviation factors. The formulation of the problem:

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left[ \mathbf{x}^T \bar{\mathbf{r}} + \min_{s_i \in \Omega} (\mathbf{z}^{s_i})^T (\hat{\mathbf{r}}\mathbf{x}) \right] \quad (15)$$

The robust counterpart formulation is obtained by adding as new constraints as the number of scenarios. The *minimax* program is then transformed into a one-level linear program of the form:

$$\begin{aligned} \max_{\mathbf{x} \in X, \theta} \quad & \mathbf{x}^T \bar{\mathbf{r}} + \lambda (\mathbf{x}^T \bar{\mathbf{r}} + \theta) \\ \text{s.t.} \quad & -\sum_{j=1}^J z_j^{s_t} \hat{r}_j x_j \leq -\theta, \quad t=1, \dots, T \\ & A\mathbf{x} \leq b \end{aligned} \quad (16)$$

**Proposition 1** *The robust portfolio obtained from a discrete uncertainty model (15) is a solution of the following convex program:*

$$\max_{\mathbf{x} \in X, A\mathbf{x} \leq b} \mathbf{x}^T \bar{\mathbf{r}} + \lambda \left[ \mathbf{x}^T \bar{\mathbf{r}} + \min_{\mathbf{z} \in \text{conv}(U')} \mathbf{z}^T (\hat{\mathbf{r}}\mathbf{x}) \right] \quad (17)$$

where  $\text{conv}(U')$  is the convex hull of the uncertainty set ( $U'$ ).

This proposition sets up the connection between discrete and interval-based uncertainty models.<sup>6</sup> In this regard, the convex hull is the smallest convex polytope that contains all scenarios and commonly defined as the intersection of affine half-spaces as in the program (11). Accordingly, an alternative way to solve program (14) is to identify the affine relationships between deviation factors by constructing the convex hull (Ben Tal et al, 2008).

<sup>6</sup> The proof of proposition 1 can be found in Bertsimas and Gupta (2011).

### 3. Problem formulation

#### 3.1 Extensions of the discrete uncertainty model

The main results of the robust optimization theory discussed so far are related to the mathematical link between discrete and polytope uncertainty models and to the effect of the size of uncertainty set on robustness. Clearly, discrete models based on historical observations seem to better describe the joint behavior of returns, even if they are not enough flexible to control the conservativeness of robust solutions. To circumvent this shortcoming, we develop a new technique that gradually modifies the lower tail dependence structure between multivariate scenarios. Similar to Bertismas and Sim (2004), this technique controls the number of parameters that will take their worst-case realizations at the same time.

In the basic example presented in Table 1, we illustrate how to modify the structure of discrete uncertainty sets. This methodology can easily be extended to higher dimensional cases. Without loss of generality, let  $A_0$  denotes the set of historical returns for three stocks over four successive periods. It is important to underline the intrinsic dependence structure between returns, especially for extreme values. For instance, note that lowest returns occur over separate periods. Hence, to protect his portfolio against adverse scenarios by considering the uncertainty set  $A_0$ , an investor has to deal with four moderate adverse scenarios.

**Table 1** Gradual structural break of tail dependencies between multivariate scenarios

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B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
A <sub>0</sub>				A <sub>1</sub>				A <sub>2</sub>							
	r <sub>1</sub>	r <sub>2</sub>	r <sub>3</sub>		r <sub>1</sub>	r <sub>2</sub>	r <sub>3</sub>		r <sub>1</sub>	r <sub>2</sub>	r <sub>3</sub>		r <sub>1</sub>	r <sub>2</sub>	r <sub>3</sub>
1	-0.1	-0.02	0.05		0.2	-0.05	-0.01		0	-0.02	-0.03		0	-0.02	-0.03
2	0	-0.05	-0.01		-0.1	-0.02	-0.01		0	0.15	-0.03		0	0.15	-0.03
3	0.2	-0.075	-0.03		0	-0.02	0.05		0.2	-0.05	0.05		0.2	-0.05	0.05
4	0.12	0.15	-0.2		-0.1	0.15	-0.2		0.2	-0.02	-0.01		0.2	-0.02	-0.01
					0	-0.05	-0.2		0.2	0.15	0.05		0.2	0.15	0.05
					-0.1	-0.02	-0.03		-0.1	-0.075	-0.01		-0.1	-0.075	-0.01
					0.2	-0.075	-0.01		0.12	-0.075	-0.01		0.12	-0.075	-0.01
					-0.1	-0.05	-0.01		-0.1	0.15	-0.01		-0.1	0.15	-0.01
					0.12	-0.05	-0.2		0.2	-0.02	-0.2		0.2	-0.02	-0.2
					0.2	-0.075	-0.2		0.12	-0.05	-0.03		0.12	-0.05	-0.03
					0	-0.02	-0.01		0.12	-0.05	0.05		0.12	-0.05	0.05
					0.12	-0.075	-0.03		0	-0.075	-0.2		0	-0.075	-0.2
					0	-0.05	-0.03		-0.1	-0.05	-0.2		-0.1	-0.05	-0.2
					0.2	-0.02	-0.03		0.12	-0.02	-0.03		0.12	-0.02	-0.03
					0.2	0.15	-0.2		-0.1	-0.075	-0.2		-0.1	-0.075	-0.2
					0	-0.075	-0.01		0	-0.075	0.05		0	-0.075	0.05
					0.12	0.15	0.05		0	0.15	0.05		0	0.15	0.05
					-0.1	-0.075	0.05		0.12	-0.075	0.05		0.12	-0.075	0.05
					0.12	0.15	-0.03		0.2	-0.05	-0.2		0.2	-0.05	-0.2
					0.12	0.15	-0.01		0.2	0.15	-0.01		0.2	0.15	-0.01
					-0.1	-0.05	0.05		-0.1	-0.05	-0.03		-0.1	-0.05	-0.03
					0	0.15	-0.2		0.12	-0.02	-0.01		0.12	-0.02	-0.01
					0	-0.05	0.05		-0.1	0.15	-0.03		-0.1	0.15	-0.03
					0	-0.075	-0.03		0	-0.02	-0.2		0	-0.02	-0.2
					0	0.15	-0.01								
					0.12	-0.075	-0.2								
					-0.1	-0.075	-0.03								
					0.12	-0.05	-0.01								
					0.2	-0.075	0.05								
					0.12	-0.02	0.05								
					0.2	-0.02	0.05								
					0.2	-0.05	-0.03								
					0.12	-0.02	-0.2								
					-0.1	-0.02	-0.2								
					-0.1	0.15	0.05								
					0.2	0.15	-0.03								

Table 1 illustrates the structural break technique to construct new uncertainty set of returns.  $A_0$  is the initial set of scenarios reflecting observed return of three assets over four successive periods. Sets  $A_1$  and  $A_2$  include scenarios obtained with one and two structural breaks in the initial dataset, respectively. Worst-case realizations are highlighted in grey.



