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Abstract. This paper proposes a hybrid two-horizon risk premium model with one- and two-period maturity debts, among which the risky asset and the riskless one depend on agents' investment horizon. A representative investor compares at each horizon the ex-ante premium offered by the market with the value they require to take a risky position, with the aim of choosing between a riskless and a risky strategy. Due to market frictions, the premium offered adjusts gradually to its required value determined by the portfolio choice theory. The required market risk premium is defined as a time-varying weighted average of the required 1- and 2-period horizon premia, where the weights represent the degree of preference of the market for each of the horizons. Our framework is more general than the standard model of the term structure of interest rates where it is assumed that the 1-period rate is the riskless rate at any time and for all agents. Setting one period equal to three months, we use 3-month ahead expected values of the US 3-month Treasury Bill rate provided by Consensus Economics surveys to estimate our 3- and 6-month horizon risk premium model using the Kalman filter methodology. We find that both 3- and 6-month maturity rates represent the riskless and the risky rates with a time-varying market preference for the former rate of about two-thirds. This result strongly rejects the standard model and shows the importance of taking into account the market preference for alternative horizons when describing risky strategies in interest rate term structure modelling.

Keywords : interest rates, risk premium, survey data.

JEL Classification : C51, D84, E43, G11, G14

1 Introduction

According to the theory of the term structure of interest rates, the spread between the long term rate and the short term rate equals the expected changes in the short rate plus a risk premium. Consequently, any empirical examination of this theory involves testing a joint hypothesis of the term structure relation and of hypotheses representing expected changes in the short rate and the risk premium, which are not directly observable variables. In the

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literature, interest rate expectations are either assumed to be rational or determined by the historical values of observed rates, while the specification of the risk premium is either derived from an intertemporal equilibrium condition of the representative investor (portfolio choice model) or from an ad-hoc representation (constant or time-varying premium represented by an ARCH-in-mean model).

In fact, when the joint hypothesis mentioned above is rejected, it is not possible to conclude whether the rejection comes from the term structure relationship or from the hypotheses on expectations and risk premium. This is why, in order to solve these indeterminacies, some authors have used interest rate expectations provided by financial experts' surveys. Such survey data allow avoiding assumptions both on expectation formation and on the representation of the *ex-ante* risk premium required by experts.[1] Concerning the formation of interest rate expectations, using data from various surveys and from various countries and periods, authors found evidence against the unbiasedness of expectations and thus rejected the rational expectation hypothesis (REH) (Friedman, 1979, 1980; Froot, 1989; Simon, 1989; Kim, 1997; MacDonald, 2000; Greer, 2003; Jongen and Veschoor, 2008; Prat and Uctum, 2010). These results highlight the relevance of the question of how interest rate expectations are formed. On this topic, some studies have reported that each of the three traditional standard expectation rules – namely the extrapolative, the adaptive and the regressive rules - can partially explain interest rate expectations. Using survey data, Kane and Malkiel (1967) found support for extrapolative (bandwagon) and regressive expectations while Malkiel and Kane (1969) and Colletaz (1986) found evidence of adaptive expectations. More recently, using Consensus Economics survey data, Prat and Uctum (2010, 2017) showed that experts form their forecasts by combining four limited-information-based rules: the three traditional extrapolative, adaptive and regressive rules and a forward-market rule. The authors argue that such results are consistent with the economically rational expectations theory according to which information costs and agents' aversion to misestimating future interest rates determine the optimal amounts of information on which they base their expectations (Feige and Pearce, 1976).

Overall, the mainstream results show that, whatever the maturity of the debt, interest rate expectations based on survey data are not rational and are well represented by a mixture of traditional rules, which make the *ex-ante* risk premium a more relevant concept than the *ex-*

[1] However, because the term structure model includes market expectations and not those of the experts involved in the survey, the requirement that survey expectations be a valuable approximation of the market expectations must be satisfied. This issue will be discussed later in section 3 devoted to data.

post premium based on the REH. Since survey data makes that the expected change in the short term rate is directly measurable, the risk premium becomes observable, so that the term structure relationship can be tested using an appropriate modeling of the risk premium. Froot (1989) and MacDonald & Macmillan (1994) found that the risk premium is significantly time-varying and concluded that the term structure model based on the pure expectations theory should be rejected.[2] In this line, Artus (1990) and Prat and Uctum (2010) validated the term structure relationship in the 3-month maturity Eurofranc market by using a specific time-varying risk premium representation based on the portfolio choice model.

In the economic literature, the riskless interest rate is given by the yield of a debt security with a one-period maturity, while the rate of the risky asset is the yield of a debt security with a multiple period-maturity (absence of default risk is supposed). In fact, this hypothesis is arbitrary, because the riskless rate is in principle given by the debt security whose duration is equal to the investor's horizon. Consequently, while the riskless rate is given by assets with one-period maturity for some investors (those whose horizon is one period), assets with *n*-period maturity (n>1) are considered as riskless assets for other investors (those whose horizon is *n* periods). In this respect, the main novelty of this paper is to account for these different ways investors can define the riskless and risky rates through a hybrid model where we assume that debt securities with 1 and *n* period maturities can be held both as riskless ~~assets~~ and risky assets according to investors' preferred horizons. Using three-month horizon expectations of the US three-month maturity Treasury Bills rate provided by Consensus Economics surveys (London) over the period November 1989 – May 2015, we estimate such a hybrid model of the term structure of interest rates where the rate of the riskless asset and the rate of the risky asset can be represented by the 3-month and the 6-month maturity Bills, respectively. Because interest rate expectation~~s~~ values are revealed by surveys, our approach will focus on the determination of the *ex-ante* risk premium required by the representative agent who invests simultaneously over the two horizons with the aim of maximizing their expected future real wealth. To our knowledge, such a hybrid model is novel in the literature.

In Section 2 we describe the theoretical foundations of the proposed model. Section 3 outlines the data used and provides some stylized facts. In Section 4 we present our empirical state-space model and discuss the Kalman filter estimation results. Section 5 concludes.

