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Abstract

We propose to re-read Ricardo's theory of rent and its modern versions. Ricardo's dynamic approach follows the transformations of a long-term equilibrium with demand. Sraffa adopted the same framework while substituting a value criterion for a physical criterion to determine the incoming marginal method, but he did not state the law of succession of methods explicitly. This prevented him to realize that his critique opens the door to all complications of capital theory, with the consequence that the Ricardian dynamics fail when a divergence appears between profitability and productivity. Contemporary studies have cast doubts on the validity of some of Ricardo's and Sraffa's over-optimistic conclusions, but the abandonment of the dynamic approach does not allow them to explain the ultimate reason of the phenomena they have pointed at. Ricardo's method has been recently rediscovered by mathematicians.

Keywords. Classical theory, land, rent, Ricardo, Sraffa

JEL classification. B12, B51, C61, D33

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1 Introduction

The[1] publication of five essays by Malthus, Torrens, West and Ricardo in February 1815 constituted a decisive step in the development of the Classical theory of rent (details in Sraffa, 1954, p. 5). The question of lands and rent, which had been analyzed by Smith (also by Anderson, 1777a and 1777b), was at the core of the struggle for economic and political power between landowners and the rising capitalist class, and the discussions on the corn laws at the Parliament triggered the economists' interest on rent and its policy implications. Ricardo's *Essay on Profits* is a reply to Malthus's two pamphlets (*Inquiry* and *Grounds*) in which Malthus developed protectionist arguments. By contrast, Ricardo used his theory as a plea in favour of free trade, which planes rents and improves the general rate of profit. He held a similar position in the *Principles* (1817), in which he made use of a more precise theory of value and described the processes of extension and intensification of cultivation in more details.

On the bicentennial of these publications, we propose to return to the analytical side of Ricardo's construction regarding production with lands and its modern legacy in the Classical tradition (see Pasinetti (2014) on that tradition). Ricardo adopted the labor theory of value for reproducible commodities and used the property that the marginal capital pays no rent to extend its field of application to agricultural products: the labor values of commodities are defined by the industrial methods and the marginal agricultural methods; these values once known, the conditions of production on intra-marginal lands (those which are already fully cultivated, as they are of a better quality than the marginal lands) determine the rents of those lands as differential costs. The price of corn being independent of its conditions of production, the owners of the best lands are indeed in a position to demand a rent from their farmers. Sraffa (*PCMC*, 1960) referred to prices of production instead of labor values and criticized some of Ricardo's statements, but his analysis constitutes the true line of descent of Ricardo's construction. His work initiated a field of researches which we identify as 'post-Sraffian'. Among other valuable results, we stress three significant conclusions: a general existence theorem (Salvadori, 1986), a necessary and sufficient criterion for uniqueness (Erreygers, 1990, 1995) and a number of results illustrating

[1] A longer version of the same paper, entitled "The Ricardian Rent: an Introduction", gives more references to Ricardo's and Sraffa's works and provides technical details on the proofs of the properties stated in Appendix B.

the general idea that the behavior of productive systems with lands is much more complex than suggested by Ricardo and Sraffa. For instance, there may exist several long-run equilibria sustaining a given level of final demand, while uniqueness is ensured for single-product systems without lands. Conclusions of that type cast doubt on the feasibility of Ricardo's program, which was also adopted by Sraffa in its modernized version.

The specialized literature on lands is often quite technical and involved. Rather than reviewing it directly, we follow a more roundabout (and more critical) path, starting from a return to Ricardo and Sraffa. Our main point is methodological. Sraffa's methodology is faithful to Ricardo, in that he adopted the dynamic approach which consists in following the transformations of a long-term equilibrium when demand increases. The dramatic moment of those dynamics occurs when some marginal method (outgoing marginal method) is replaced by another (incoming marginal method). We characterize that approach in Section 2 and identify its central questions as a search in two directions: (i) the law of succession of methods when demand changes; and, (ii) the reduction of the study of a productive system with lands to that of a system without lands ('reduction', in short). We propose a re-reading of Ricardo and Sraffa in view of the answers they provide to these questions. Concerning the succession of methods, Sraffa determined the incoming marginal method by means of a value criterion instead of Ricardo's physical criterion, and this led him on the verge of an explicit statement of a general law, but he failed to do so. That law is stated in Section 3. In Section 4, we look at the interactions between the two aspects of Ricardo's and Sraffa's researches. It turns out that, for two independent reasons, the value criterion may prevent the reduction, a failure illustrated by the violation of the trade-off property between wages and profits. Moreover, Sraffa did not notice that, once the value criterion is put forward, the critique he addressed to capital theory has its counterpart in the analysis of production with land, with the effect that the working of the dynamics themselves is not guaranteed (Section 5).

By contrast with Ricardo and Sraffa, most of the post-Sraffian literature on lands follows a static approach, the question becoming that of the existence of a cost-minimizing system for a given level of demand. Though the conclusions are formally correct, it is argued in Section 6 that the abandonment of the dynamics does not allow to explain the origin of some apparent paradoxes. We reinterpret the results relative to existence and uniqueness in terms of the dynamic approach. The final Section 7 shows Ricardo as an

unexpected precursor of a contemporary mathematical tool.