To introduce a first break on tail dependencies, let us assume that returns of any couple of assets observed at time  $(t)$  may combine with that of a third asset observed at time  $(s)$ ,  $(t \neq s)$ . As such, the set  $A_1$  contains all new combinations obtained following this condition that modifies dependencies between scenarios compared to the initial dataset. Only two of the three assets keep the same temporal occurrence so that it becomes possible that two stocks simultaneously reach their worst-case levels. We pursue the example by building more adverse scenarios assuming that each return observed at a time  $(t)$  may combine with other returns over distinct periods. The set  $A_2$  displayed in Table 1 contains all new combinations obtained following this condition. To link these results with the model of Bertsimas and Sim (2004), we define the following sets:  $U_0 = A_0$ ,  $U_1 = A_0 \cup A_1$  and  $U_2 = A_0 \cup A_1 \cup A_2$ . The set  $U_2$  indicates the presence of two structural breaks compared to the initial set of scenarios. The set  $U_2$  contains more adverse scenarios to the investor than  $U_1$ . With  $(J)$  uncertain parameters and  $(I)$  initial scenarios, we may generate new scenarios by fixing the number of  $(K)$  unchanged parameters and introducing  $(J-K)$  structural breaks.

This technique introduces, however, additional difficulties regarding the explicit definition of new uncertainty sets. As shown in Table 1, the size of the uncertainty sets increases tremendously with the number of structural breaks. It becomes challenging to list all the scenarios and to make the link with the polytope uncertainty model through the construction of the convex hull. For such case, obtaining a robust counterpart formulation leads to an increase in computational complexity and suggests the development of an appropriate framework.

### 3.2 Economic interpretation of the problem

Because of the implicit form of uncertainty sets induced by structural breaks, we use a *minimax* formulation as suggested by Bienstock (2007) which can be viewed as a sequential zero-sum game between two players: a leader and a follower.<sup>7</sup> In our case, the investor can be seen as the leader and the nature as the follower. An interesting question addressed concerns the interaction between the two players. Specifically, how a player can limit the loss induced by the adversary choice? In this sense, we assume that the investor has two control mechanisms. First, he fixes the number of structural breaks in the dataset. The second mechanism is to impose an exogenous constraint that explicitly sets the level of protection and control the conservatism of the robust solution. We summarize the structure of the game using the following assumptions:

*Assumption 1:* The investor and the nature act on a non-cooperative way to reach their goals.

*Assumption 2:* The investor fixes the uncertainty model to limit the "nature" power.

*Assumption 3:* As a second control mechanism, the investor may impose an explicit constraint to limit the "nature" power.

*Assumption 4:* The level of protection is always obtained from the smallest number of structural breaks.

The first assumption presents the general structure of the game. Assumptions 2 and 3 describe the control mechanisms used by the investor, which entail a cost in terms of robustness discussed

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<sup>7</sup> Unlike the standard form of the game introduced by Stackelberg, the players have the same objective function which is maximized for the first and minimized for the second. The pessimistic form of the Stakelberg model better describe our problem. The follower (nature) responds to the investor portfolio decision by choosing the worst-case multivariate scenario, while the investor tries to minimize the loss resulting from the nature choice.

next. Assumption 4 reflects a priority order in the activation of control mechanisms. The investor first sets the number of structural breaks and then varies the level of protection.

### 3.3 Formulation and problem solution

Our model attempts to characterize a range of optimal levels of protection for various numbers of structural breaks. In a second step, we determine the intermediate values between two consecutive numbers of structural breaks by activating a specific constraint that will fix the degree of conservatism. Solutions obtained from this problem will be robust to all scenarios belonging to the set of uncertainty. Minimax problems are special instances of bi-level problems. Structurally, a bi-level problem is an optimization problem that contains a constraint which requires a subset of the variables to optimize a subordinate optimization problem. The bi-level problem can be defined as follows:

$\max_{\mathbf{x}} \quad \sum_{j=1}^J \bar{r}_j x_j + \lambda \left( \sum_{j=1}^J x_j \left( \bar{r}_j + \hat{r}_j \sum_{t=1}^T y_{t,j} z_{t,j} \right) \right)$	(18)
$\text{s.c} \quad \mathbf{a}_i \mathbf{x} \leq b_i \quad \forall i=1, \dots, I,$	(19)
$\mathbf{x} \geq \mathbf{0}$	(20)
$\min_{\mathbf{y}, \mathbf{w}} \quad \sum_{j=1}^J x_j \left( \bar{r}_j + \hat{r}_j \sum_{t=1}^T y_{t,j} z_{t,j} \right)$	(21)
$\text{s.c} \quad \sum_{t=1}^T y_{t,j} = 1 \quad \forall j=1, \dots, J,$	(22)
$\sum_{t=1}^T w_t = 1$	(23)
$\sum_{j=1}^J y_{t,j} - K \leq 0 \quad \forall t=1, \dots, T,$	(24)
$K w_t - \sum_{j=1}^J y_{t,j} \leq 0 \quad \forall t=1, \dots, T,$	(25)
$-\sum_{j=1}^J x_j \left( \bar{r}_j + \hat{r}_j \sum_{t=1}^T y_{t,j} z_{t,j} \right) \leq -\text{PL}_K$	(26)
$y_{t,j} = \{0,1\} \quad \forall t, j, \quad w_t = \{0,1\} \quad \forall t.$	(27)

The problem (18-27) belongs to the class of bi-level mixed integer linear programs. The upper level program (18-20) is continuous with  $(J)$  variables and  $(I+J)$  constraints. The lower level program (21-27) is discrete with  $T(J+1)$  binary decision variables and  $(2T+J+2)$  constraints. In this program, constraint (26) corresponds to the second control mechanism that will be used by the investor to limit the "nature" power.

To solve the bilevel problem, we use a cutting-plane algorithm as in Bienstock (2007). This algorithm refines iteratively the feasible set by means of linear inequalities (cuts). Such procedure is commonly used to find solutions to MILP programs. The principle is that the original problem is relaxed by ignoring the follower's minimization. During the procedure, each cut should eliminate as much as possible of the unnecessary part of the feasible region and new vertices are generated. The computation of a worst-case return vector and the robust portfolio can be done iteratively with an update of the set containing the optimal deviation factors as follows:

**Step 1** Maximize the minimum of upper level problem with an initial feasible solution  $\mathbf{z}_{[0]}$ , to obtain the optimal solution  $\mathbf{x}_{[0]}^*$ .

**Step 2** Solve the lower level problem with  $\mathbf{x}_{[0]}^*$  to get the optimal solution  $\mathbf{z}_{[1]}^*$ . If  $\|\mathbf{z}_{[1]}^* - \mathbf{z}_{[0]}\| \leq \varepsilon$ , the solution  $(\mathbf{x}_{[0]}^*, \mathbf{z}_{[0]}^*)$  is optimal, otherwise, go to Step 3. (*Epsilon is a small positive real number*).

**Step 3** Maximize the minimum of the upper level problem over the updated set  $[\mathbf{z}_{[1]}^*, \dots, \mathbf{z}_{[i]}^*]$  to obtain the optimal solution  $\mathbf{x}_{[i]}^*$  and go to Step 2.<sup>8</sup>

It is worth noting that the level of protection depends on the uncertainty model is chosen by the investor. In the literature, there is not a specific rule for selecting a priori uncertainty model. It is necessary to find an additional condition which may reflect the degree of conservativeness and robustness.