[2] The countries analyzed in the former study are the U.S., Germany, Japan and Australia, while the latter study exploits data from U.K. and uses individual survey data. MacDonald (2000) gives an overview of the related literature.

2 Theoretical issues

2.1 Time horizons of investors, debt security maturities and the *ex-ante* market risk premium

Consider a debt security market without default risk offering to agents the possibility of investing on a τ-month period with yield ${}_{\tau}r_t$ or on a $n\tau$-month period with yield ${}_{n\tau}r_t$ ($n>1$). For a risk-adverse investor *i* willing to realize a τ-month period investment, the riskless interest rate is ${}_{\tau}r_t$. At time *t*, this investor is faced with two choices: (a) buy the τ-month debt security that ensures the yield ${}_{\tau}r_t$ between *t* and $t+\tau$, or (b) buy the $n\tau$-month debt security in the prospect of selling it at $t+\tau$, given that its return ${}_{n\tau}h_{t,t+\tau}$ between *t* and $t+\tau$ is random since its future price at $t+\tau$ is unknown at *t*. The investor will choose strategy (b) only if their expected return in purchasing the long-maturity asset over τ months exceeds ${}_{\tau}r_t$ by an amount deemed sufficient to compensate the risk incurred. When the debt securities are "zero coupon" bonds with continuous and compound interest[3], the τ-month ahead expected return of the long-maturity debt security is written, on a monthly basis, as[4] :

$$\frac{1}{\tau}E_{it}({}_{n\tau}h_{t,t+\tau}) = n\,{}_{n\tau}r_t - (n-1)E_{it}({}_{(n-1)\tau}r_{t+\tau}) \tag{1}$$

where $E_{it}({}_{(n-1)\tau}r_{t+\tau})$ stands for the τ-month horizon expected return of the debt security that has a residual maturity of $(n-1)\tau$ months. For an investment horizon of τ months, the risk premium ${}_{\tau}\phi_{it}^{(n)}$ is by definition equal to the difference between the τ month-ahead expected return of the $n\tau$-maturity debt security (*n* is thus the ratio between the maturities of the long

[3] Recall that a property of a zero-coupon instrument is that residual maturity equals duration. This is an interesting feature in that what matters in risk management strategies is duration and not maturity. For example, an investor with a given investment horizon takes no risk if their portfolio duration is equal to their horizon.

[4] The price of a zero coupon debt security with maturity $n\tau$ months equals the discounted value of future receipts, so that ${}_{n\tau}C_t = F / e^{n\tau\,{}_{n\tau}r_t}$, where F is the nominal value, known at time *t*, which will be paid at maturity (interests plus repayment), and ${}_{n\tau}r_t$ the interest rate (expressed in decimal-monthly basis) of the $n\tau$-months debt security. Accordingly, and noting that the $n\tau$-month security at time *t* becomes a $(n-1)\tau$-month security at time $t+\tau$, the expected return of the long term security between *t* and $t+\tau$ is written as $E_{it}({}_{n\tau}h_{t,t+\tau}) = E_{it}[\log({}_{(n-1)\tau}C_{t+\tau} / {}_{n\tau}C_t)]$, where E_{it} is the conditional expectation for investor *i*. Reporting the former equation into the latter and dividing by τ leads to Eq.(1).

and the short term debt securities) $E_{it}({}_{n\tau}h_{t,t+\tau})/\tau$, and the interest rate of the τ -month maturity debt security, ${}_{\tau}r_t$. Using Eq. (1), this premium writes

$$ {}_{\tau}\phi_{it}^{(n)} = n_{n\tau}r_t - (n-1)E_{it}({}_{(n-1)\tau}r_{t+\tau}) - {}_{\tau}r_t \qquad (2)$$

Due to limitations implied by the availability of survey data on interest rate expectations, we consider the case $n = 2$ from now on. We thus do not specify it in the risk premium for the sake of notational simplicity. Accordingly, Eq.(2) reduces to ${}_{\tau}\phi_{it} = 2_{2\tau}r_t - E_{it}({}_{\tau}r_{t+\tau}) - {}_{\tau}r_t$, where ${}_{\tau}\phi_{it}$ is the *ex-ante* risk premium offered by the market to agent *i* considering horizon τ if they choose the risky strategy (b). Assume now that agent *i* considers the 2τ -month horizon. The riskless interest rate is now given by the debt security yield with a maturity of 2τ . At time *t*, the investor can choose between (a) purchasing the 2τ -month maturity debt security with a risk-free return ${}_{2\tau}r_t$, then earning the total return $2_{2\tau}r_t$, and (b) purchasing the τ month maturity debt in view of repeating the purchase in τ months, given that the operation is risky since the price of the τ -month asset at $t+\tau$ is unknown at *t*. The 2τ -month ahead expected total return for the speculative strategy (b) is then ${}_{\tau}r_t + E_{it}({}_{\tau}r_{t+\tau})$, which must be compared to the secure strategy (a) yielding $2_{2\tau}r_t$. Accordingly, the *ex-ante* risk premium ${}_{2\tau}\phi_{it}$ offered by the market to agent *i* for selecting the risky strategy can be written as ${}_{2\tau}\phi_{it} = {}_{\tau}r_t + E_{it}({}_{\tau}r_{t+\tau}) - 2_{2\tau}r_t$. Note that at time *t* the values of *ex-ante* premia ${}_{\tau}\phi_{it}$ and ${}_{2\tau}\phi_{it}$ are known to any investor *i* since the two market rates are observable and the individual expected values are obviously known to them. It must be noted that whatever their preferred horizon of τ or 2τ months, an investor can a priori decide to hold assets with a maturity of τ or 2τ months and choose among strategies of type (a) or (b), depending on their perceived uncertainty and risk aversion. Of course, the whole panel of investors consists in a multitude of agents *i*. We now describe the *ex-ante* market risk premium using the representative agent concept. If the aggregate τ and 2τ -month debt securities market is comprised of *p* investors, the representative agent's expectation is given by the market belief that is $E_{Mt}({}_{\tau}r_{t+\tau}) = \frac{1}{p}\sum_{i=1}^{p}E_{it}({}_{\tau}r_{t+\tau})$. It hence follows from the aggregation of each of the two individual risky strategies mentioned above that the *ex-ante* risk premia offered by the market to the representative agent for the τ and 2τ -month horizons can be written as

$$ {}_{\tau}\phi_{M,t} = 2\,{}_{2\tau}r_t - E_{M,t}({}_{\tau}r_{t+\tau}) - {}_{\tau}r_t \tag{3} $$

and

$$ {}_{2\tau}\phi_{M,t} = {}_{\tau}r_t + E_{M,t}({}_{\tau}r_{t+\tau}) - 2\,{}_{2\tau}r_t \tag{4} $$