In the whole of the paper, except in Section 4.2, the model we consider is a corn-land model, with corn as the unique produced good in the economy. Corn can be cultivated on different lands (theory of extensive rent) or on the same land by several methods (theory of intensive rent), or both. The hypothesis of a unique good suffices to understand the structure of rent theory and the difficulties it meets, and the simplicity of the framework aims at discarding the common opinion that rent theory would be a complex matter or that its main difficulties would start with the multiplicity of agricultural products. Rent theory is first a question of method: the law of succession of methods is the Ariadne's thread of the whole construction.

2 Aim and methodology

For Ricardo (1817), the labor theory of value provides the tool for understanding the working of the forces at stake in a capitalist economy and, in particular, for explaining prices. That theory allowed him to state the trade-off property between wages and profits (the higher the real wage, the lower the rate of profit). However, it only applies to reproducible commodities and not, *a priori*, to land or commodities produced by means of lands. This is why, immediately after having introduced the notion of labor value, Ricardo examined the case of production with land. His extension of the labor theory of value to agricultural products is based on the analytical possibility of 'getting rid of rent' (letter to McCulloch, 13 June 1820) thanks to the property that 'the capital last employed pays no rent' (*Principles*, Chapter 2). As a consequence, the labor theory of value still applies to the industrial methods and the marginal agricultural methods. At prices determined by these methods, a land of a higher grade yields a rent equal to the differential cost of production with the marginal land. For a given level of demand, the trade-off property between wages and profits still holds and, in the face of an increasing demand and a given real wage, rents rise at the expense of profits.

Sraffa substituted the reference to prices of production for that to labor values but, as far as prices and distribution are concerned, and in the absence of any opposite hint, he shared the opinion that the results proved for single-product systems often apply: 'In the case of a single quality of land, the multiplicity of agricultural products would not give rise to any complications' (*PCMC*, Section 89). The most significant hint of Sraffa's agreement

with Ricardo's global project is of a methodological nature. Two distinct approaches to the study of production with land can be conceived. The static approach consists in writing down a system of equalities and inequalities for a given level of demand. Then a long-term position is defined as a solution to these equations, which involve both physical and value conditions which will be examined in Section 6 and Appendix B. Ricardo did take such conditions into account and, for instance, set the nullity of rent on non-fully cultivated lands. But the dynamic approach he privileged is different: it is based on the study of the transformations of equilibrium with demand. The basic property is that, most of the time, a slight change in final demand is met by a slight adaptation of activity levels with, on the physical side, no changes in the list of operated methods and cultivated lands and, on the value side, no changes in prices and rents. The adaptation of activity levels only concerns the methods already in use and consists in extending cultivation on a partly cultivated land or extending the use of a more productive method on a fully cultivated land. A limit to that adaptation is reached when a scarcity constraint is met. Then the price of corn jumps to a higher level and a new marginal method is introduced. The rents on cultivated lands rise suddenly with the price of corn. After that shock, a new equilibrium is found and another period of calm opens again, with a smooth adaptation to changing physical requirements.

Sraffa's adopting the same scheme in his study of rent is all the more noteworthy that the dynamic approach he follows contradicts the explicit warning according to which '[n]o changes in output [...] are considered' (Preface of *PCMC*). As the adaptation of activity levels to demand during calm periods sets no difficulties, the main point of the dynamic approach is the study of the phenomena which occur under critical circumstances. Even if the spasmodic change of method then involved goes with a discontinuity in prices and rents, the dynamics are not chaotic. First, when a scarcity constraint is met, there is no complete reorganisation of production, as the economy reacts by changing only one marginal method. Second, activity levels vary continuously with demand during calm periods and, we stress, also at breaking points: whether a new method is introduced on another field or on the same field, it is introduced at a low activity level while the previously operated methods either keep the same activity levels (extension of cultivation) or reduce them slightly to leave room to the new method (introduction of a more intensive method). The smooth adaptation of activity levels is a universal property which reduces the core of the dynamics to the identification of the outgoing marginal method and the incoming marginal method at critical

moments: we call that phenomenon the law of succession of methods.

3 The law of succession of methods

3.1 Physical vs. value vs. criterion

The outgoing marginal method is always identified by means of a physical criterion. In the case of extensive cultivation, the limit of an equilibrium is reached when some land becomes fully cultivated. The corresponding method of cultivation is the outgoing marginal method, i.e. it is marginal in the present equilibrium but will become intra-marginal in the next. In that of intensive cultivation, the limit is reached when the activity evel of the less productive method drops to zero. All other presently operated industrial and agricultural methods will be operated in the next equilibrium.

The point on which Sraffa criticized Ricardo concerns the determination of the incoming method. Let us first follow Sraffa's critique concerning extensive cultivation. Ricardo assumed that lands can be classified according to their fertility, and the extension of cultivation follows that natural order. This is indeed the case if cultivation on a land of a lesser grade requires more of any input than on a better land, but the hypothesis is unduly restrictive. Sraffa (*PCMC*, Section 86) substituted a value criterion for Ricardo's physical criterion and showed that the order of cultivation is dictated by costs of production, the cheapest lands being cultivated first. Since relative costs depend on distribution, the order may vary with it and is not given by Nature.