#### 4. Probability guarantee

The uncertainty set is not necessarily defined to cover the whole uncertain space. In this case, a question naturally arises about the size of the uncertainty set necessary to ensure that the degree of protection level violation does not exceed a certain level. Upon solution of the robust optimization problem, what is the degree of constraint violation? The answers to those questions are related to the probabilistic guarantee on the protection level satisfaction called a priori probability bound. Chen et al. (2007), Bertsimas and Sim (2004) and Ben-Tal and Nemirovski (2000) construct probabilistic index for interval-based model. To our knowledge, this issue has not been addressed for discrete uncertainty models. Using the nonparametric predictive inference (NPI) developed by Coolen (2010), we construct a model to assess the robustness of assets allocation. This technique has proved to be efficient for measuring the probability of outcomes that cannot be done using precise probabilities. It relies on the  $\mathcal{A}_{(n)}$  assumption developed by Hill (1968), which gives the probability on the realization of a random quantity, conditional to a set of past observed values. The use of  $\mathcal{A}_{(n)}$  together with lower and upper probabilities enable inference without prior information on the dataset. However, this method is not sufficient to derive precise probabilities and provides only a probability interval. This imprecise probability characterized by a lower and an upper bounds is close to Bertsimas and Sim (2004) model. Its advantages include the possibility to deal with conflicting evidence, to base inferences on weaker assumptions than needed for precise probabilistic methods, and to allow for simpler and more realistic elicitation of subjective information.

To introduce NPI technique, let us consider a sequence of  $(n + m)$  exchangeable Bernoulli trials, each with “success” and “failure” as potential outcomes. Let  $(s)$  denotes the observed number of successes in the  $(n)$  first trials. Let  $Y_1^n$  denotes the random number of successes in trials 1 to  $n$ . Because of the assumed exchangeability of trials, a sufficient representation of the data is  $Y_1^n = s$ . Let  $Y_{n+1}^{n+m}$  denotes the random number of successes in future trials  $(n + 1)$  to  $(n + m)$ .

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<sup>8</sup> Bienstock (2007) introduces some refinements to the basic algorithm described above that reduce the number of iterations needed for convergence.

Under these assumptions, Coolen (2010) defines the conditional probability interval of the specific event of having at least  $(k)$  success within the  $(h)$  future trials :

$$P^U(h, k | T, s) = P^U(Y_{T+h}^{T+h} \geq k | Y_1^T = s) = \binom{T+h}{T}^{-1} \left[ \binom{s+k}{s} \binom{T-s+h-k}{T-s} + \sum_{l=k+1}^h \binom{s+l-1}{s-1} \binom{T-s+h-l}{T-s} \right] \quad (28)$$

$$P^L(h, k | T, s) = P^L(Y_{T+h}^{T+h} \geq k | Y_1^T = s) = 1 - \binom{T+h}{T}^{-1} \left[ \sum_{l=0}^{k-1} \binom{s+l-1}{s-1} \binom{T-s+h-l}{T-s} \right] \quad (29)$$

where  $P^U$  and  $P^L$  are the upper and lower bounds of the probability interval, respectively. For the case where the horizon  $(h)$  is equal to one the probability interval is equivalent to:

$$P^U(1, k | T, s) = P^U(Y_{T+1}^{T+1} \geq 1 | Y_1^T = s) = \frac{s+1}{n+1} \quad (30)$$

$$P^L(1, k | T, s) = P^L(Y_{T+1}^{T+1} \geq 1 | Y_1^T = s) = \frac{s}{n+1} \quad (31)$$

As discussed earlier, the robust solution obtained for a given set of uncertainty will not allow violation of protection level from realized scenarios belonging to this set. Following this condition, we are interested in the particular event of non-violation of the protection level by a future scenario. In the absence of structural breaks in the dataset, applying (NPI) framework to our problem is straightforward. More specifically, knowing that none of the  $(T)$  observed scenarios violate the protection threshold, we aim to determine the probability interval for a future scenario to fulfill this condition. To link with the previous developments, we assume that a success refers to a non violation of the level of protection. To determine the probability interval, one has to apply the formulas (28) and (29) by assuming  $(T)$  as the number of scenarios,  $(s)$  the number of non-violation in the initial set,  $(h)$  is the number of future periods and  $(k)$  is the minimum number of non violation.

Applying the NPI technique in the presence of structural breaks requires first to find the number of hypothetic scenarios that do not violate the protection threshold. To this end, let  $T_1$  denotes the number of scenarios obtained with one structural break in the dataset. Based on assumptions (1-4), we assume that  $T_1$  satisfies the following relationship:

$$T_1 = T_0 + N_{0,1} \quad (31)$$

where  $N_{0,1}$  is the number of all intermediate scenarios between the levels of protection obtained respectively from the initial set of scenarios and in the presence of one structural break in the dataset. For  $(M)$  structural breaks, the recursion formula gives the following result:

$$T_M = T_{M-1} + N_{M-1,M} \quad \forall M = 1, \dots, J - K \quad (32)$$

Listing all intermediate scenarios between two consecutive structural breaks will be obtained through the activation of constraint (26). More precisely, we apply the pseudo-code presented in Appendix (1). The intermediate number of scenarios is expected to be sensitive to the dependencies between tail returns, the size of the original dataset, the number of uncertain parameters and the number of structural breaks assumed. The probabilistic guarantee is expected to raise as the size of the dataset increases. A similar result applies for the dependence structure and the number of uncertainty parameters. Having large historical information improves the lower NPI bound if new dependencies do not deviate significantly from those on the original dataset. Unlike Bertsimas et al. (2004), our measure is solution-dependent. This advantage is fundamental especially when new deterministic constraints are added to the initial portfolio problem.

## 5. Computational experiments

This section presents the numerical experimentation of the robust portfolio optimization. First, we describe briefly the data used as input in the uncertainty model. Then, we measure the lower tail dependence between returns for different quantiles.

### 5.1 Data

Data used in this study are collected from DataStream database. The data contains weekly return of nine international stock market indices (*as presented in Table 2*). The sample period ranges from 01/23/1998 to 12/16/2010 for a total of 669 observations. The first 554 observations (01/23/1998 to 08/29/2008) refer to the in-sample period and are used as input for the uncertainty set. The last 115 observations (09/05/2008 to 12/16/2010) refer to the out-of-sample period and are used for the ex-post effectiveness analysis.

### 5.2 Summary statistics

Table 2 presents summary statistics of indices returns over the in-sample and out-of-sample periods. Skewness values show that the distributions exhibit negative skew which is indicative of a high frequency of negative returns. On the other hand, positive excess of kurtosis illustrate the greater likelihood of extreme values. From the *Kolmogorov-Smirnov* test results, assets returns seem to deviate largely from the normal distribution. There are some differences however that should be noted between in-sample and out-of-sample periods. Equity markets are more volatile and their distributions are more skewed to the left over the second period, which covers the global financial crisis of 2008.