It can be straightforwardly seen that each premium can take any sign, that they both have the same absolute value but are of opposite signs, that is ${}_{\tau}\phi_{Mt} = -{}_{2\tau}\phi_{Mt}$.[5]

To describe the decision making mechanisms for both horizons, it is now necessary to consider the values ${}_{\tau}\phi^*_{Mt}$ and ${}_{2\tau}\phi^*_{Mt}$ of the premia required by the τ and 2τ-month investors to adopt the risky strategies (b) described above. These are, for each horizon, the solutions of the optimal portfolios composed by τ and 2τ-month maturities debt securities (see section 2.2). For example, consider at time *t* that the representative investor is concerned by the τ month horizon. If their expectation $E_{Mt}({}_{\tau}r_{t+\tau})$ and the interest rates ${}_{\tau}r_t$ and ${}_{2\tau}r_t$ are such that the condition ${}_{\tau}\phi_{Mt} \geq {}_{\tau}\phi^*_{Mt}$ holds[6] (i.e. the market offers not less than what they require), then they will have incentive to adopt a risky strategy by purchasing the 2τ-month risky asset[7] with the intention of reselling it τ months later; as a result, ${}_{2\tau}r_t$ should decline, implying a decrease in ${}_{\tau}\phi_{Mt}$. If now the condition ${}_{\tau}\phi_{Mt} < {}_{\tau}\phi^*_{Mt}$ holds, the investor will abandon the strategy (b) and purchase the τ-month debt offering a riskless return, i.e. choose the strategy (a). In this case, ${}_{\tau}r_t$ should decrease, implying a fall in ${}_{\tau}\phi_{Mt}$. Similarly, consider that the representative agent has a preference for the 2τ-month investment horizon. If ${}_{2\tau}\phi_{Mt} \geq {}_{2\tau}\phi^*_{Mt}$, they will be prompted to adopt a risky strategy (b) by purchasing the τ-month risky asset with a view to reinvest in this debt security three months later. This should imply a decrease in ${}_{\tau}r_t$, leading in turn to a decrease in ${}_{2\tau}\phi_{Mt}$. By contrast, if ${}_{2\tau}\phi_{Mt} < {}_{2\tau}\phi^*_{Mt}$, the investor will prefer to buy today the 2τ-month asset offering a riskless return, so that ${}_{2\tau}r_t$ will

[5] Note that such symmetry holds only when the ratio *n* between the long and the short-term maturities equals 2.

[6] Since both observed and required values may take positive or negative values (for the case of required values see section 2.2), the arbitrages described below apply whatever the sign of the premia.

[7] The investor may have acquired the 2τ-month asset by selling their holding of the τ-month security, if any. The resulting increase in ${}_{\tau}r_t$ would then contribute to the decrease in ${}_{\tau}\phi_{Mt}$. More generally, any substitution in either direction between the τ- and the 2τ-month maturity assets is prone to reinforce the adjustment of the observed premium to its equilibrium value.

decrease and ${}_{2\tau}\phi_{Mt}$ rise. Consequently, as long as inequalities between observed and required premia prevail, the investor's decisions trigger adjustments in interest rates, which help to restore equilibrium.

2.2 The required market risk premium

Since horizons τ and 2τ are considered at the same time for the determination of the market risk premium ϕ_{Mt}, the representative agent is supposed to make separately for each horizon a "mental accounting" [8] to assess the value of the equilibrium risk premium corresponding to the optimal portfolio composed by τ - and 2τ -month debt securities. These values are those required to adopt a risky behavior. Because our investor is a representative agent, regardless of whether they wish to invest at 3 or 6 months the two mental accountings involve at time *t* the same total wealth, the same aggregate 3-month rate expectation, the same expected volatility and the same preference parameters.

The τ -month horizon investment strategy

Let ${}_{\tau}N_t$ be the amount of the face value of the τ -month maturity debt priced at $1/(1+{}_{\tau}r_t)$, ${}_{2\tau}N_t$ the amount of the face value of the 2τ -month maturity debt priced at $1/(1+{}_{2\tau}r_t)^2$, W_t the investor's wealth and ρ the relative risk aversion coefficient. Putting the expected utility of real wealth in the mean-variance form, the program of the investor writes[9] :

$$Max\, E_t\left[U(\frac{P_t}{P_{t+\tau}}W_{t+\tau})\right]=Max\, E_t(\frac{P_t}{P_{t+\tau}}W_{t+\tau})-\frac{\rho}{2}V_t(\frac{P_t}{P_{t+\tau}}W_{t+\tau}) \tag{5}$$

subject to the budget constraint

$$W_t={}_{\tau}N_t\,/(1+{}_{\tau}r_t)+{}_{2\tau}N_t\,/(1+{}_{2\tau}r_t)^2 \tag{6}$$

At $t+\tau$, investor's real wealth from the same portfolio composed at *t* is given by

[8] Thaler (1999) defines mental accounting as "cognitive operations used by individuals and households to organize, evaluate, and keep track of financial activities".
[9] For similar approaches, see Artus (1990) and Prat and Uctum (2010). In the same vein, Shiller (1990) considers a consumption-based modelling where the representative agent maximizes their expected utility of consumption. Referring to the CAPM, Roll (1971) also adopts the portfolio choice theory to analyze the term structure of interest rates.