The same distinction between a physical and a value criterion underlies the difference between Ricardo's and Sraffa's conceptions of the intensification process. Ricardo considered that the intensification process consists in the investment of an additional layer of capital on an already fully cultivated land: more seeds, more manure and/or more labor are deposited on a part of that land. No supplementary rent is paid because that investment takes place on a fully cultivated land, for the use of which farmers have already paid. By contrast, Sraffa characterizes the intensification process by the coexistence of two agricultural methods, the intensive method being more productive per acre. Let us clarify the distinction by means a numerical example suggested by Ricardo. Let the initial method be

$$a_1 \text{ qr. corn} + l_1 \text{ labor} + 1 \text{ acre land} \rightarrow 100 \text{ qrs corn} \tag{1}$$

For Sraffa, the intensification process consists in the coexistence of method 1 with another method 2 on the same land

$$a_2 \text{ qr. corn} + l_2 \text{ labor} + 1 \text{ acre land} \rightarrow 185 \text{ qrs corn} \tag{2}$$

while, for Ricardo, the additional investment (which takes place after method 1) per acre of land is written

$$\Delta a \text{ qr corn} + \Delta l \text{ labor} \rightarrow 85 \text{ qrs corn} \tag{3}$$

Both formalizations are equivalent on the peculiar hypothesis $a_2 \geq a_1$ and $l_2 \geq l_1$ ($\Delta a = a_2 - a_1 \geq 0$, $\Delta l = l_2 - l_1 \geq 0$). That hypothesis is analogous to the physical criterion sustaining the ranking of lands on a natural basis. No condition of that type, however, is required when one refers to a value criterion, the only restriction set by Sraffa to the coexistence of two methods being the nonnegativity of rent .

3.2 The law of succession

What is the law determining the incoming method? For the sake of simplicity, we retain Sraffa's hypothesis of a given rate of profit and assume that wages are paid *post factum*, though these assumptions are inessential.[2] Labor serves as the numéraire. When a scarcity constraint is met, the price of corn starts rising. A first effect is that the rents on already fully lands increase (otherwise, farmers on these lands would get more than the ruling rate of profit). The second effect is that the profitability of any agricultural method which is still

[2]Sraffa's hypotheses simplify analysis because, at a given rate of profit, prices and rents are the solution to an affine system of equations. In particular, we shall use the following property (in its multisector version). A price-and-rent equation being associated with each of the $\overline{n}$ operated methods, be they marginal or not, the present equilibrium price-and-rent vector x_0 is the solution of a linear system $Ax = b$ with $\overline{n}$ equations and $\overline{n}$ unknowns. Let us increase demand and reach a breaking point. The outgoing marginal method once identified by the scarcity constraint, we delete the corresponding price equation, which leaves room for another equation (the one associated with to the still unknown incoming method). Whatever the missing equation is, the new price-and-rent vector x_1 is of the type $x_1 = x_0 + \lambda\pi$, where x_0 is the previous price-and-rent vector while vector π, which represents the direction of the change in that vector, is entirely determined by the $\overline{n} - 1$ remaining methods. Therefore the only unknown magnitude is the scalar λ, the intensity of the change. Determining the incoming method and the new price-and-rent vector amounts to choosing the right value of λ.

not operated increases.[3] The profitability of these methods being initially low (this is why they were not used), that profitability reaches the normal level at some point. Hence:

Law of succession of methods (corn model). Given an equilibrium and the evolution of demand, the outgoing method is determined by a scarcity constraint. The incoming marginal method is the first previously non-operated method which yields the ruling rate of profit when the price of corn rises, taking into account the effect on rents.

As long as the rise in the price of corn is smaller than the critical level defined by the law, there is no incentive to introduce a new method; were the rise greater, the first method we are considering would yield more than the ruling rate of profit. The level of the rise is therefore the minimum compatible with the introduction of a new method. The law could alternatively be stated as a rule of minimum rise, which follows from competition between farmers.

The important lesson is that the outgoing marginal method is determined by a physical side of the problem while the incoming method is determined by its value side. The next two Sections examine some consequences of that duality.

4 Reduction to single-product systems?

It was Ricardo's aim to extend the labor theory of value to production with lands, by getting rid of rent. The modern interpretation of the idea is to reduce the study of a productive system with lands to that a system without land. That program may not work, as shown by a possible violation of the trade-off property between wages and profits. We first examine a corn-land model with an intensification process, then a multisector model. In both cases, the difficulties stem from the value criterion referred to in the law of succession of methods.

[3] Except, possibly, for a method which might be operated on an already fully cultivated land and for which the negative effect due to the rise of rent supersedes the positive effect due to the rise of corn.

