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Abstract

The parametric Lemke algorithm is used to show the existence of an odd number of solutions of a generalized bimatrix game in a certain domain. These solutions are classified into two types according to the relative sign of two determinants. The British economist David Ricardo made an implicit use of that algorithm at the beginning of the nineteenth century.

Keywords. Complementarity problems, Generalized bimatrix game, Oddity, Parametric Lemke algorithm, Ricardo.

AMS classification. 01A55, 90-03, 90C33, 91B60

1 Introduction

The[1] Lemke algorithm was first elaborated to find solutions of bimatrix games. One of its variants, the parametric Lemke algorithm, follows the deformations of a solution $(w(t), z(t))$ of a linear complementarity problem $LCP(q, M)$ when $q = q(t)$ varies with a parameter t. An important property is that the distribution of the zeroes in $w(t)$ and $z(t)$ remains the same by intervals. A critical point is reached when some positive component of $w(t)$ or $z(t)$ vanishes and, then, a switch in one basic variable is required. As long as a new basic variable can indeed be defined at critical points, a solution obtained for $q = q(0)$ allows us to obtain a solution for $q = q(1)$. We return to that algorithm for cross-dual games, a nonlinear extension of bimatrix

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games (Sections 2 and 3). We show the connection between its working and the existence of an odd number of solutions in a certain domain, with a classification of the solutions into two types according to the relative sign of two determinants (Section 4). That (generic) oddity result holds in particular for bimatrix games. In the historical Section 5, we attribute the idea of the algorithm to the economist David Ricardo who applied it to the land problem two centuries ago and we illustrate the economic problem linked to the presence of antitone moves.

2 The cross-dual game

By cross-dual game, we mean a complementarity problem of the type

$$f(x) \geq c \quad [y] \tag{1}$$
$$g(y) \geq d \quad [x] \tag{2}$$
$$x \geq 0, y \geq 0 \tag{3}$$

where x is an $n \times 1$ vector in an open set $\Omega_f \subset R^n$, with $R^n_+ \subset \Omega_f$, $f : \Omega_f \rightarrow R^l$ a continuous function, c an $l \times 1$ vector, y an $l \times 1$ vector in an open set Ω_g, with $R^l_+ \subset \Omega_g$, $g : \Omega_g \rightarrow R^n$ a continuous function, and $d \in R^n$ is an $n \times 1$ vector (notation ≥ 0 means nonnegativity, > 0 semipositivity, $>> 0$ positivity). The bimatrix game is obtained when f and g are respectively represented by $l \times n$ and $n \times l$ matrices A and B. It is assumed that c is negative

$$c << 0 \tag{4}$$

The following Theorem generalizes a well known existence result for bimatrix games, the extension of assumption $A + B^T \geq 0$ being

$$\forall x \geq 0 \;\; \forall y \geq 0 \;\; y^T f(x) + x^T g(y) \geq 0 \tag{5}$$

The proof makes use of the (simplified) Gale-Nikaido-Debreu lemma:

Lemma 1 *(GND Lemma). Let S be a compact convex set in R^m and $z : p \in S \rightarrow z(p) \in R^m$ be a continuous function satisfying the Walras identity $p^T z(p) = 0$. There exists a vector $p_0 \in S$ such that $p^T z(p_0) \geq 0$ for any $p \in S$.*

If S is the unit simplex, the conclusion is that p_0 is a solution of the complementarity problem $z(p) \geq 0$, $p \geq 0$, $p^T z(p) = 0$. The following proof (same idea as in Bidard (2011)) forces the Walras identity by considering the function $\overline{z}(p,t) = (tz(t^{-1}p), -pz(t^{-1}p))$

Theorem 1 *Let $f : R^n_+ \to R^l$ and $g : R^l_+ \to R^n$ be continuous functions satisfying condition (5). Let f be homogeneous of degree one, vector $c \in R^l_-$ be negative and vector $d \in R^n$ be such that*

$$\{x > 0, f(x) \geq 0\} \Rightarrow d^T x < 0 \tag{6}$$

Then the cross-dual game (1)-(2)-(3) admits a solution.