$$W_{t+\tau} = {}_{\tau}N_t + {}_{2\tau}N_t / (1 + {}_{\tau}r_{t+\tau}) \,. \qquad (7)$$

Solving for $W_{t+\tau}$ by eliminating ${}_{\tau}N_t$ between (6) and (7) leads to approximate the argument of the expectation and the variance operators in (5) as follows[10]:

$$\frac{P_t}{P_{t+\tau}} W_{t+\tau} = \frac{P_t}{P_{t+\tau}} W_{\tau}(1 + {}_{\tau}r_t) + {}_{2\tau}N_t (2\,{}_{2\tau}r_t - {}_{\tau}r_t - {}_{\tau}r_{t+\tau}) \qquad (8)$$

Reporting (8) into (5) and maximizing for ${}_{2\tau}N_t$ leads to the following required value of the market risk premium**:**

$${}_{\tau}\phi^{*}_{M,t} = \rho\, W_t \; (\omega_t \; V_t \; + Cov_t) \qquad (9)$$

where $V_t = V_t({}_{\tau}r_{t+\tau})$ represents the expected variance of the future interest rate, $Cov_t = \mathrm{cov}({}_{\tau}r_{t+\tau}, \pi_{t+\tau})$ the expected covariance between the future interest rate and upcoming inflation (defined as $\pi_{t+\tau} = \log(P_{t+\tau} / P_t))$ and $\omega_t = \frac{{}_{2\tau}N_t / (1 + {}_{2\tau}r_t)^2}{W_t}$ the share of the 2τ -month asset in the portfolio. Eq. (9) provides the required value to which the observed τ -month market premium given by (3) converges.

The* 2τ *- month horizon investment strategy

The program of the representative agent is now:

$$Max\; E_t\left[U\left(\frac{P_t}{P_{t+2\tau}} W_{t+2\tau}\right)\right] = Max\; E_t\left(\frac{P_t}{P_{t+2\tau}} W_{t+2\tau}\right) - \frac{\rho}{2} V_t\left(\frac{P_t}{P_{t+2\tau}} W_{t+2\tau}\right) \qquad (10)$$

subject to the same budget constraint (6) as above. The wealth at time $t+2\tau$ is now defined as $W_{t+2\tau} = {}_{\tau}N_t(1 + {}_{\tau}r_{t+\tau}) + {}_{2\tau}\, N_t$. A similar derivation as above leads to the following required value of the market risk premium:

[10] To obtain expression (8) the following approximation has been used:
$(P_t / P_{t+\tau})(2\,{}_{2\tau}r_t - {}_{\tau}r_t - {}_{\tau}r_{t+\tau}) \big/ [(1 + {}_{\tau}r_{t+\tau})(1 + {}_{2\tau}r_t)^2] \approx 2\,{}_{2\tau}r_t - {}_{\tau}r_t - {}_{\tau}r_{t+\tau}$. For τ =3, the empirical correlation between the two sides of this proxy is found to be 0.998, which makes the approximation admissible.

$$ {}_{2\tau}\phi^{*}_{M,t} = \rho\, W_t \; \left[(1-\omega_t)\, V_t - Cov_t \right] \qquad (11) $$

where we assumed that $Cov_t({}_{\tau}r_{t+\tau}, \pi_{t+2\tau}) \approx Cov_t({}_{\tau}r_{t+\tau}, \pi_{t+\tau}) = Cov_t$.[11] Eq. (11) provides the required value to which the observed 2τ-month premium given by (4) converges. It can be seen from Eqs. (10) and (11) that the values of the τ and 2τ-month required premia are not opposed to each other, contrary to the observed premia (3) et (4). This is due to the fact that planning, at time *t*, to sell a 2τ-month Bill at time $t+\tau$ (i.e., implementing a τ-month horizon risky strategy) does not imply a symmetric risk exposure than planning, at time *t*, to buy a τ-month Bill at time $t+\tau$ (i.e., following a 2τ- month horizon risky strategy). Indeed, redeeming the long term asset before maturity and scheduling the reinvestment of the short term asset involve uncertainty in the next τ-month return and the one after which imply different management approaches over subsequent periods.

We can infer from Eq.(9) that if the expected covariance between interest rate and inflation is positive, then the required premium ${}_{\tau}\phi^{*}_{M,t}$ is positive. Conversely, if this covariance is negative, it suffices that the expected variance of interest rate be smaller than or equal to the absolute value of the covariance for the premium to be negative. For a τ- month period investment, a negative required risk premium has the economic sense that the assessment of inflation risk adds up to interest rate risk, or in other words that the actual interest rate ${}_{\tau}r_t$ is perceived as not compensating future inflation, which affects negatively the expected real wealth. In this case, the agent willing to invest for a τ- month period might accept to pay a premium to purchase the 2τ-month maturity debt security in view of selling it τ months later. This strategy would allow the agent to reduce their loss, and even to expect profit if the price of the 6-month asset were to increase significantly to offset upcoming inflation. In the same manner, Eq.(11) suggests that according to the values of the variance and of the covariance, the required 2τ-month premium can be of any sign.

The aggregate equilibrium risk premia

At the aggregate supply side, it follows from the symmetry between (3) and (4) that, at any time, representing one of the premia amounts to representing the opposite of the other one

[11] It seems indeed unlikely that agents might make a clear distinction between the two expected covariances, especially since the horizon of $\pi_{t+\tau}$ overlaps by 50% the one of $\pi_{t+2\tau}$.

by using the same set of information. We can then express the *ex-ante* market premium using indistinguishably the same measure for the two horizons, provided that this measure can take any sign. We arbitrarily choose to specify the ex-ante market premium using the τ-month horizon measure, that we call the "two-horizon ex-ante premium offered by the market" and that we define as:

$$\phi_{M\,t} \;=\; 2\,{}_{2\tau}r_t - E_{M\,t}({}_{\tau}r_{t+\tau}) - {}_{\tau}r_t \tag{12}$$

According to Eq. (12), a positive (negative) value of $\phi_{M,t}$ means that the market premium is positive (negative) at the τ-month horizon but negative (positive) at the 2τ-month one.