4.1 Intensification in a corn-land model

Consider the model (1)-(2). The cheapest method 1 being first used on the totality of land, Ricardo imagines that a further layer of capital represented by method Δ is deposited on a part of that land. Since method Δ pays no rent, land can be ignored and the properties of a simple corn model without land hold. In Sraffa's more general case, the intensification of production is characterized by the coexistence of two methods (1) and (2). With labor as numéraire, the price-and-rent equations associated with the simultaneous use of these methods are written

$$(1+r)a_1p + l_1 + \rho = 100p \tag{4}$$
$$(1+r)a_2p + l_2 + \rho = 185p \tag{5}$$

A rent-free relationship is obtained by subtraction

$$(1+r)(\Delta a)p + \Delta l = 85p$$

With corn as numéraire, the same equality is written as a relationship between the real wage w and the rate of profit r:

$$(1+r)\Delta a + w\Delta l = 85 \tag{6}$$

If Δa or Δl are both positive (Ricardo's hypothesis), the rent-free relationship (6) is the wage-profit equation associated with the additional investment Δ described by relation (3) and the trade-off property is obvious. But if Δa and Δl have opposite signs (Sraffa's generalisation), equality (6), which still holds, is not attached to a method of production, and it is immediately seen that the real wage and the rate of profit are positively correlated! Clearly enough, Sraffa did not see that consequence of his theory of intensive rent.

4.2 Multisector models

In multisector models, the reduction may also fail for another reason. Consider a bisector model, one corn method and one iron method being operated in the present equilibrium. When the price of corn rises because land has become scarce, the price of iron also rises as soon as corn enters into the production of iron. In its general form, the previous statement of the law of succession must be modified in order to take into account the effect of the

rise of corn on the prices of the other commodities, not only on rents (see Note 1).

Because of the rise in iron, the profitability of alternative iron methods may well improve provided that they are corn saving, and industrial methods of that type participate in the run for profitability described by the law. Suppose it wins the run, i.e. a corn-saving iron method is the first among the previously non-operated methods to get the ruling rate of profit. By the minimum rule, that iron method will be operated in the next equilibrium (in more concrete terms, the high price of oil triggers the adoption of oil-saving methods in industry). The new equilibrium sees the coexistence of two iron methods, with the progressive transfer of the production of iron from the previous method to the new one. The price equations are those associated by these two iron methods. Even if no joint production is involved, the corresponding system is not of the single-product type, since the conditions of agricultural production do not intervene in the determination of prices, even in that of corn. The prices of corn once determined by the coexistence of two iron methods, the price-and-rent equation associated with corn only defines the level of the rent on land. Such a 'non-Ricardian' equilibrium is characterized by the absence of a one-to-one correspondence between the commodities and the rent-free equations. It does not behave like a single-product system and there are no analytical grounds for a trade-off property between wages and profits.

5 Dynamics and capital theory

5.1 Sraffa's argument

We now return to a corn model. When the cultivation of corn is extended to a new land, the scarcity constraint at the origin of the change of methods is solved. Is that result also guaranteed if land is homogeneous, i.e. does the incoming method designated by the law of succession always help to increase the net product? The point deserves attention because the incoming method is chosen by means of a value instead of a physical criterion. Sraffa's argument (*PCMC*, Section 87) is that the positivity of rent ensures that the more expensive method is also more productive, the cost being calculated 'at the ruling levels of the rate of profits, wages and prices', but rent apart.

To discuss the argument, consider a corn model with intensive rent and

three methods of cultivation on a homogeneous land of total area 100 acres. Method 1 is land-intensive but the cheapest when rent is zero and its net product amounts to 40 quarters per acre. The productivity of method 2 is lower (20 quarters per acre), that of method 3 higher (60 quarters), but method 3 is costly because it is highly labour intensive. When final demand increases and reaches 4,000 quarters, the question of the incoming method arises. The obvious solution on the physical side consists in introducing the more productive method 3, not method 2. The economic problem stems from the application of the law of succession as it results from profitability considerations. Sraffa suggests, or seems to suggest, that the value criterion would exclude the introduction of the less productive method 2 jointly with method 1, because it would lead to a negative rent. Though the 'economic condition' he states does hold at any given equilibrium, the argument does not apply to the succession of equilibria: the relative cost of two methods depends on the price-and-rent vector, so that it may well be the case that method 1 is cheaper when rent is zero and, simultaneously, more expensive than method 2 when both methods are operated jointly. This is what happens in the above example: when the price of corn rises, the law of succession leads to the introduction of method 2 rather than the more costly method 3. This does not result in a contradiction on the value side, as the intensive rent stemming from the coexistence of methods 1 and 2 is positive. The difficulty lies on the physical side: the coexistence of methods 1 and 2 does not solve the scarcity problem at the origin of the change of equilibrium, because method 2 which should be progressively substituted for 1 has a lower productivity per acre.

The phenomenon at stake is strongly connected with capital theory: there is no systematic relationship between profitability and productivity and, for instance, a fall in the rate of profit which, in a neoclassical approach, is deemed to favour the introduction of more capitalistic techniques, may not lead to an increase in the product per head. The same in a corn-land model: the absence of an *a priori* connection between profitability and productivity explains why the incoming method defined by the law of succession may not solve the scarcity problem.

We arrive at a contradiction between the value side and the physical side of the problem. A first attempt to overcome it would be to modify the law of succession and to apply a minimum rule restricted to those methods which meet the physical constraint. The modified rule would imply that, once land is fully cultivated by method 1, method 3 is introduced. But the use of the

labor-intensive method 3 requires a high price of corn, so high that the non-operated method 2 would yield extra-profits, and the contradiction remains. A more systematic attempt is to proceed by drawing up the exhaustive list of all technical combinations (this is the way privileged in the static approach). As one can discard the joint use of three methods (the price of corn would be overdetermined) and that of methods 1 and 3 (method 2 would yield extra-profits), three possibilities remain:

(i) If only one method is operated, it must be the cheapest method 1. The long-term equilibrium E_1 sustained by that method can produce between 0 and 4,000 quarters.