Proof. For $\varepsilon \geq 0$, let the simplex S_ε in $R^n_+ \times R^l_+ \times R_+$ be defined by

$$S_\varepsilon = \left\{ (x,y,t); x \geq 0, y \geq 0, t \geq \varepsilon, \sum_i x_i + \sum_j y_j + t = 1 \right\}$$

For any $\varepsilon > 0$, the continuous function $z : S_\varepsilon \to R^n \times R^m \times R$ defined by

$$z(x,y,t) = (tg(t^{-1}y) - td, f(x) - tc, x^T d + y^T c - x^T g(t^{-1}y) - y^T f(t^{-1}x)) \tag{7}$$

satisfies the Walras identity $(x,y,t)^T z(x,y,t) = 0$ on S_ε. According to the GND lemma, there exists a point $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in S_\varepsilon$ such that inequality

$$\forall (x,y,t) \in S_\varepsilon \quad (x,y,t)^T z(x_\varepsilon, y_\varepsilon, t_\varepsilon) \geq 0 \tag{8}$$

holds. Taking into account assumption (5), we obtain in particular that

$$\forall (0,y,t) \in S_\varepsilon \;\; y^T (f(x_\varepsilon) - t_\varepsilon c) + t(x_\varepsilon^T d + y_\varepsilon^T c) \geq 0 \tag{9}$$

Consider a cluster point $(x_0, y_0, t_0) \in S_0$ of $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ when ε tends to zero. By continuity, property (9) also holds for $\varepsilon = 0$, therefore $f(x_0) - t_0 c \geq 0$ and $x_0^T d + c y_0^T \geq 0$. Suppose $t_0 = 0$. If $x_0 > 0$, a contradiction is obtained with assumption (6) and $c << 0$. If $x_0 = 0$, then $y_0 > 0$ and a contradiction is again obtained with assumption $c << 0$. Therefore $t_0 \neq 0$ and property (8) can be extended by continuity to $\varepsilon = 0$. This shows that $g(t_0^{-1} y_0) - d \geq 0, f(t_0^{-1} x_0) - c \geq 0, x_0^T d + y_0^T c - x_0^T g(t_0^{-1} y_0) - y_0^T f(t_0^{-1} x_0)) \geq 0$, and $(t_0^{-1} x_0, t_0^{-1} y_0)$ is a solution of the cross-dual game. ■

In the remaining of the paper, we shall not use Theorem 1 directly but proceed to a constructive proof. It will be assumed that f is linear and represented by matrix A. After introducing an asymmetric treatment of inequalities (1) and (2), inequality (2) will be replaced by an equality. Vector d can then be interpreted as a parameter in the problem $CD(d)$. Note first that, when d is negative, the solution of $CD(d)$ is unique:

Lemma 2 *Under hypotheses (4) and (5), the unique solution of $CD(d)$ for $d << 0$ is $x = 0$, $y = 0$.*

Proof. Complementarity implies that $y^T c + x^T d = y^T f(x) + x^T g(y) \geq 0$, therefore $x = 0$ and $y = 0$. Conversely, $(x = 0,\ y = 0)$ is indeed a solution since inequality (5) implies $f(0) \geq 0$ and $g(0) \geq 0$. ■

Let the functions $\overline{f} : \Omega_f \to R^l \times R^n$ and $\overline{g} : \Omega_g \times R^n \to R^n$ be respectively defined by

$$\overline{f}(x) = \begin{pmatrix} f(x) \\ x \end{pmatrix} \tag{10}$$

$$\overline{g}(y,z) = g(y) - z \tag{11}$$

and $\overline{c}$ be the $(l+n) \times 1$ vector c once completed by n zeroes. By setting $\overline{y}^T = (y^T, z^T)$, we have $\overline{y}^T \overline{f}(x) + x^T \overline{g}(\overline{y}) = y^T f(x) + x^T g(y)$, assumption (5) implies

$$\forall x \geq 0 \;\; \forall \overline{y} \geq 0 \;\; \overline{y}^T \overline{f}(x) + x^T \overline{g}(\overline{y}) \geq 0 \tag{12}$$