At the aggregate demand side, the representative agent who reflects all investors in the market is concerned at any time by both horizons τ and 2τ, given that any investor must choose between a riskless and a risky strategy. Accordingly, the two-horizon required market premium $\phi^*_{M,t}$ can be represented by a weighted average of ${}_{\tau}\phi^*_{M,t}$ and ${}_{2\tau}\phi^*_{M,t}$:

$$\phi^*_{M,t} = \lambda_t\,{}_{\tau}\phi^*_{M,t} + (1-\lambda_t)\,{}_{2\tau}\phi^*_{M,t} \qquad\qquad 0 \le \lambda_t \le 1 \tag{13}$$

where λ_t stands for the weight granted by the representative agent to the τ-month horizon in their required premium, thus measuring the agent's degree of preference for this horizon vis-à-vis risky strategies. Note that at any time *t*, the sign of the premium required by the market depends both on the signs of ${}_{\tau}\phi^*_{M,t}$ and of ${}_{2\tau}\phi^*_{M,t}$ and on the value of λ_t. In the literature, it is generally assumed that $\lambda_t = 1\ \forall t$, that is $\phi^*_{M,t} = {}_{\tau}\phi^*_{M,t}$, which means that ${}_{\tau}r_t$ and ${}_{2\tau}r_t$ are at any time the risk-free and the risky rates, respectively. This implies that the horizon of the market is assumed to be unchangingly τ months. While in some cases this restrictive assumption may prove compatible with the term structure model, the restrictive hypothesis that λ_t equals 1 at each point in time was strongly rejected with our data. This is notably due to the fact that the required risk premium ${}_{\tau}\phi^*_{M,t}$ is not sufficient on its own to account for the alternating signs of the observed market risk premium $\phi_{M,t}$ (see figure 1). This shortcoming of the literature can be addressed by using our two-horizon required market premium to characterize the demand-side of the debt market. Accordingly, the equilibrium condition $\phi_{M,t} = \phi^*_{M,t}$ leads to equalize the right hand sides of equations (12) and (13), hence

$$2\,{}_{2\tau}r_t - E_{M\,t}({}_{\tau}r_{t+\tau}) - {}_{\tau}r_t = \lambda_t\,{}_{\tau}\phi^*_{M,t} + (1-\lambda_t)\,{}_{2\tau}\phi^*_{M,t} \qquad (14)$$

Regarding Eq.(14), it seems useful to discuss the limit cases when λ_t reaches the values 0 and 1. If λ_t=1, the market offers to the representative agent a premium that directly identifies with the τ -month premium (3); at equilibrium, Eq.(14) writes $2\,{}_{2\tau}r_t - E_{M,t}({}_{\tau}r_{t+\tau}) - {}_{\tau}r_t$ = ${}_{\tau}\phi^*_{M,t}$. If now λ_t=0, then the market offers to the representative agent a premium which is given by the 2τ -month premium (4); at equilibrium the state of the nature is given by ${}_{\tau}r_t + E_{M,t}({}_{\tau}r_{t+\tau}) - 2\,{}_{2\tau}r_t$ = ${}_{2\tau}\phi^*_{M,t}$. However, given the symmetry between the observed market premia, the equilibrium condition (14) is still consistent with this limit case in the sense that any value of the required 2τ -month premium generates the opposite value in the dependent variable $2\,{}_{2\tau}r_t - E_{M\,t}({}_{\tau}r_{t+\tau}) - {}_{\tau}r_t$. For example, a τ -month required premium of 1% implies an observed value of 1% when λ_t=1, while a 2τ -month required premium of 1% produces an observed value of -1% when λ_t=0. The appropriateness of the two-horizon market premium in representing these extreme cases can be generalized to any value of the weighting coefficient λ_t. Reporting (9) and (11) into (13), the value of the two-horizon equilibrium risk premium characterizing the representative agent writes:

$$\phi^*_{M,t} = \rho\, W_t\,\left[\left((2\lambda_t - 1)\omega_t + 1 - \lambda_t\right)V_t + (2\lambda_t - 1)\,Cov_t\right] \qquad (15)$$

where W_t is given by Eq.(6).

2.4 The adjustment of the observed market premium to its equilibrium value

When the deviation $z_t = \phi_{M\,t} - \phi^*_{M\,t}$ between observed and equilibrium market premia is zero, there are no arbitrage opportunities, but when $z_{it} > 0$ ($z_{it} < 0$), agents can improve their utility by selling or buying Bills (see section 2.1 above). A non-zero deviation can result from market frictions due to liquidity constraints, to search costs (especially asset selection costs in portfolio optimization) and to information costs related to the determination of the equilibrium value. As shown by Anderson (1997) for the US Treasury bill rate, transaction costs can also cause deviations between price and equilibrium value. Moreover, by influencing the market volatility, noise traders' behaviour may lead to mispricing the equilibrium value of an asset; this generates uncertainty about the true equilibrium value and then contributes to make arbitrage risky (Shleifer and Summers (1990)). Consequently, both transaction costs and risky arbitrage make arbitrage unfeasible at some point and generate adjustment delays of market asset prices towards their equilibrium values (Jawadi and Prat (2012)).[12] Accordingly, assuming the adjustment process is linear, the dynamics of the spread between observed $\phi_{M,t}$ and equilibrium premia $\phi^*_{M,t}$ is represented by the following error correcting model (ECM):

$$\Delta z_t = -\kappa\, z_{t-1} + \sum_{q=1}^{m} a_q\, \Delta z_{t-q} + \zeta_t \qquad 0 \le \kappa \le 1 \tag{16}$$

Rearranging the terms, we get:

$$\phi_{M,t} = \phi^*_{M,t} + \gamma\,(\phi_{M,t-1} - \phi^*_{M,t-1}) + \sum_{q=1}^{m} a_q\, \Delta z_{t-q} + \zeta_t \tag{17}$$

where $\gamma = 1 - \kappa$ and where $\phi^*_{M,t}$ is given by Eq.(15).