**Table 2** Summary statistics of returns

**Panel A:** In-sample

	<i>Mean</i>	<i>JBS</i>	<i>Min</i>	<i>Max</i>	<i>Std Dev</i>	<i>Skew</i>	<i>E-Kurt</i>	<i>K-S</i>	<i>p-val</i>
<i>S&amp;P 500</i>	4.14	4.29	-11.60	7.78	16.70	-0.35	5.17	119.95	0
<i>CAC40</i>	5.99	6.04	-11.42	11.67	20.26	-0.02	4.21	33.76	0
<i>DAX30</i>	6.87	6.87	-13.13	13.75	23.23	-0.08	4.73	70.08	0
<i>FTSE100</i>	1.89	2.17	-8.48	10.59	15.74	-0.12	4.51	54.25	0
<i>OMX.S</i>	6.76	6.77	-10.90	18.34	21.15	0.06	5.70	168.70	0
<i>OMX.H</i>	14.21	13.8	-20.68	18.34	31.23	-0.22	5.28	124.56	0
<i>OAS</i>	13.03	12.68	-13.37	11.78	19.72	-0.48	5.29	142.12	0
<i>SMI</i>	3.15	3.36	-13.62	17.69	18.94	0.10	9.18	883.14	0
<i>IBEX35</i>	6.05	6.1	-10.98	14.55	20.26	-0.01	5.19	111.13	0

**Panel B:** Out of sample

	<i>Mean</i>	<i>JBS</i>	<i>Min</i>	<i>Max</i>	<i>Std Dev</i>	<i>Skew</i>	<i>E-Kurt</i>	<i>K-S</i>	<i>p-val</i>
<i>S&amp;P 500</i>	1.24	1.35	-18.20	12.03	28.92	-0.53	6.42	61.37	0
<i>CAC40</i>	-1.38	-1.13	-22.16	13.24	32.87	-1.00	7.16	101.80	0
<i>DAX30</i>	7.98	7.72	-21.61	16.12	33.12	-0.64	7.63	110.65	0
<i>FTSE100</i>	5.84	5.7	-21.05	13.41	29.48	-0.87	9.22	199.85	0
<i>OMX.S</i>	14.96	14.31	-20.59	11.61	30.07	-1.06	7.91	136.78	0
<i>OMX.H</i>	-2.15	-1.85	-16.45	10.94	30.67	-0.75	4.62	23.39	0
<i>OAS</i>	3.04	3.05	-20.88	14.66	36.14	-0.75	5.91	51.20	0
<i>SMI</i>	-0.50	-0.3	-22.28	14.07	28.68	-0.97	12.36	438.02	0
<i>IBEX35</i>	-0.14	0.05	-21.20	11.74	33.68	-1.05	6.40	76.83	0

Table 2 presents summary statistics of the nine indices used in the dataset over the in-sample and the out-of-sample periods. Expected returns are estimated using the arithmetic mean and the Jorion, Bayes and Stein (JBS) estimator, respectively (see Appendix 1). The values in the first five columns are given in percentages. The means and standard deviation are annualized.

There are two parameters that reach their lowest values at the same time during the in-sample period. Therefore, the maximum number of structural breaks that will be assumed for our example is fixed to seven. Table 3 presents the matrix of nonparametric left tail dependence among major stock indices at 1% and 5% quantiles. Tail dependence refers to the co-movement among extreme events, which is not necessarily similar to that among ordinary observations. The results on pairwise tail dependence suggest that extreme returns are positively dependent at the 1% quantile. The FTSE 100 has the highest lower tail dependence with the SMI suggesting that the U.K. market suffers the least co-crashes with the Swiss stock market. To check the robustness of these results and that they do not correspond to outliers, we perform the same analysis at the 5% quantile. The values of dependencies increase significantly and confirm the trend highlighted at the 1% quantile. Between October 2008 and December 2008 the S&P500, the plunged 33%. European stock market suffered similar losses: the FTSE-100 lost over the same period about 23%, the DAX-30 dropped about 25% and the CAC-40 lost about 30%. The dramatic drop of the stock markets was triggered by the financial crisis.

**Table 3.** Lower tail dependence between returns over the in-sample period

**Panel A:** (In sample) Quantile(1%)

	<i>CAC40</i>	<i>DAX30</i>	<i>FTSE100</i>	<i>OMX.S</i>	<i>OMX.H</i>	<i>OAS</i>	<i>SMI</i>	<i>IBEX35</i>
<i>S&amp;P 500</i>	0.333	0	0.5	0.333	0.167	0.167	0.5	0
<i>CAC40</i>		0.333	0.556	0	0	0.111	0.444	0.222
<i>DAX30</i>			0.105	0.053	0	0.053	0.105	0.211
<i>FTSE100</i>				0.2	0	0.2	0.8	0.4
<i>OMX.S</i>					0.077	0.231	0	0
<i>OMX.H</i>						0	0	0
<i>OAS</i>							0.071	0
<i>SMI</i>								0.429

**Panel B:** (In sample) Quantile(5%)

	<i>CAC40</i>	<i>DAX30</i>	<i>FTSE100</i>	<i>OMX.S</i>	<i>OMX.H</i>	<i>OAS</i>	<i>SMI</i>	<i>IBEX35</i>
<i>S&amp;P 500</i>	0.571	0.464	0.536	0.5	0.357	0.214	0.5	0.321
<i>CAC40</i>		0.419	0.419	0.302	0.186	0.256	0.488	0.302
<i>DAX30</i>			0.232	0.232	0.089	0.143	0.304	0.196
<i>FTSE100</i>				0.542	0.292	0.375	0.75	0.542
<i>OMX.S</i>					0.18	0.22	0.28	0.2
<i>OMX.H</i>						0.075	0.05	0.1
<i>OAS</i>							0.308	0.231
<i>SMI</i>								0.429

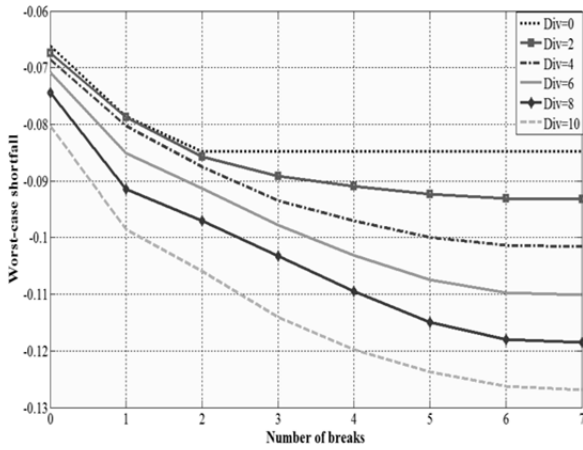
*Table 3 presents the nonparametric lower tail dependence matrix of returns over the in-sample period at the quantile of 1% and 5%. Dependence coefficients are calculated using the formula defined in appendix 2.*

## 6. Results

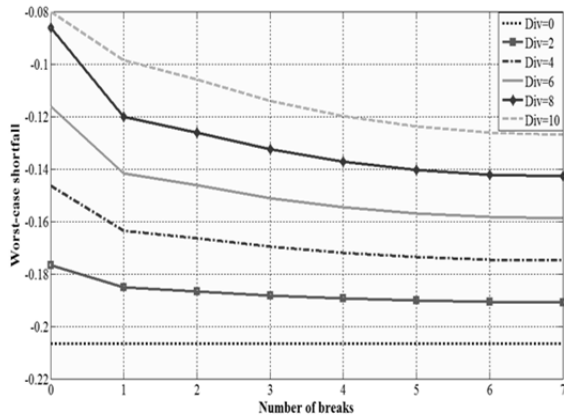
### 6.1 Sensitivity analysis

This section reports and discusses results obtained from numerical experimentations. Figures (1-3) illustrate for various levels of risk aversion the effect of the uncertainty model and the diversification constraints on the protection level. Regardless of the investor's risk attitude, worst-case shortfall decreases as the number of structural breaks in the dataset increases. With an infinite lambda and no diversification constraints, the protection level is (-6.62%). When the number of structural breaks is the highest, the minimum guaranteed return is (-8.48%). All other things being equal, when the parameter  $\lambda$  increases, the protection level decreases. Note that for large  $\lambda$ , the lowest worst-case shortfall is obtained with a portfolio without diversification constraints. In contrast, for small  $\lambda$ , we have the opposite result. Investors with low risk aversion are more willing to choose high-return assets, which generally are more volatile and with more extreme negative returns.