A solution (x, y) of the cross-dual game $CD(d)$ generates a solution $(x, \overline{y})$ of the system

$$\overline{f}(x) \geq \overline{c} \quad [\overline{y}] \tag{13}$$

$$\overline{g}(\overline{y}) = d \tag{14}$$

$$\overline{y} \geq 0 \tag{15}$$

where $\overline{y}$ is the $(l+n) \times 1$ vector obtained by stacking vectors y and $g(y) - d$. Conversely, a solution $(x, \overline{y})$ of (13)-(14)-(15) generates a solution (x, y) of the initial system, where vector y consists of the first n components of $\overline{y}$.

An n-set $K \subset \{1, ..., l+n\}$ is said to sustain a solution of $CD(d)$ if it is the support of a nonnegative vector $\overline{y}$ (i.e., the set of its nonzero components) in a solution of the equivalent problem (13)-(14)-(15). Let R_K^{l+n} is the set of the $(l+n)\times 1$ vectors with support K. Given a subset $S\subset\{1,...,l+n\}$ with cardinal s, $\overline{f}_S:\Omega_f\rightarrow R^s$ is the projection of $\overline{f}$ on R^S, i.e. it is the function defined by its components $\overline{f}_i$ for $i\in S$. Function $\overline{g}_K$ is the restriction of function $\overline{g}$ to vectors with support K. $\left[J\overline{f}_K\right]$ (respectively $[J\overline{g}_K]$) is the Jacobian matrix of $\overline{f}_K$, $J\overline{f}_K$ (respectively $J\overline{g}_K$) its determinant (f and g are now assumed to be of class C^1). When f is linear and represented by an $l\times n$ matrix A, $\overline{f}$ is represented by $(l+n)\times n$ matrix $\overline{A}$ obtained by stacking A and the identity matrix I_n, and $\left[J\overline{f}_K\right]=\overline{A}_K$ is the $n\times n$ sub-matrix of $\overline{A}$ made of the rows corresponding to the set K.

3 The parametric Lemke algorithm

We consider the generic case, with no specific algebraic relationship between the data $(\overline{A},\overline{g},c,d)$. Flukes apart, the complementarity relationship in inequality (13) requires that the number n of components of x is at least equal to the number of positive components of $\overline{y}$, while equality (14) requires that the number of positive components of $\overline{y}$ is at least equal to n. The number of positive components of $\overline{y}$ is therefore generically equal to n (if $\overline{g}(0)=0$, as in a bimatrix game, the point $d=0$ is an exception, since $\overline{y}=0$ works) and the number of strict inequalities in (13) is generically equal to l. If $\overline{A}_K$ and $\overline{g}_K$ are injective, each arbitrary n-set K sustains at most one solution of (14) and of (13), therefore the number of solutions is finite: finding a solution amounts basically to identifying the n-set K of the positive components of vector $\overline{y}$. For that purpose, we make use of the parametric Lemke algorithm (Cottle *et al.*, 1992), where vector d is a parameter. The algorithm follows the deformations of a solution $(x(d),\overline{y}(d))$ of the cross-dual game $CD(d)$ when d varies, with a specific attention given to the support of $\overline{y}(d)$. An important property is that the support remains the same by intervals. A critical point is reached when some positive component of $\overline{y}(d)$ vanishes and, then, a switch in one basic variable is required. As long as a new basic variable can indeed be defined, a solution obtained for $d_0=d(0)$ allows us to obtain a solution for another vector d. The following Lemma reinterprets assumption (6) as a condition for the working of the algorithm.

Lemma 3 *Assume (5). Let d belong to the set $\mathcal{D}$*

$$\mathcal{D}=\left\{d;\exists\widehat{z}\geq 0\quad d<<-A^T\widehat{z}\right\}\tag{16}$$

and let $\overline{y}$ be a solution of $CD(d)$. If $\det \overline{A}_K \neq 0$ and $J\overline{g}_K(\overline{y}) \neq 0$, the parametric Lemke algorithm works in a neighborhood of d, except by fluke.