(ii) The joint use of methods 1 and 2 sustains a long-term equilibrium, denoted E_{12} (method 3 is too costly at the associated price and rent). That equilibrium can produce between 4,000 and 2,000 quarters according to the proportions of the two methods.

(iii) The joint use of methods 2 and 3 sustains a long-term equilibrium E_{23} (the land-intensive method 1 is not profitable when rent is high) which can produce between 2,000 and 6,000 quarters.

It might seem that the last combination provides the solution to the scarcity problem when demand exceeds 4,000 quarters. This, we claim, only holds from the formal point of view retained in the static approach. The Ricardian dynamics follow the sequence of equilibria when the demand for corn increases from low to high levels. When it is low, method 1 is operated and progressively extended to the whole land, until demand reaches 4,000 quarters. A switch from equilibrium E_1 to E_{23} at that point would mean that the cultivation of the whole land by method 1 would suddenly be replaced by the joint use of methods 2 and 3, each on half of the land. That phenomenon is not consistent with Ricardo's views, which assumed a smooth physical transition at breaking points and the progressive introduction of one new method. The overall conclusion is that, in that example, the Ricardian dynamics fail.

5.2 An algebraic criterion

The condition for the working of the dynamics is that productivity goes with profitability. Let us give it an algebraic expression in the corn model defined by methods (1) and (2). Isolate first the value side. Assume that method 1 is cheaper when demand is so low that rent is zero. When land is fully cultivated, the price of corn rises and the rent becomes positive. Method 2 is

introduced if these changes improve its profitability and, at equilibrium, the price p of corn and the rent ρ per acre associated with the simultaneous use of both methods satisfy equalities

$$(1+r)a_1p + l_1 + \rho = 100p \quad (7)$$
$$(1+r)a_2p + l_2 + \rho = 185p \quad (8)$$

It is easily checked that condition

$$185 - (1+r)a_2 > 100 - (1+r)a_1 \quad (9)$$

guarantees simultaneously the positivity of rent and the existence of an incentive to let method 2 operate. Suppose that this profitability condition holds. It differs from the productivity condition ensuring that the incoming method has a higher productivity per acre:

$$185 - a_2 > 100 - a_1 \quad (10)$$

The lessons are:

- When the rate of profit is zero, the incoming method is always more productive and the Ricardian dynamics work.

- When it is positive, the working of the dynamics is submitted to a condition on the relative sign of two scalars.

The first statement expresses the duality property, or golden rule, between the quantity side and the value side when the rate of profit is zero. The second property is a further aspect of the paradoxes in capital theory (see Harcourt, 1969, for an overview of the debates)

Figures 1 and 2 illustrate the discussion for three methods on homogenous land. Method 1 is operated when rent is zero (equilibrium E_1), then method 2 jointly with method 1 (equilibrium E_{12}) and finally method 3 jointly with method 2 (equilibrium E_{23}). On the horizontal axis the activity level of the incoming method is increasing. The vertical axis shows the corresponding net products. In Figure 1 the dynamics work: 'The increase [of production] takes place through the gradual extension of the method that produces more corn at a higher cost, at the expense of the method that produces less. As soon as the former method has extended to the whole area, the rent rises to the point where a third method which produces still more corn at a still higher cost can be introduced to take the place of the method that has just been superseded' (*PCMC*, Section 88). This is a precise description of the

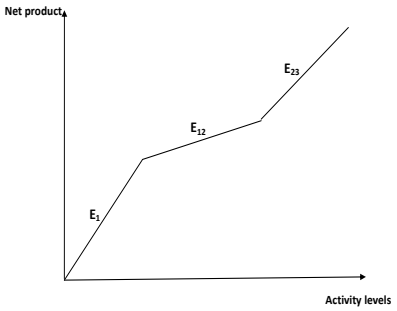

FIGURE 1. The Ricardian dynamics

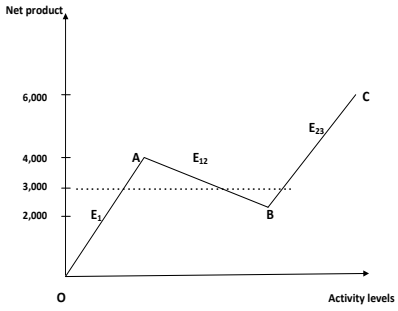

FIGURE 2. Failure of the dynamics

succession of methods *if* the incoming method is indeed more productive. Sraffa's mistakenly thought that the nonnegativity of rent suffices to ensure that property, while the condition is of another nature. Figure 2 illustrates a case when the Ricardian dynamics fail, because the incoming method 2 is less productive than the method 1 it replaces progressively.