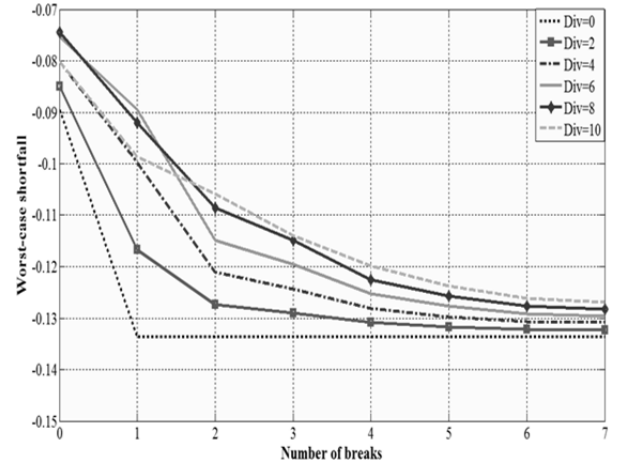
**Figure 1** Protection level sensitivity to breaks and diversification for  $\lambda=10^9$



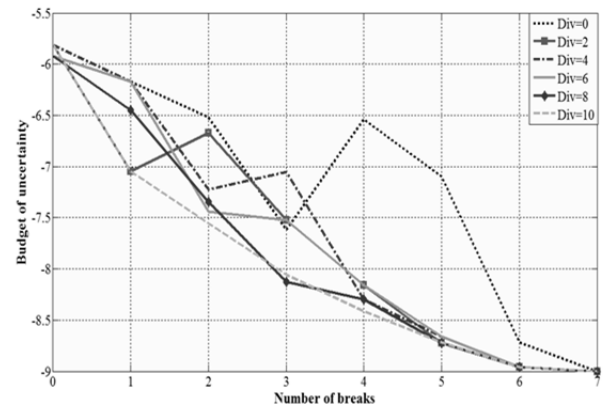
**Figure 3** Protection level sensitivity to associated to breaks and diversification for  $\lambda=0$



**Figure 2** Protection level sensitivity breaks and diversification for  $\lambda=1$



**Figure 4** Induced Budget of uncertainty the worst-case shortfall, for  $\lambda=10^9$



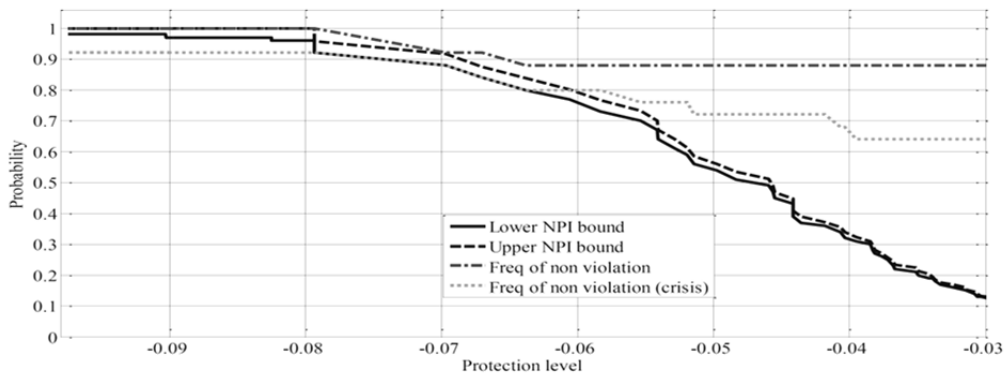
To make connection with Bertsimas and Sim (2004) model, Figure 4 depicts the evolution of the budget of uncertainty associated to the worst-case shortfall for  $\lambda=10^9$ . In the absence of

breaks in the dataset, the value of this function is 5.85. Not all the curves in figure 4 are monotonic decreasingly, as they should be. This pattern is reduced as the diversification and/or the number of structural breaks increases. This result shows the convergence of our model with Bertsimas and Sim (2004). Note that for the same level of protection, there are several robust portfolios obtained for different levels of diversification. Two criteria will be used to evaluate the performance of these portfolios. The first one is the probability of non-violation of the protection threshold. The second criterion is the cost of robustness or the sacrifice of expected return against robustness. These two criteria are examined for our model and that of Bertsimas and Sim (2004), respectively.

## 6.2 Ex-post effectiveness

Figures 5-8 present the probability guarantees and frequencies of non-violation of protection level for two future investment horizons (25 and 50 weeks). We test the performance of robustness measures including and excluding to the investment horizons the period of the financial crisis of 2008. We compare results obtained from NPI and from the probability measure of Bertsimas and Sim (2004). First, we note that the frequency of non-violation for both models is high and it is close to 90%. This result tends to improve as the level of protection becomes more conservative or the investment horizon increases. The number of violations increases significantly when the financial crisis phase is included in the test periods. Overall, there is little difference between the results of the two models compared to the frequency of violations. In terms probability guarantee, we focus mainly on the lower bound.<sup>9</sup> Note that an increase in the absolute level of protection results improves the probability guarantees. The lower bounds of the two probability models systematically frame and all investment horizons frequency of non-infringement. However, that obtained from the non-parametric technique (NPI) seems more accurate. Unlike Bertsimas and Sim, our measure decreases more gradually following the protection and it is sensitive to the investment horizon.

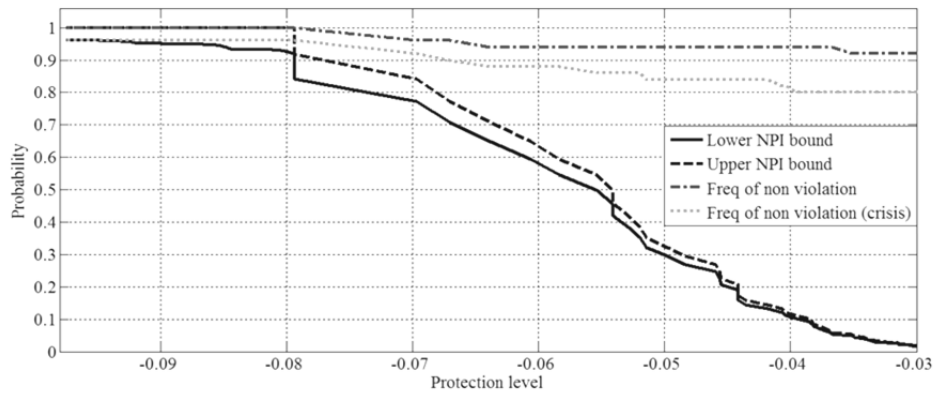
**Figure 5** Probability lower bound and frequency of non-violation obtained from the new model over an investment period of 25 weeks



<sup>9</sup> For the Bertsimas and Sim (2004) model, the upper limit of the probability of non-violation of the protection level is one.