3 Data and stylized facts

3.1. Observed and expected US Treasury Bills rates

Our study is concerned with the US Treasury Bills market**.** T-Bills are the most marketable debt and are a way for the U.S. government to raise money from the public. They are short-term securities whose maturities range from a few days to a maximum of 52-weeks, but common maturities are 1-, 3-, 6- and 12-months. T-Bills rates have many advantages. First, a T-Bill is a simple zero-coupon debt: it is purchased for a price that is less than its face

[12] For a group of investors, the absolute deviation may exceed their arbitrage costs (defined as the sum of transaction costs plus a risky arbitrage premium), so that they will push forward a mean-reverting mechanism towards their equilibrium values. For other investors, arbitrage costs may be larger than absolute deviation. The larger the proportion of mean-reverting agents, the higher the strength of the market adjustment, implying nonlinearity in the adjustment process if the proportion of the two groups is time-varying. According to the linear Eq.(16) below, this proportion is supposed to be stable.

value, the government paying the holder the full par value when it matures; interest paid is thus the difference between the purchase price of the Bill and what the investor gets at maturity. This implies that the duration equals the residual maturity; this is an important feature, since a riskless claim is a claim whose duration equals the horizon of the investment. Second, T-Bills are affected by a very low default risk because the U.S. government guarantees to the investor a return on the invested capital. Third, income from Treasury bonds is generally exempt from state and local taxes (although they are subject to federal income taxes).

Our data covers the period November 1989 – May 2015. At the beginning of each month, *Consensus Economics* (CE, London) asks about 200 economists, financial market operators and executives in various institutions (commercial and investment banks, forecasting agencies and industrial corporations) in over 30 countries to forecast future values of principal macroeconomic variables for the three and the twelve month horizons. In particular, the CE newsletter publishes every month the “consensus” corresponding to the arithmetic average of individual expected values of the 3-month Treasury Bill rates that we denote $E_t({}_3r_{t+\tau})$, where τ stands for the 3 month horizon.[13] About 30 financial institutions are asked to predict this variable. These institutions are, by their own activity, directly concerned by forecasting US interest rates and include essentially major American banks (Bank of America, Goldman Sachs, Barclays, Wells Fargo, JP Morgan, Northern Trust…), investment advisory firms (First Trust Advisor, Wells Capital Management…), research organizations or academic institutions (The Conference Board, Moody’s Analytics, RDQ Economics, Georgia State University, University of Michigan, University of Maryland…), and industrial companies (General Motors, Eaton Corporation…). The experts answer only when they think they have a good knowledge about the variable of interest, and this allows assuming that those who respond are informed agents. Since the individual answers are confidential (only the consensus is disclosed to the public, with a time lag) and since each individual is negligible within the consensus, it is difficult to claim that, for reasons which are inherent to speculative games, individuals might not reveal their « true » opinion. For all these reasons, one can reasonably assume that the expectations provided by the respondent experts are representative of market expectations. Considering the panel of experts, it can be noted that about half of the respondents remain unchanged over the period. The turnover in the other

[13] CE provides also the expected value of ${}_3r_t$ for a 12-month horizon over our period. Given the theoretical model described in section 2, our study only requires the 3-month ahead expected values.

half can therefore lead to a bias due to a lack of homogeneity in the average responses over time. However, this bias can be considered as being negligible regarding the dispersion of the opinions. Indeed, the coefficient of variation (i.e., at time *t*, the ratio of the standard-deviation of the responses to their mean) of experts' 3-month ahead interest rate forecasts fluctuates around an average of 0.067. This implies that the dispersion of individual expectations is limited enough (i.e. strongly lower than 1) to assume that no serious statistical problem arises from the aggregation of forecasts, so that the "consensus" values can be viewed as reflecting the representative agent's expectations.

The CE requires a very specific day for the answers. As a rule, this day is the same for all respondents.[14] Accordingly, we consider the 3-month and 6-month maturity interest rates ${}_3r_t$ and ${}_6r_t$ released at the same day as the consensus of the expected values $E_t({}_3r_{t+3})$. Actual values of the 3-month bills rate are directly published in the CE bulletin while interest rates for the 6-month maturity are extracted from the Board of Governors of the US Federal reserve System at a daily frequency.

3.2 Stylized facts : the term spread, the expected change in interest rate and the risk premium

The term spread has classically two components that are the expected change in interest rate and the risk premium. Concerning our data, we have the identity ${}_6r_t - {}_3r_t = \frac{1}{2}\left[E_t\,({}_3r_{t+3}) - {}_3r_t\right] + \frac{1}{2}\,\phi_{M\,t}$ with $\phi_{Mt} = 2{}_6r_t - E_t\,({}_3r_{t+3}) - {}_3r_t$. As shown in Figure 1, the term spread ${}_6r_t - {}_3r_t$ and its two components $\frac{1}{2}\left[E_t\,({}_3r_{t+3}) - {}_3r_t\right]$ and $\frac{1}{2}\,\phi_{M\,t}$ vary around zero with comparable magnitudes. Table 1 shows that the spread is equally correlated with the expected change and the risk premium, but that these two components are weakly correlated between each other. In other words, the risk premium represents more than half of the variability of the spread, which confirms the importance of modeling the premium to explain the term structure of interest rates.

[14] This day is the first Monday of the month until March 1994, and the second Monday since April 1994, except the closed days (in this last case, the survey is dated at the following day). The effective horizons however always remain equal to 3 and 12 months. If, for instance, the answers are due on the 3rd of May (which was the case in May 1993), the future values are asked for August 3, 1993 (3 months ahead expectations) and for January 3, 1994 (12 months ahead expectations). The individual responses are then concentrated on the same day.