6 The static approach

6.1 The static problem

Sraffa's work on the treatment of lands in long-term equilibria initiated many contemporary studies, starting from Quadrio-Curzio (1966). For extensive cultivation, Montani (1972, 1975) showed that the order of cultivation coincides with that of the (decreasing) wages sustained by the agricultural methods in the absence of rent. The treatment of intensive rent set more questions, some of them (e.g., is the intensive rent linked to a monopoly power of landowners?) being nowadays outdated. Kurz (1978) and Abraham-Frois and Berrebi (1980) are representative of early post-Sraffian researches. It is worth noting that post-Sraffian scholars do not identify the dynamic approach as a significant component of Ricardo's and Sraffa's methodology and that the bulk of their works adopts the static point of view: a long term-equilibrium is the solution of a system of equalities and inequalities for a given demand. Some of these equalities refer to the physical side (scarcity and demand constraints) and others to the value side (e.g., the uniformity of the rates of profit). The transformations of equilibria with demand play no role in that approach: this explains why the law of succession of methods, though at the core of the Classical approach, is not stated in the modern literature.

As a matter of fact, the Ricardian dynamics are faithfully described in most books devoted to history of economic thought but, paradoxically, are missing in those attempting to develop modern versions of the Ricardian theory (e.g. Schefold, 1989; Kurz and Salvadori, 1995; Bidard, 2004) and from papers on rent theory (e.g., collective book edited by Bidard, 1987).[4] We see two main reasons for that puzzling situation: first, Sraffa's formalization, undoubtedly a useful tool for a rigorous study of rent, remained incomplete because Sraffa did not write down the constraints relative to the 'requirements for use' (the same for multiple-product systems in general). The completion of the formalization with the explicit introduction of a demand vector provided a valuable guide for further analysis. Second, that formalization proved its efficiency by allowing the researchers to study new questions and also by casting doubts on some of Sraffa's statements: e.g., Saucier (1981) discovered the existence of non-Ricardian equilibria ('external differential rent', in his terminology) and D'Agata (1983) provided a first numerical example with multiple equilibria. Though not stated explicitly, the idea that the complexities of rent theory are linked to intersectoral relationships prevented the authors to study the corn-land model (Freni (1991) is an exception) and favored the use of heavy mathematical procedures, only partly alleviated by the use of geometrical figures. The main critique we address to modern studies is to present themselves as in line with Ricardo and Sraffa's approach, which is at least partly disputable.

6.2 Existence and uniqueness

Existence and uniqueness are typical static problems, which were treated as such respectively by Salvadori (1986) and Erreygers (1990, 1995). We would like to reinterpret these results in the light of the Ricardian dynamics.

As soon as scarce resources are required for production, the levels of demand sustained by long-term equilibria admit an upper bound. Can an estimate of that limit be guessed directly from the initial data (list of methods, areas of lands and distribution)? This is an existence problem, as it amounts

[4] A recent exception is Bidard (2010), who applied the dynamic approach to the theory of intensive rent. However, the determination of the incoming method by considering the upper envelope of a family of curves ignores the possible occurrence of non-Ricardian equilibria. This, incidentally, shows that the minimum rule may not result in a maximum wage property. Montet (1979) and Steedman (1982) have looked more specifically at the distributional aspects of rent theory, another Ricardian theme not considered here.

to ensuring the existence of an equilibrium when demand is smaller than a certain level. It was solved by Salvadori (1986; the existence set is defined formally in Appendix B) who transposed a mathematical result relative to linear complementarity problems, but that result is disconnected from dynamics. In the dynamic approach, the existence problem is linked to the law of succession which, we recall, defines the incoming method as the first non-operated method which becomes profitable when the price-and-rent vector moves in a certain direction. Obviously, the rule only applies if the move improves the profitatibility of one non-operated method at least. It can be shown that this is indeed the case when demand is not too high, and the upper limit of the demand level thus found is the one which ensures the existence of a long-term equilibrium (Bidard, 2011).

Uniqueness means that any admissible level of demand is sustained by a unique long-term equilibrium. In the dynamic approach with several commodities, infinitely many trajectories lead from a low level of demand to a given demand vector, and the uniqueness property amounts to stating that the final equilibrium is path-independent. The connection between uniqueness and dynamics becomes clear in Figures 1 and 2 associated with a corn-land model. When the Ricardian dynamics work (Figure 1), the incoming method always sustains an increase in production, and therefore there is a one-to-one correspondence between an equilibrium and a range of levels of demand. Figure 2 illustrates the opposite case: the dynamics fail because the incoming method in equilibrium E_{12} leads to a decrease of production and, then, intermediate levels of demand are sustained by multiple equilibria (three equilibria for $d = 3,000$). In multisector models, the general condition for local uniqueness (i.e., when comparisons are restricted to techniques which differ by one method α or β) is expressed as a sign equality (E) between two magnitudes $\varepsilon_\alpha = e_\alpha(r)/e_\alpha(0)$ and $\varepsilon_\beta = e_\beta(r)/e_\beta(0)$. That condition, first stated by Erreygers (1990), is more easily found in a dynamic approach and its interpretation is clearer: as in equalities (9) and (10) above, the scalars involving r are relative to profitability, while the same expressions for $r = 0$ concern productivity. The assumed sign equality amounts to setting that both phenomena go together. Moreover, the criterion admits a global version (Erreygers, 1990, 1995). It turns out that global uniqueness holds if and only if the Ricardian dynamics work everywhere, i.e. if the productivity condition holds at every change of method. Therefore, the dynamic approach also shows that the underlying cause of the multiplicity of equilibria at a given level d of demand is the failure of the Ricardian dynamics at some point

when one tries to link a low-level equilibrium with an equilibrium of level d. (Sceptical readers are invited to check that statement on any numerical example of multiplicity.)