Proof. Let $(x_0, \overline{y}_0)$ be a solution of (13)-(14)-(15) at d_0, and assume first that $\overline{y}_0$ has n positive components (set K). Since $\overline{g}_K$ is a local diffeomorphism, a slight change in d is met by an adaptation of the positive components of $\overline{y} = \overline{y}(d)$, with no change in x_0. Assume now that some positive component k of $\overline{y}(d) = \overline{g}_K^{-1}(d)$ vanishes at point $d_1 \in \mathcal{D}$ for $\overline{y} = \overline{y}_1$ ('critical point'). The component k which vanishes is unique, flukes apart. For $L = K \setminus \{k\}$, let $\overline{A}_L$ be the corresponding rows of $\overline{A}$. $\overline{A}_L$ has dimension $(n-1) \times n$ and the set of solutions to equation $\overline{A}_L x - \overline{c}_L = 0$ is $x = x_0 + \lambda x'$ where λ is an arbitrary scalar and x' a nonzero vector in the kernel of $\overline{A}_L$. Choose x' such that $\overline{A}_k x' > 0$ (otherwise, replace x' by $-x'$), so that the kth inequality in (13) with $\overline{f} = \overline{A}$ is met for any nonnegative scalar λ. Consider the complementary subset K' of K, for which the strict inequality $\overline{A}_{K'} x_0 >> \overline{c}_{K'}$ holds except by fluke, and assume for a moment that inequality

$$\overline{A}_{K'} x' \geq 0 \tag{17}$$

also holds. Then, by definition of the nonzero vector x',we have $\overline{A}x' \geq 0$, hence $Ax' \geq 0$ and $x' \geq 0$. Since x' is nonzero, we obtain from (16) that $d_1^T x' < -\widehat{z}^T A x' \leq 0$, and from $\overline{g}_L(\overline{y}_1) = d_1$ that $\overline{g}_L(\overline{y}_1)^T x' < 0$. Let x'' be the semipositive vector x' completed by l zeroes. According to property (12) applied to $x = x''$ and $y = \overline{y}_1$, we have $\overline{y}_1^T \overline{A} x'' \geq -\overline{g}(\overline{y}_1)^T x'' = -\overline{g}_L(\overline{y}_1)^T x' > 0$, therefore $\overline{y}_1^T \overline{A}_L x' > 0$. A contradiction being obtained with the definition of x', inequality (17) does not hold, i.e. there exists at least one row m outside K such that $\overline{A}_m x' < 0$. Consider the smallest positive value of λ for which one of the strict inequalities $\overline{A}_{K'} x_0 >> \overline{c}_{K'}$ is turned into an equality when x_0 is replaced by $x_1 = x_0 + \lambda x'$ (flukes apart, that inequality is uniquely determined). Then $(x_1, \overline{y}_1)$ is another solution of $CD(d_1)$. When the positive components of $\overline{y}$ remain close to those of $\overline{y}_1$, but its mth component $\overline{y}_m$ becomes positive (half-neighborhood of $\overline{y}_1$,with support $M = L \cup \{m\}$), $(x_1, \overline{y})$ sustains a solution of $CD(d)$ for d varying in a half-neighborhood of d_1. ■

We now examine whether the half-neigborhoods associated with each of the n-sets K and M coincide or are complementary. These n-sets have the $(n-1)$-set L in common.

Lemma 4 *The determinants of the $n \times n$ matrices $\overline{A}_K$ and $\overline{A}_M$ have opposite signs.*

Proof. $\overline{A}_K$ and $\overline{A}_M$ have the submatrix $\overline{A}_L$ in common, which is such that $\overline{A}_L x' = 0$. They only differ by one row, respectively $\overline{A}_k$ or $\overline{A}_m$ and, by construction, $\overline{A}_k$ and $\overline{A}_m$ are such that $\overline{A}_k x' = a > 0$ and $\overline{A}_m x' = b < 0$, therefore $(b\overline{A}_k - a\overline{A}_m)x' = 0$. There follows that $(b\overline{A}_K - a\overline{A}_M)x' = 0$, hence $\det(b\overline{A}_K - a\overline{A}_M) = 0$ and the result. ■