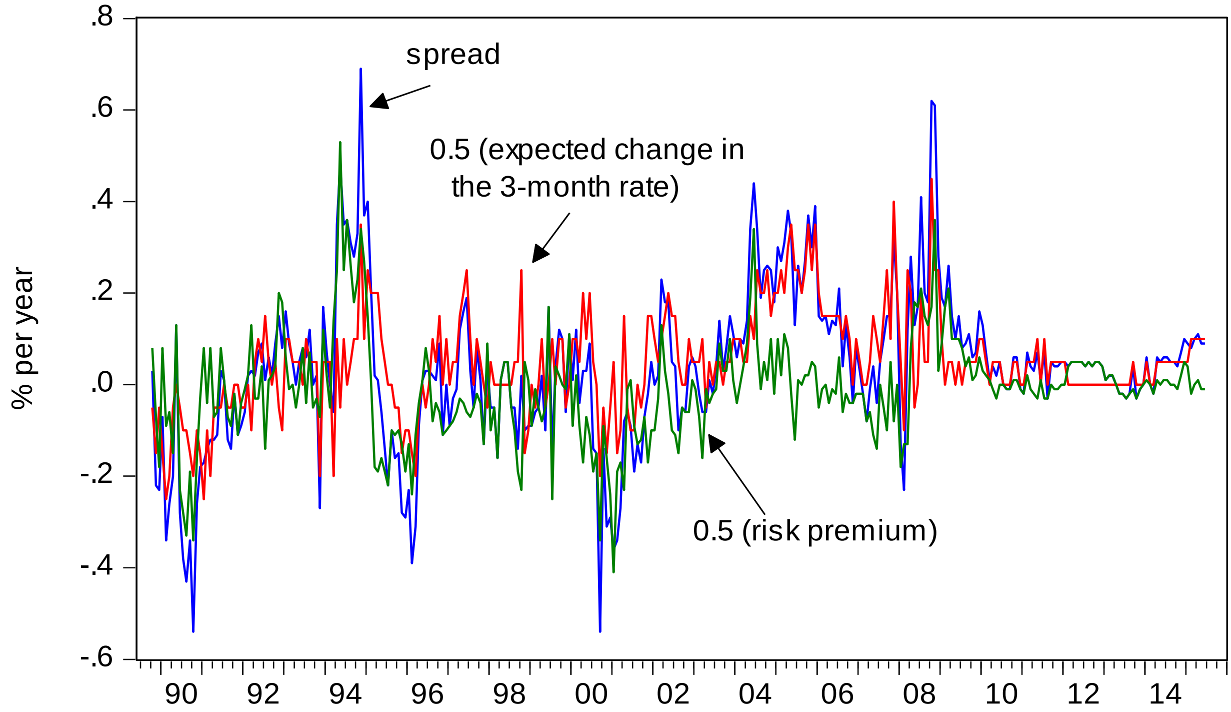
Table 1. Correlations between the term spread and its two components

	Spread $_6r_t - _3r_t$	*Expected change* $E_t[_3r_{t+3}] - _3r_t$	*Risk premium* $\phi_{M\,t}$
$_6r_t - _3r_t$	1.00	0.75	0.78
$E_t[_3r_{t+3}] - _3r_t$		1.00	0.16
$\phi_{M\,t}$			1.00

Table 1 indicates that expectations and risk are complementary to describe the dynamics of the spread; of course, when the *ex-ante* market risk premium $\phi_{M\,t}$ is not explained by a structural model, the term structure $_6r_t - _3r_t$ remains also unexplained. ADF tests have shown that the spread and its two components are stationary at the 1% level of significance, while the Breusch-Godfrey serial correlation LM test indicated that they are all three highly autocorrelated, which suggests the existence of deterministic factors[15]. Especially, the risk premium fluctuates around zero (mean=-0.02) between a maximum of 1.06% per year and a minimum of -0.82% with a standard-error of 0.26%; the proportions of positive, negative and zero values are respectively 42%, 54% and 4%. In particular, the sharp rise of the premium observed in 1994 could be conditioned by the restrictive policy pursued by the Fed, which considered that the growth rate was too fast, implying a risk of overheating and therefore of inflation. The risk premium also exhibits a substantial increase during the Global financial crisis. The premium dampens during 2009 and remains around zero since then, which is likely due to the Federal Reserve's forward guidance of the short term interest rates at the post-2008 period that led to lower the uncertainty in expectations and thus to shrink risk premium.

[15] These tests are not reported but are available upon request.

Figure 1. Term structure of the 6- and 3-month T-Bill rates: spread, expected change in the 3-month rate and risk premium



4. Empirical issues

4.1. Specifying the unobservable variables and the time-varying parameter model

To make the structural equation (17) along with (15) operational, some additional assumptions must be formulated concerning the expected variance $V_t = V_t({}_3r_{t+3})$, the expected covariance $Cov_t = \mathrm{cov}({}_3r_{t+3}, \pi_{t+3})$, the amounts ${}_3N_t$ and ${}_6N_t$ of the face values of the claims in 3- and 6-month T-Bills respectively, the share of the 6-month asset in the portfolio ω_t and the time-varying weight λ_t.

The expected variance of the 3-month interest rate was estimated using several alternative specifications such as the inter-day variance during the last week or during the last month (and their monthly lagged values), the rolling variance of ${}_3r_t$ (or of the change in ${}_3r_t$)

over alternative windows of 1 to 9 month-widths, an unrestricted weighted average of the past values of the quadratic change in ${}_3r_t$, and an ARCH approach. The best estimation results from our structural model were obtained using the latter, and more specifically an AR(1)-GARCH(1,1) process. As a result, we assume V_t to be simply given by this conditional variance equation.[16] The AR(1) process characterizing the mean equation was found to be insignificantly different from a random walk, hence suggesting that the expected variance of ${}_3r_t$ can equivalently be written as the expected variance of the *change* in ${}_3r_t$. This formulation will be useful as it is comparable with the specification of the expected covariance below.