7 Ricardo and modern mathematics

Ricardo is an unknown pioneer of a fruitful modern mathematical method commonly used to solve linear complementarity problems.

Complementarity problems are a family of problems frequently met in theoretical and applied mathematics: thousands of papers have been written on the topic in the last fifty years, and potentially every field of science has its own complementarity problems (Facchinei and Pang, 2003). The unknowns of a complementarity problem are nonnegative variables: in economics, they are typically activity levels and quantities on the physical side, prices on the value side. Complementarity means that the problem is expressed in terms of equalities (e.g., all operated methods yield the same rate of profit) and inequalities (e.g., the cultivated areas do not exceed the available areas) and, when some inequality is strict, the dual variable attached to it is zero (rent is zero on non fully cultivated lands). General equilibrium is the most famous complementarity problem in economics: the inequalities express that the excess supply on each market is nonnegative and, if the inequality is strict for some good, its price is zero. Though the existence of a solution can be dissociated from its calculation (e.g., the existence of a general equilibrium is proved independently of the convergence of the tâtonnement process, which may fail), the question of the effective determination of a solution remained open for a long time, even in the simplest case of linear complementary problems, i.e. when the equalities and inequalities are linear functions of the unknowns (Cottle *et al.*, 1992). A long-term equilibrium with lands is the solution of a bimatrix game, which is a specific type of linear complementarity problem (Salvadori, 1986).

It is in 1965 that Lemke found an algorithm to calculate a solution of a bimatrix game. Lemke's method is close to the famous simplex algorithm used in linear programming and the discovery drew immediately the specialists' attention. Several extensions and variants were soon found. One of its variants, called the parametric Lemke algorithm, consists in making the problem one considers depend on a parameter. Though the original Lemke algorithm and its parametric version were elaborated for mathematical purposes only,

it turns out that the parametric method admits an economic interpretation when applied to the land problem. The parameter then considered is the demand vector d. The mathematical strategy to find an explicit solution for a given vector d consists in starting from a simple solution corresponding to another vector d_0 (for instance, a level of demand so low that the scarcity of lands can be ignored) and to follow the transformations of the initial solution when the parameter moves along a path joining d_0 to d. If the transfer works, the problem is solved for vector d. It is immediately recognised that the stategy used in the parametric Lemke algorithm coincides with Ricardo's dynamic approach. The reason of the mathematical efficiency of the method is that, most of the time, small changes in the demand vector only need minor adaptations of the solution (adaptation of activity levels). It is only at a finite number of points that, in mathematical terms, a 'change of basis' is required, the new basis being obtained by 'pivoting', a procedure involving a change in one 'basic variable', the new basic variable being identified mechanically by applying a minimum rule. In the economic interpretation, one marginal method is changed when a scarcity constraint is met, and the change obeys the law of succession.

There exists, however, a difference between the Ricardian dynamics and the parametric algorithm, as mathematicians have recognized the existence of a potential difficulty in the working of the algorithm: the new basis may not allow to go further on the path initially drawn from d_0 to d. In the land model, this occurs when the new method is less productive than the one it replaces: the dynamics stop at this stage. By contrast, the rule adopted by the mathematicians let the algorithm make a U-turn on the path ('antitone move'). In Figure 2, this means that the parametric Lemke algorithm starts by following path OA, reaches a local maximum (4,000 quarters) at point A, then continues by reducing demand along path AB (equilibrium E_2) down to 2,000 quarters, and eventually follows path BC (equilibrium E_3). The reduction of demand at an intermediate stage allows the algorithm to find solutions for high levels in a further step. The method is mathematically powerful, but the temporary reduction of demand has no economic interpretation.

8 Conclusion

Since its elaboration two centuries ago, the Ricardian theory of rent has been the subject of controversial readings. The present overview of the topic is in

line with Sraffa's interpretation of Ricardo but shows that the economic phenomena are more complex than these authors themselves thought. Its main messages are: (i) the reaction of the economic system to a physical scarcity is dictated by the law of succession of methods, which only takes into account the evolution of prices and rents; (ii) the associated physical phenomena may be complex: a shortage of corn may lead to the introduction of a new iron method or to the cultivation of barley on a land already cultivated with oat; and all these reactions may, or may not, fit the evolution of final demand; (iii) the difficulties of rent theory occur even in simple frameworks; (iv) Sraffa did not draw all consequences of his analysis and did not see that the use of a value criterion opens the door to all complications of capital theory; finally, (v) the dynamic approach followed by Ricardo and Sraffa is richer than the static framework mainly adopted in the last fifty years.

9 Appendix A: Fallowing and dynamics

The physical limit which marks the end of an equilibrium when demand increases seems to take different forms: for extensive cultivation, it is reached when some land becomes fully cultivated (upper bound); for intensive cultivation with several agricultural products, when the activity level of some method vanishes (lower bound). That duality is rather awkward and we propose to refer to a unique criterion, which is a nonnegativity condition (lower bound). This is achieved by considering fallowing as an agricultural method.