Lemma 5 *Let d be close to a critical point $d_1 = \overline{g}(\overline{y}_1)$. The components of $\overline{g}_K^{-1}(d)$ (respectively, $\overline{g}_M^{-1}(d)$) other than the kth (respectively, the mth) are positive. The components $(\overline{g}_K^{-1}(d))_k$ and $(\overline{g}_M^{-1}(d))_m$ have the relative sign as $J\overline{g}_K(\overline{y}_1)$ and $J\overline{g}_M(\overline{y}_1)$.*

Proof. By continuity, the components of $\overline{y} = \overline{g}_K^{-1}(d)$ are close to those of $\overline{g}_K^{-1}(d_1)$, therefore they are close to zero for the kth component and positive for the others. For $i \in K$, let b_i be the gradient of $\overline{g}_i$ at $\overline{y}_1$. The algebraic equality $d - d_1 = \sum_{i \in K} (\overline{y} - \overline{y}_1)_i b_i$ holds up to a first order approximation, from which there follows that the scalar $\det(b_i, i \in L; d - d_1)$ has the same sign as $(\overline{y} - \overline{y}_1)_k \det(b_i, i \in L; b_k) = (\overline{g}_K^{-1}(d))_k J\overline{g}_K(\overline{y}_1)$. The same scalar has also the sign of $(\overline{g}_M^{-1}(d))_m J\overline{g}_M(\overline{y}_1)$. The result follows. ■

Definition 1 *Let K be an n-set sustaining a solution of the cross-dual game $CD(d)$. That solution is called white if $\det(-\overline{A}_K)$ and $J\overline{g}_K(\overline{y})$ have the same sign, black if they have opposite signs.*

For instance, the unique solution obtained for $c << 0$ and $d << 0$ corresponds to the subset $K = \{l+1, ..., l+n\}$ for which $\overline{A}_K = I_n$ and $\overline{g}_K = -I_n$, therefore that solution is white.

Consider a path $d = d(t)$ in the set $\mathcal{D}$, a solution $(x_0, \overline{y}(t))$ of $CD(d(t))$ for $t \in S = [t_1 - \eta, t_1[$ for which the support of $\overline{y}(t)$ is the set K, with $\overline{y}_k(d(t))$ vanishing at $d(t_1) = d_1$. By Lemma 3, there exists an n-set M obtained from K by replacing k by an adequately chosen m, such that M is the support of another solution $(x_1, \overline{y}')$ of $CD(d(t))$ for t varying in one of the semi-intervals S or $T =]t_1, t_1 + \eta]$ (the semi-interval for which $\overline{y}'_m(d(t))$ is positive). Both values $\overline{y}_k(t) = \overline{y}_k(d(t))$ and $\overline{y}'_m(t) = \overline{y}'_m(d(t))$ change their sign at $t = t_1$ when $d(t)$ crosses the hypersurface corresponding to $\overline{y}_k = \overline{y}'_m = 0$, therefore $\overline{y}_k(t)$ and $\overline{y}'_m(t)$ have either always opposite signs or always the same sign on $]t_1 - \eta, t_1 + \eta, [$. Assume first they have opposite signs. For the first solution with support K, $\overline{y}_k(t)$ is positive when t is smaller than t_1, therefore $\overline{y}'_m(t)$ is negative on S and positive on T. This means that M sustains a solution when t continues to move in the same direction after t_1. By Lemma 5, this case occurs when the Jacobians $J\overline{g}_K(\overline{y}_1)$ and $J\overline{g}_M(\overline{y}_1)$ have opposite signs or, taking into account Lemma 4 and Definition 1, when the successive sets K and M have the same color. On the contrary, if $\overline{y}_k(t)$ and $\overline{y}_m^{'}(t)$ have the same sign, M sustains another solution on the same half-interval as K, i.e. when $d = d(t)$ makes a U-turn on the path (antitone move). That change of direction occurs when K and M have opposite colors.