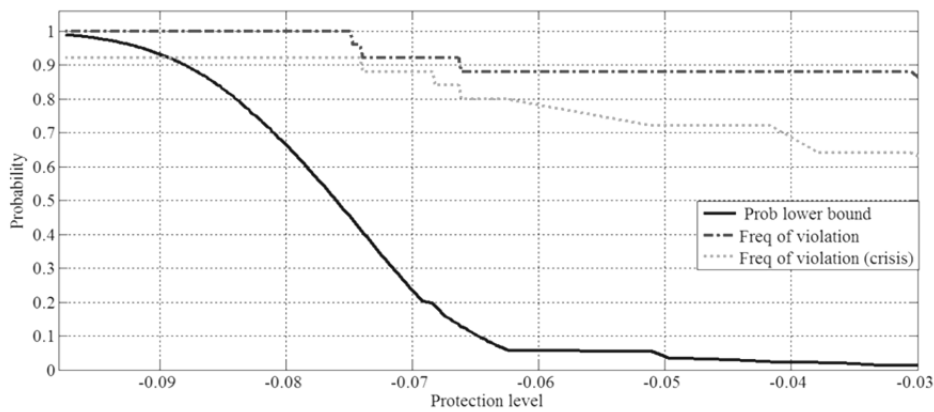


**Figure 6** Probability lower bound and frequency of non-violation obtained from the new model over an investment period of 50 weeks

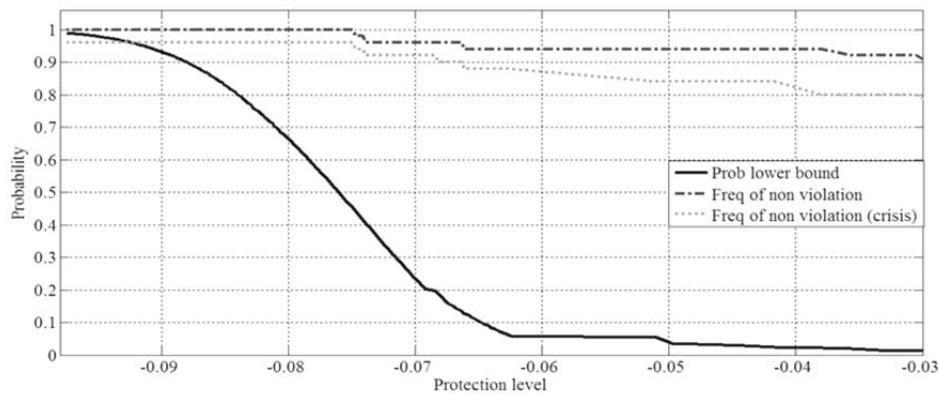


Figures 5 and 6 present the probability intervals obtained from the method (NPI) and the frequency of non-violation of protection level. These measurements cover respectively the investment horizons of 25 and 50 weeks.

**Figure 7** Probability lower bound and frequency of non-violation obtained from Bertsimas and Sim model over an investment period of 25 weeks



**Figure 8** Probability lower bound and frequency of non-violation obtained from Bertsimas and Sim model over an investment period of 50 weeks

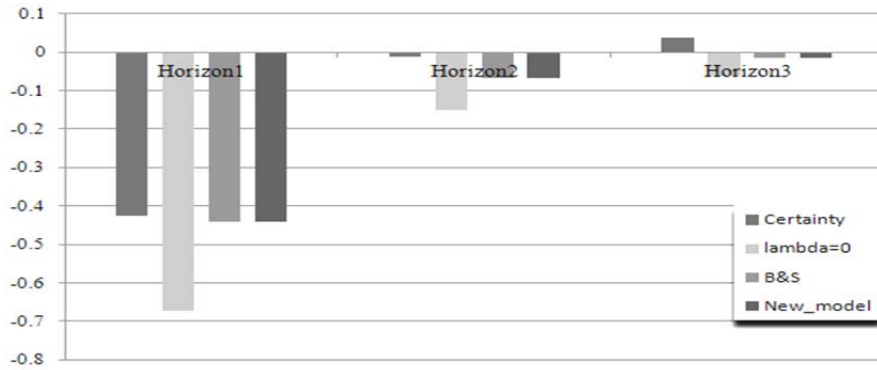


Figures 7 and 8 present the probability lower bound and the frequency of non-violation of the protection level. These measurements are obtained from the model of Bertsimas and Sim (2004) and respectively cover investment horizons of 25 and 50.

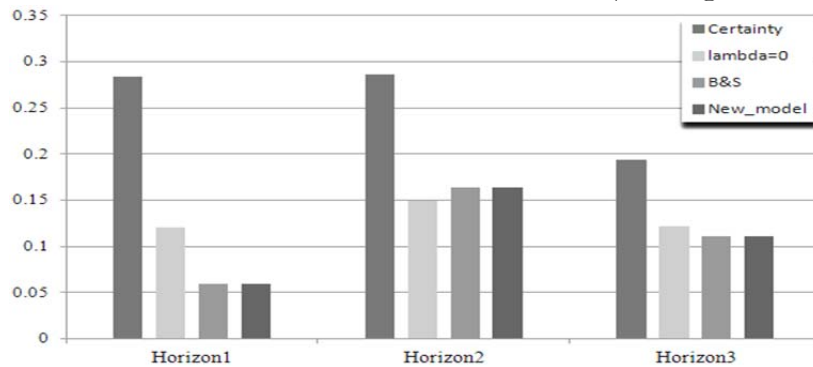
### 6.3 Cost of robustness

We compare the realized returns related to four investment models for three horizons periods of (25, 50 and 75 weeks), which include and then exclude the crisis period. The first model describes a situation of perfect certainty where one seeks a portfolio that maximizes return having all observations. The second model considers the case of an investor how is only interested by the maximization of performance using expected return estimations. The last two models integrate the protection of the portfolio with a lambda of 10. They correspond to the model of Bertsimas and Sim and the one we propose. Results are obtained with a threshold diversification set to three and a number of structural breaks of seven. Overall, there is no significant difference between the last two models. In times of crisis, they give similar results to the certainty case. The investor who focuses on maximizing reward function ( $\lambda = 0$ ) has the lowest realized return. For the non-crisis period, the two models of uncertainty are outclassed by the other two models. All these results show that the loss in terms of profitability remains relatively low. We can conclude on the potential of the discrete model of uncertainty to determine the robust solution and to provide a good estimate of probability of violation of the protection threshold.

**Figure 9** Realized returns over different investment horizons (*including the crisis period*)



**Figure 10** Realized returns over different investment horizons (*excluding the crisis period*)



Figures 9 and 10 exhibit realized return obtained from four models over the investment horizons of (25, 50 and 75 weeks) including and excluding the crisis period (September - December 2008.). The first model describes a situation of perfect certainty where the investor maximizes its portfolio using realized returns. The remaining models use sample mean of returns estimated from the in-sample period. The second model assumes that investors are only interested in maximizing returns. The last two models are based on a lambda equal 10 and correspond to the model of Bertsimas and Sim and our model.