Fallowing is a method with the following characteristics: with some area of land as the only input, its product consists in the same amount of land and no final good. The price equation associated with it shows that fallowing is operated only if rent is zero. With the convention regarding fallowing as an agricultural method, a non fully cultivated land is now seen as a land on which fallowing is operated, and a land which becomes fully cultivated as a land on which the activity level of fallowing vanishes. Then the universal criterion to identify the physical limit of an equilibrium is that the activity level of some operated method, be it fallowing or another method, drops to zero.

Considering fallowing as a specific method is consistent with the representation of the Ricardian dynamics as a process of substitution of a new method for another when demand changes. According to the usual approach, the extension of cultivation means that more and more lands are cultivated,

with more and more operated methods. But when non-cultivated lands are considered as those on which fallowing is operated, the same phenomenon is interpreted as the replacement of fallowing by another method. The number of operated methods remains constant all along the dynamical process, equal to the sum of the total number of commodities and that of lands ('squareness'). This is why the knowledge of the operated methods and distribution allows us to determine all prices and rents, including the zero rents. The only exception to squareness occurs at breaking points, when the activity level of some method vanishes and therefore the number of operated method falls by one of its usual value.

10 Appendix B: Long-term equilibrium

Let there be m methods of production with constant returns, n produced commodities, k types of lands and homogenous labour. Method i ($i = 1,...,m$) makes use of a vector $a_i \in R^n_+$ of material inputs, a vector $\Lambda_i \in R^k_+$ of lands ($\Lambda_i = 0$ for industrial methods and Λ_i has but one positive component for agricultural methods, but these facts play no role) and an amount $l_i \in R_+$ of labour (it is assumed that labour is directly or indirectly necessary for the production of a net product). The product, obtained after one period, is a basket of commodities represented by the vector $b_i \in R^n_+$. These technical data can be stacked as (A, Λ, l, B), where A and B are matrices of dimension (n, m), Λ a matrix of dimension (k, m) and l is an m-vector. Let $\overline{\Lambda} \in R^k_+$ the k-vector representing the available areas of the various qualities of lands. Under Sraffa's hypotheses, the rate of profit per period r is given and wages are paid at the end of the period.

Let the final demand be represented by a vector $d \in R^n_+$. The unknowns which characterise a long-term equilibrium sustaining that final demand are: (i) the vector $y \in R^m_+$ of activity levels of the various methods; (ii) the price vector $p \in R^n_+$ of commodities, with labour as numéraire; (iii) the rent vector $\rho \in R^k_+$ per acre of the various types of lands. A long-term equilibrium is a solution to the system of linear inequalities with complementarity relationships

$$(B - A)y \geq d \quad [p] \tag{11}$$

$$\Lambda y \leq \overline{\Lambda} \quad [\rho] \tag{12}$$

$$(1 + r)A^T p + \Lambda^T \rho + l \geq B^T p \quad [y] \tag{13}$$

Inequality (11) means that final demand is met by the net product of the economy, the goods in excess being zero-priced. Inequality (12) expresses the scarcity constraints on lands, with a zero rent on non-fully cultivated lands. Inequality (13) means that no method yields more than the ruling rate of profit, and that only methods which yield that rate may be operated. Formalization (11)-(12)-(13) is the one referred to in Section 6.1. An alternative but equivalent formalization consists in inserting free disposal and fallowing in the list of available methods: then inequalities (11) and (12) are transformed into equalities. The choice of the more adequate formalisation is merely a matter of convenience, but the analytical aspects of the dynamic approach are more easily managed in terms of equalities: the only significant difference between the static and the dynamic points of view lies in the way to solve the system, not in the way to write it down.

The set D of demand vectors for which existence is ensured (Salvadori, 1986) and which, according to the argument developed in Section 6.2, coincides with the set on which the working of the law of succession is guaranteed, is

$$D = \left\{ d; \exists y > 0 \;\; d << (B - (1+r)A)y \text{ and } \Lambda y << \overline{\Lambda} \right\} \tag{14}$$

(symbol $<<$ means strict inequality componentwise). An economic interpretation is that D would be the set of strictly feasible final demand vectors if the rate of accumulation were r.

In the dynamic approach, an equilibrium corresponding to final demand d in D admits generically $n+k$ operated methods, the corresponding columns $(\widehat{A}, \widehat{\Lambda}, \widehat{l}, \widehat{B})$ describing the active part of the economy. Let $e(r)$ be the determinant of the square matrix $(\widehat{B} - (1+r)\widehat{A}, -\widehat{\Lambda}, -\widehat{l})$. The color, white or black, of an equilibrium, is defined by the relative sign $\varepsilon = \pm 1$ of $e(r)$ and $e(0)$. Uniqueness is guaranteed if all equilibria are white (Erreygers, 1990). In the dynamic approach, the dynamics work locally when the two neighboring equilibria, i.e. which differ by one method, have the same color. It can be shown that, generically, the number of white equilibria exceeds by one that of black equilibria (Bidard, 2011; that property extends the result established by Bidard and Erreygers (1998) for multiple-product systems with no scarce resources, and is clearly of the same nature as the one concerning general equilibrium in a differentiable framework (Dierker, 1972)).

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