4 Main result

Consider $CD(d)$ and let $d = d(t)$ move on an oriented curve in $\mathcal{D}$ which does not cross itself, starting from $d_0 << 0$, going to a given point d, and coming back to $d_1 << 0$. Starting from the unique solution for d_0, that solution is transferred along the path. Even in the presence of antitone moves, the important property of the algorithm is that the same solution is not found twice (the law of succession of the sets K, M, ..., being uniquely defined along the path and reversible, a contradiction would be obtained by considering the first set which appears twice in that sequence), therefore the algorithm starting at d_0 goes first to d, then reaches d_1. (The algorithm starting from d_1 follows the reverse path.) Each time the algorithm reaches d, a new solution of $CD(d)$ is obtained. Taking into account the connection between the change of color and the change of direction on the path, we obtain that the solutions thus defined, which we call the accessible solutions (i.e., they are reached by following the given path $d(t)$), satisfy the following existence and oddity result:

Definition 2 *Functions $f = A$ and g are called regular if, for any n-set $K \subset \{1, ..., l+n\}$, $\overline{A}_K$ and $\overline{g}_K$ are diffeomorphisms.*

Theorem 2 *Let A and $\overline{g}$ be regular and such that condition (5) holds. Let $c << 0$. For d in the set $\mathcal{D}$ defined by (16), and flukes apart, the number of white solutions of the cross-dual game $CD(d)$ exceeds by one the number of black solutions.*

Proof. Let us start from a given solution at d, which is sustained by a set K. By Lemma 3, it can be transferred along the path. If either d_0 or d_1 is reached ultimately, the solution belongs to the unique sequence of accessible solutions, for which the oddity result holds. If not, the sequence starting from K makes a loop and returns to K, hence to d. Considering the corresponding move on the path $d(t)$, that loop sustains as many white as black solutions at d. The non accessible solutions are therefore partitioned into subsets containing as many solutions of each color. Hence the result when $n \geq 2$. The case $(n = 1, d > 0)$ is treated apart as any curve from $d_0 < 0$ to $d > 0$ then to $d_1 < 0$ makes a to-and-fro movement and crosses itself. Then the inequalities (1) are written $a_1 x \geq c, ..., a_l x \geq c$ and the only constraint which matters is the one corresponding to the smallest a_i, therefore we may assume $l = 1$. The system is reduced to the scalar inequalities

$$\begin{aligned} ax &\geq c \quad [y] \\ g(y) &\geq d \quad [x] \\ x &\geq 0, y \geq 0 \end{aligned}$$

with $c < 0$, $d > 0$, $ay + g(y) \geq 0$ for any $y \geq 0$ and $a < 0$ (otherwise, no scalar $d > 0$ belongs to $\mathcal{D}$). If $g(0) < d$, the set of solutions is $S = \{(x, y); x = c/a, g(y) = d)\}$; if $g(0) > d$, it is $(x = 0, y = 0) \cup S$. Since the color of a solution in S depends on the sign of $g'(y)$ when $g(y) = d$, the result holds in both cases (the limit case $g(0) = d$ corresponds to a degeneracy). ■

One may wonder if the proof of Lemma 3 and the oddity result can be extended to the case when f is nonlinear (then, one must distinguish between the local behavior of f associated with its Jacobian matrix and its global behavior). The proof.of the Lemma makes use of the following property: consider the curve $(C) = \left\{x; , \overline{f}_L(x) = \overline{c}_L\right\}$ which we also parameterize as $x = x(\lambda)$ with $x(0) = x_0$, and choose a direction on that curve such that $\overline{f}_k(x(\lambda))$ is locally increasing when λ becomes positive. Then, for any i, function $\overline{f}_i$ varies monotonously on the oriented curve and, if it is decreasing, its decreases to $-\infty$.That property holds when f is linear (then (C) is a straight line) but its extension to the nonlinear case seems rather artificial.