## 7. Conclusion

Financial crisis are generally accompanied by stunning decreases in stocks prices that adversely affect even well-diversified portfolios. For an investor optimizing a portfolio using a safety-first criterion, this paper provides a robust control approach to mitigate the impact of parameter uncertainty by allowing to vary the protection level in a smooth way across the uncertainty set. This issue is of great relevance to portfolios managers since uncertainty leads to unstable portfolio weights and low risk-adjusted returns. Unlike many other robustification approaches, our model makes no assumptions about the distributions of the unknown parameters. As a second advantage, it captures multivariate tail dependence between returns by introducing progressive structural breaks in the dataset to controls the robustness of the solution. Furthermore, our approach gives probabilistic bounds of robustness that depend on intrinsic characteristics of the uncertainty model, optimal solution and investment horizon.

Numerical results highlight the importance of taking into account the uncertain model in the optimization process. The concept of structural breaks fairly describes the behavior of financial assets and reflects the shift of correlations that may occur during bear markets. Single extreme asset movement is often less adverse to investors seeking to guarantee the minimum return than a negative variation of all securities. Obviously, in the second case diversification is no longer effective. The nonparametric probability measure  $NPI$  provides accurate prediction intervals and illustrates high *ex post* performance of robust portfolios. The analysis of trade-off between performance and robustness of the solution is also explored. It shows that robust decisions have relatively low cost on the objective function.

The evidence provided in the paper, based on a real data application, suggests that scenario-based models work well in practice and provide a viable and a simple alternative to interval-based ones. Our model could be applied to other applications related for instance to the solvency issue, which should be fulfilled regardless risks occurrence. Accordingly, it would be possible to determine capital requirement following a worst-case framework. This setting is expected to reduce the estimation risk induced by a flawed choice of probability distributions. We are aware, however, of some weaknesses of our approach. Clearly, much work has to be done to effectively address the problems of outliers. It would be interesting in a future study to examine a robust multistage setting for the portfolio protection problem.

### Appendix 1: Algorithm of intermediate scenarios enumeration

**Set**  $(b)$  the number of structural breaks

**Find** the robust solution  $(\mathbf{x}_b^*)$  for a level of breaks  $(b)$  and determine the protection level  $PL_b(\mathbf{x}_b^*)$  (by solving the program (18-27))

**Find** at the node  $(b-1)$  the worst-case shortfall  $PL_{b-1}(\mathbf{x}_b^*)$  for  $(\mathbf{x}_b^*)$  (by solving the program (21-27))

**Set**  $Total\_number = T$

**Set**  $i=0$

**While**  $b>0$

$$PL_b(\mathbf{x}_b^*)_{[0]} = PL_b(\mathbf{x}_b^*)$$

solve (21-27) for  $(b)$  and with the additional constraint:

$$-PL_b(\mathbf{x}_b^*)_{[i+1]} \leq -PL_b(\mathbf{x}_b^*)_{[i]} - \varepsilon \quad (\text{where } \varepsilon \text{ is a small positive real number}).$$

**If**  $PL_b(\mathbf{x}_b^*)_{[i+1]} < PL_{b-1}(\mathbf{x}_b^*)$

**Set**  $i=i+1$

**Else**

**Set**  $Nbre=i$

**Set**  $i=0$

**Set**  $b=b-1$

**Find** at the node( $b-1$ ) the protection level  $PL_{b-1}(\mathbf{x}_b^*) = (\mathbf{x}_b^*)^T \mathbf{r}_{b-1}^*$  for  $\mathbf{x}_b^*$  (by solving the program (21-27))

**End**

**Set**  $Total\_number = Total\_number + Nbre$

**End**

## Appendix 2: Jorion-Bayes-Stein estimator

The Jorion-Bayes-Stein estimator was proposed by Stein (1956) and further elaborated by Jorion (1986). It relies on the shrinkage technique and the Bayesian framework. It is defined as:

$$r_j^{JSB} = \omega r_g + (1 - \omega) \bar{r}_j \quad (2.1)$$

where  $r_j^{JSB}$  the adjusted asset is mean,  $\bar{r}_j$  is the original asset sample mean,  $r_g$  is the global mean (approximated by the *MSCI* global index) and  $\omega$  is the shrinkage factor. Jorion (1986) estimates the shrinkage factor as:

$$\omega = \frac{(J + 2)}{(J + 2) + T(\hat{\mathbf{r}} - r_g \mathbf{e})' \hat{\Sigma}^{-1} (\hat{\mathbf{r}} - r_g \mathbf{e})} \quad (2.2)$$

where  $T$  is the sample size,  $J$  is the number of assets,  $\hat{\Sigma}$  is the sample covariance matrix,  $\mathbf{e}$  is a unit vector and  $\hat{\mathbf{r}}$  is a vector of the sample means of returns.

## Appendix 3: Lower tail dependence index

Let us consider a pair of random variables  $(x, y)$  whose realizations are observed over  $T$  periods,  $x_t$  and  $y_t$  for  $t = 1, \dots, T$ . The nonparametric dependency estimator  $\tau_{y/x}$  between the extreme values is calculated as the ratio between the number of observations where  $x$  and  $y$  are jointly extreme and those where only the variable  $x$  is extreme. More precisely, one has to set a positive small integer ( $k$ ) and find the  $k$  lowest values for the two variables satisfying the following formula:

$$\hat{\tau}_{y|x} = \frac{\sum_{t=1}^T I_{y,t} \text{ and } I_{x,t}}{\sum_{t=1}^T I_{x,t}} \quad (3.1)$$

where  $I_{y,t}$  et  $I_{x,t}$  are indicator variables taking 1 if the observation at time ( $t$ ) for the variables  $x$  and  $y$  is strictly inferior to the quantile at the level  $(k/T)$ .