It follows from Theorem 2 that global uniqueness is obtained if and only if any solution is white (Erreygers, 1995). A reason why the oddity property has not been stated previously for bimatrix games is that it is not apparent, as bimatrix games may have an even number of solutions: some of them are located outside the set $\mathcal{D}$ on which the working of the parametric Lemke algorithm and the existence of a solution are both guaranteed.

5 A pioneer

5.1 Ricardo on lands

We would like to draw attention on the connection between the work of the British economist David Ricardo (1772-1823) who, in the *Principles* (1817), elaborated a theory of long-term prices and rents, and the parametric Lemke algorithm. We follow here a modern formalization of Ricardo's theory mainly due to Sraffa (1960). The data are the set of methods of production, the rate of profit r per year ($r \geq 0$), the vector δ ($\delta \geq 0$) of demand for consumption and the vector $\overline{\Lambda}$ of scarcity constraints on lands. Let there be g goods and h qualities of lands. The ith method of production is represented by a vector $u_i \in R_+^g$ of material inputs, a vector $\lambda_i \in R_+^h$ of land inputs and a quantity $-c_i \in R_{++}$ of labor ($c << 0$), and the product obtained one year after investment is represented by a vector $v_i \in R_+^g$. Nonnegative combinations of these l methods are allowed. The unknowns are the price vector (the wage being set equal to one) $p \in R_+^g$, the vector $\rho \in R_+^h$ of rents per acre of lands and the vector $y \in R_+^m$ representing the activity levels of the methods. A long-term equilibrium is a solution of the system

$$\begin{aligned} -V^T p + (1+r)U^T p + \Lambda^T \rho &\geq c \quad [y] && (18) \\ -\Lambda y &\geq -\overline{\Lambda} \quad [\rho] && (19) \\ (V-U)y &\geq \delta \quad [p] && (20) \end{aligned}$$

Condition (18) states that all operated methods yield the ruling rate of profit, while non-operated methods do not yield more. Condition (19) expresses the scarcity constraints on lands, and the complementarity relationship means that non fully cultivated lands yield a zero rent (competition between landowners). Condition (20) means that the demand δ for consumption is met by the net product, the overproduced commodities being zero priced. By introducing vector x obtained by stacking p and ρ, the problem is transformed into a bimatrix game with $A = \left[(1+r)U^T - V^T, \Lambda^T\right]$ and $B = [V-U, -\Lambda]$ for which condition $A + B^T \geq 0$ is met. The parallel was pointed out by Salvadori (1986), who applied the existence result to the economic problem.

Ricardo was interested in the effect of an increase of demand δ (especially an increase in the demand for corn) due to the increase in the number of workers. He stressed that, starting from an equilibrium for the present level of demand, it suffices to increase the activity levels on partially cultivated lands, with no effect on prices and rents, as long as no new scarcity constraint on lands is met. He also expressed the idea that, when a scarcity constraint is met, the price of corn rises up to the point where some new agricultural method, which was not operated before because it was too expensive when corn was cheap, becomes profitable (law of succession of methods). A new equilibrium, with higher prices and rents, is obtained. Ricardo's ideas, even if they remain informal (at that time, economists did not write equations) are clearly the same as those sustaining the parametric Lemke algorithm (Bidard, 2014).

Ricardo considered two ways, which he considered as basically equivalent, to extend the production of corn. One consists in extending cultivation on a new land (extensive cultivation), the other in changing a method already in use on some land and adopting a more productive one (intensive cultivation). He did not see, however, that in the second case the incoming method designated by the law of succession of methods, which is only based on a profitability criterion, may in fact be less productive than the present method it replaces. In other words, Ricardo missed the possibility of an antitone move: from a formal point of view, the 'Ricardian dynamics' are the parametric Lemke algorithm with no antitone move. Contemporary economists have pointed at the possible multiplicity of equilibria but missed the link between the multiplicity phenomenon and the failure of the Ricardian dynamics on the path leading from low to high levels of demand.