## References:

- Averbakh, I. and Zhao, Y. (2008). “Explicit reformulations for robust optimization problems with general uncertainty sets”. *Siam Journal on Optimization*, 18, 1436 – 1466.
- Avis, D. (1999). “Computational Experience with the Reverse Search Vertex Enumeration Algorithm”. *Working paper*, School of Computer Science, McGill University.
- Ben-Tal, A., El Ghaoui, L. and Nemirovski, A. (2008). “Robust optimization”. *Princeton University Press Princeton and Oxford*.
- Ben-Tal, A., Bertsimas, D., Brown, D.B. (2010). “A soft robust model for optimization under ambiguity”. *Operations Research*, 58, 1220–1234.
- Ben-Tal, A. and Nemirovski, A. (2000). “Robust solutions of linear programming problems contaminated with uncertain data”. *Mathematical Programming*, 88: 411 – 424.
- Bertsimas, D. and Doan, X. (2010). “Robust and data driven approaches to call centers”. *European Journal of Operational Research*, 207 (2) : 1072 – 1085.
- Bertsimas, D. and Sim, M. (2004). “The price of robustness”. *Operations Research*, 52(1): 35 – 53.
- Bertsimas, D. and Thiele, A. (2006). “Robust and Data-Driven Optimization: Modern Decision-Making Under Uncertainty”. *Working paper*, Sloan School of Management.
- Bertsimas, D. and Brown, D. (2009). “Constructing Uncertainty Sets for Robust Linear Optimization”. *Operations Research*, 57 (6): 1483 – 1495.
- Bertsimas, D., Brown, D. and Caramanis, C. (2010). “Theory and Applications of Robust Optimization”. *Working paper*, Sloan School of Management and Operations Research Center.
- Bertsimas, D., Pachamanova, D. and Sim, M. (2004<sup>b</sup>). “Robust linear optimization under general norms”. *Operation research letters*, 32: 510 – 516.
- Bienstock, D. (2007). “Experiments with robust optimization”. *Journal of computational finance*, 1: 1 – 48.
- Brogan, A.J. and Stidham, S. (2005). “A note on separation in mean-lower-partial-moment portfolio optimization with fixed and moving targets”. *IEEE Transactions*, 37 (10): 901 – 906.
- Chen, X., Sim, M. and Sun, P. (2007). “A robust optimization perspective to stochastic programming”. *Operations Research*, 55(6): 1058 – 1071.
- Chopra, V.K. and Ziemba, W.T. (1993). “The effect of errors in means, variances, and covariances on optimal portfolio choice”. *Journal of Portfolio Management*, 23: 6 – 11.
- Coolen, F.A. (2010). “Nonparametric Predictive Inference”. *International Encyclopaedia of Statistical Sciences*. Springer.
- Cornuejols, G. and Tutuncu, R. (2007). “Optimization Methods in Finance “. Cambridge University Press.
- Costa O. L. and Paiva, A. C. (2002). “Robust portfolio selection using linear-matrix inequalities”. *Journal of Economic Dynamics and Control*, 26: 889–909.
- DeMiguel, V. and Nogales, F. J. (2009). “Portfolio selection with robust estimation “. *Operations Research*. 57: 560–577.
- DeMiguel, V., Garlappi, L. and Uppal. R. 2009. “Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?” *Review of Financial Studies* 22: 1915–53.
- Ding, Y. and Zhang, B. (2009). “Optimal portfolio of safety-first models”. *Journal of Statistical Planning and Inference*, 139: 2952 – 2962.
- El Ghaoui, L., Oks, M. and Oustry, F. (2003). “Worst case value-at-risk and robust portfolio optimization: a conic programming approach”. *Operations Research*, 51(4): 543 – 556.
- Erdogan, E., Goldfarb, D. and Iyengar, G. (2004). “Robust portfolio management”. Technical report, Columbia University, New York, NY.
- Fabozzi, F., Kolm, P., Pachamanova, D., Focardi, S., 2007. *Robust portfolio optimization and management*. Wiley.
- Fabozzi, F.J., Huang, D., Zhou, G., 2010. “Robust portfolios: contributions from operations research and finance”. *Annals of Operations Research* 176, 191–220.
- Gabrel V. and Murat, C. (2010). “Robustness and duality in linear programming”. *Journal of the Operational Research Society*, 61:1288 – 1296.

- Gabrel, V., Murat, C., 2010. “Robustness and duality in linear programming. *Journal of the Operational Research Society* 61, 1288–1296.
- Gabrel, V., Murat, C., Thiele A. (2013). “Recent Advances in Robust Optimization: An Overview”. Working paper, Université Paris-Dauphine.
- Goldfarb D. and Iyengar, G. (2003). “Robust portfolio selection problems”. *Mathematics of Operation Research*, 13: 28 – 38.
- Gregory, C., Dowman, K. and Mitra, G. (2011). “Robust optimization and portfolio selection: The cost of robustness”. *European Journal of Operational Research*, 212: 417 – 428.
- Gupta, S. and Bertsimas, D. (2011). “Multistage Air Traffic Flow Management under Capacity Uncertainty: A Robust and Adaptive Optimization Approach”. Massachusetts Institute of Technology.
- Harlow, W.V. (1991). “Asset allocation in a downside-risk framework”. *Financial Analysts Journal*, Septembre /Octobre: 28 – 40.
- Hill, B.M., (1968). “Posterior distribution of percentiles: Bayes’ theorem for sampling from a population”. *Journal of the American Statistical Association*, 63 (322): 677 – 691.
- Jorion, P. (1986). “Bayes-Stein estimation for portfolio analysis “. *Journal of Financial and Quantitative Analysis*, 21: 279 – 292.
- Kataoka, S. (1963). “A stochastic programming model”. *Econometrica*, 31:181 – 196.
- Kim, J.H, Kim, H.C and Fabozzi, F.G. (2013). “Recent Developments in Robust Portfolios with a Worst-Case Approach”. (forthcoming) *Journal of Optimization Theory and Applications*.
- Kouvelis P. and G. Yu. “Robust discrete optimization and its applications”. Kluwer Academic Publishers, Boston, (1997).
- Krokmal, P., Palmquist, J. and Uryasev, S. (2002). “Portfolio Optimization with Conditional Value-at-Risk Objective and Constraints”. *The journal of risk* 4: 43 – 68.
- Lobo, M. and Boyd, S. (2000). “The Worst Case Risk of a Portfolio”. *Working paper*, Stanford University.
- Markowitz, H., (1952). “Portfolio selection”. *Journal of Finance*, 7(1) : 77 – 91.
- Miao, W., Yin, H., Du, D., Han, J. (2007). “Robust Optimization Model for a Class of Uncertain Linear Programs”. *Proceedings of the First international conference on Combinatorics, Algorithms, Probabilistic and Experimental Methodologies*, 384 – 395.
- Michaud, R. (1989). “The Markowitz Optimization Enigma: Is Optimized Optimal “. *Financial Analysts Journal*, 45 (1): 31 – 42.
- Mulvey, J.M., Vanderbei, R., J. and Zenios, S. A. (1995). “Robust optimization of large-scale systems”. *Operation Research*, 43: 264 – 281.
- Natarajan, K., Pachamanova, D. and Sim, M., (2009). “Constructing risk measures from uncertainty sets”. *Operations Research*. 57(5): 1129 – 1141.
- Quaranta, A. and Zaffaroni, R., (2008). “Robust optimization of conditional value at risk and portfolio selection”. *Journal of Banking & Finance*, 32: 2046 – 2056.
- Roy, A.D. (1952). “Safety-first and the holding of assets”. *Econometrica*, 20 (3): 431 – 449.
- Soyster, A., (1973). “Convex programming with set-inclusive constraints and applications to inexact linear programming”. *Operations Research*, 21: 1154 – 1157.
- Stein, C. (1956). “Inadmissibility of the usual estimator for the mean of a multivariate normal distribution”. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1: 197 – 206.
- Telser, L.G. (1955). “Safety first and hedging”. *Revue of Economics Studies*, 23(1): 1 – 16.
- Thiele, A., Terry, T. and Epelmanz, M., (2010). “Robust Linear Optimization With Recourse”. *Working paper*, Lehigh University.
- Tütüncü, R. and Koenig, M. (2004). “Robust asset allocation”. *Annals of Operations Research*, 132: 157 – 187.
- Zhu, S. and Fukushima, M. (2009). “Worst-Case Conditional Value-at-Risk with Application to Robust Portfolio Management”. *Operations Research*, 57: 1155–1168.