5.2 Numerical examples

Simple numerical examples illustrate the difficulties of the Ricardian dynamics. Let there be a unique quality of land ($h = 1$) with total area $\overline{\Lambda}$ =100 acres, a unique commodity ($g = 1$), corn, and three available methods to produce corn in one period by means of corn and labor. By setting $l_i = -c_i$, method i ($i = 1, 2, 3$) is described as

$$(u_i \text{ quarters corn, } l_i \text{ days labor, } \lambda_i \text{ acres land}) \rightarrow v_i \text{ quarters corn}$$

with, respectively:

$$\begin{aligned} u_1 &= 5, l_1 = 5, \lambda_1 = 1, v_1 = 15 \\ u_2 &= 5, l_2 = 20, \lambda_2 = 1, v_2 = 20 \\ u_3 &= 5, l_3 = 40, \lambda_3 = 1, v_3 = 25 \end{aligned}$$

Let the rate of profit be $r = 1$ and labor be chosen as numeraire. We solve system (18)-(19)-(20) for these data, the level δ of demand being the parameter. For low levels of demand, land is not fully cultivated and the rent is zero. Method 1 is operated because it is the cheapest (competition among farmers) and then the price of corn is $p = 1$. Since the net product per acre of method 1 amounts to 10 quarters, that solution sustains a final demand for corn up to level $\delta = 1,000$ when land is fully cultivated. At this stage, the price p of corn rises and the rent ρ per acre becomes positive. As method 1 will continue to be operated in the next equilibrium, equality

$$2(5p) + 5 + \rho = 15p$$

will hold, i.e. a rise of the price of corn from its present level to $p = 1+\pi$ goes with a rise of the rent per acre from 0 to $\rho = 5\pi$. These moves improve the profitability of the alternative methods and method 2 is the first to reach the ruling rate of profit $r = 1$ for $\pi = 2$. The next equilibrium price is $p = 3$, with $\rho = 10$. Method 2 is introduced and progressively substituted for method 1. As it is more productive per acre, that equilibrium sustains any demand δ in the interval $[1,000, 1,500]$. At level $\delta = 1,500$, land is fully cultivated by method 2, the price of corn and the rent rise up to $p = 4$ and $\rho = 20$, allowing for the progressive replacement of method 2 by the more productive method 3: the Ricardian dynamics work.

Consider now the economy described by methods 1, 2 and 4, where method 4 is characterized by the data

$$u_4 = 3, l_4 = 7, \lambda_4 = 1, v_4 = 12$$

For low levels of demand, method 1 is operated, with $p = 1$ and $\rho = 0$. When the price of corn rises to $p = 1 + \pi$ and the rent to $\rho = 5\pi$, method 4 becomes profitable at level $\pi = 1$, before method 2. Method 4 should be progressively substituted for 1 but, since it is less productive per acre, its introduction would lead to a fall in the net product: the Ricardian dynamics fail. By contrast, the Lemke algorithm goes on by reducing demand from $1,000$ to 900. In the next step, method 4 is maintained and the algorithm proceeds to the introduction of method 2 which replaces method 1. For the intermediate level $\delta = 920$, two white solutions (method 1 alone, or joint use of methods 2 and 4) are found, as well as one black solution corresponding to the joint use of methods 1 and 4.

Finally, consider the economy described by methods 1 and 4 alone. At level $\delta = 920$, there are two solutions, but this is not a counter-example to

Theorem 2 since, after the disappearance of method 2, that level is outside the set $\mathcal{D}$ defined by (16).

For the economy defined by methods 1, 4 and 6

$$u_6 = 1, l_6 = 17.5, \lambda_6 = 1, v_6 = 11.5$$

the set $\mathcal{D}$ corresponds to levels of demands lower that 950. For demand starting from low levels and growing up to $\delta = 920$, the equilibrium defined by method 1 is reached, but the black equilibria made of methods 1 and 4 and the white equilibrium made of methods 4 and 6 (with $p = 3$, $\rho = 11$) are not accessible.

6 Conclusion

Oddity results are not so infrequent in mathematics. In this paper we link the property and the distinction of two types of equilibria to the working of the parametric Lemke algorithm. It is likely that, beyond the bimatrix and the cross-dual games, the strategy of proof applies to other types of complementarity problems and similar results can be obtained.

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