We now consider the representation of the expected covariance Cov_t between the 3-month interest rate and upcoming inflation. The rolling covariance between the levels of the two variables calculated over alternative windows of 1 to 9 month-widths did not allow for good fits of the structural state-space model. We then computed the rolling covariance (denoted ${}_kCov_t$) of the *changes* in the two variables using a window width of *k* months and calculated the expected covariance as the weighted average of actual and past values of these rolling covariance terms, that is $Cov_t = \sum_{i=0}^{m} b_i \cdot {}_kCov_{t-ik} \Big/ \sum_{i=0}^{m} b_i$ with $b_0 = 1$. Parameters k, m and b_i were determined in the course of the estimation of our risk premium model and optimal values were found for *k*=3 and *m*=2. The expected covariance is simplified as:

$$Cov_t = \frac{{}_3Cov_t + b_1 \cdot {}_3Cov_{t-3} + b_2 \cdot {}_3Cov_{t-6}}{(1 + b_1 + b_2)} \tag{18}$$

We now turn to the representation of the amount of face values in 3- and 6-month T-Bills. By definition, we posit ${}_\tau N_t = {}_\tau n_t\, F$ and ${}_{2\tau} N_t = {}_{2\tau} n_t\, F$ in (6), where ${}_\tau n_t$ and ${}_{2\tau} n_t$ refer to the number of securities outstanding and F to their face value. Because only the total amount TB_t of the outstanding T-bills expressed at market price and in current USD is available[17], we examined several assumptions. A first approach was to suppose that each of the face value amounts ${}_3N_t$ and ${}_6N_t$ is proportional to the total face value amount of the T-Bills measured by $N_t = d_o(1+\bar{r}_t)TB_t$, where $d_o(1+\bar{r}_t)$ represents the reverse of the average

[16] We also tested for variants with several lags in the conditional variance V_t but none were found to be significant.
[17] Source : Datastream.

market price of the T-Bills, $d_o = 1/F$ and where $\bar{r}_t = ({}_1r_t + {}_3r_t + {}_6r_t + {}_{12}\,r_t)/4$ represents the average market price of T-Bills across maturities. A second approach consisted in representing each of ${}_3N_t$ and ${}_6N_t$ by a constant plus a polynomial trend (up to the degree 4). The constant term was found to be significant but the trend polynomials were drastically rejected when introduced in the risk premium equation (17), this result strongly suggesting that ${}_3N_t$ and ${}_6N_t$ should be represented by constants, say, ${}_3N_o$ and ${}_6N_o$.[18] As a result, we assume that

$$ {}_3N_t = {}_3N_o \ , \ {}_6N_t = {}_6N_o \text{ and } \alpha = {}_3N_o \big/ {}_6N_o \tag{19} $$

Reporting the proportionality condition in (19), the share of the 6-month asset in the portfolio $\omega_t = \dfrac{{}_6N_o/(1+{}_6r_t)^2}{W_t}$ and the total wealth W_t as defined in (6) into the structural Eq.(15), we get the following equation of the two-horizon risk premium to be estimated :

$$ \begin{aligned} \phi_{M,t} = {} & \rho\, {}_3N_o \Big\{ V_t\big[\alpha(2\lambda_t - 1)/(1+{}_6r_t)^2 + (1-\lambda_t)(1/(1+{}_3r_t) + \alpha/(1+{}_6r_t)^2)\big] + (2\lambda_t - 1)\, Cov_t(1/(1+{}_3r_t) + \alpha/(1+{}_6r_t)^2) \Big\} \\ & + \gamma\phi_{M,t-1} - \rho\gamma\, {}_3N_o \Big\{ V_{t-1}\big[\alpha(2\lambda_{t-1} - 1)/(1+{}_6r_{t-1})^2 + (1-\lambda_{t-1})(1/(1+{}_3r_{t-1}) + \alpha/(1+{}_6r_{t-1})^2)\big] \\ & + (2\lambda_{t-1} - 1)\, Cov_{t-1}(1/(1+{}_3r_{t-1}) + \alpha/(1+{}_6r_{t-1})^2) \Big\} + \sum_{q=1}^{m} a_q\, \Delta z_{t-q} + \zeta_t \end{aligned} \tag{20} $$

The time-varying weight λ_t is an unobservable variable reflecting the degree of preference in the 3-month horizon relative to the 6-month horizon. We assume that λ_t follows an AR(1) process[19].

$$ \lambda_t = \lambda_o + \theta\,\lambda_{t-1} + \eta_t \tag{21} $$

where $\eta_t = N(0, \sigma_t)$ and $E(\eta_t, \zeta_t) = 0$.

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[18] To provide some insight to these results, note that the total amount at the market price of the outstanding T-bills can be written as $TB_t = \dfrac{N_t}{(1+\bar{r}_t)} = \dfrac{N_o}{(1+\bar{r}_t)} + \dfrac{N_t - N_o}{(1+\bar{r}_t)}$; we run the regression $TB_t = \dfrac{\hat{N}_o}{(1+\bar{r}_t)} + c_o + \hat{e}_t$ where the product $(c_o + \hat{e}_t)(1+\bar{r}_t)$ can be viewed as a proxy of the underlying dynamics of N_t. Regarding the ADF test, this product was found to be stationary at the 5% level of significance, hence suggesting that N_t can be supposed to be stationary. This implies that the hypotheses of constant values for the mean and the variance of N_t are acceptable, which suggests that the assumption that ${}_3N_t$ and ${}_6N_t$ are constant does not alter significantly the risk premium equation over our period.

[19] Higher orders did not appear to better fit the data. We also attempted at introducing in the state equation changes in the expected values of inflation and of the GDP growth provided by the same surveys. All the variants were found to be insignificant.

We can now estimate our model in the form of a two equations state space model, where Eq. (20) defines the measurement (or signal) equation while Eq.(21) stands for the state equation. This model with a time varying coefficient is estimated using Kalman filtering (Harvey (1992), Hamilton (1994)). The initial value of λ_t has been set by a grid search so as to minimize the information criteria (AIC, Schwarz and Hannan-Quinn). Because we are interested in a structural interpretation of the model, the values of the measurement and state variables are calculated at each time using the whole sample of observations (smoothed inference) rather than only past observations (filtered inference).

The state-space model above is designed for the risk premia required for the 3- and 6-month maturities of debt. These maturities are among the prominent ones in the T-Bills market. However, because this market offers maturities ranging from a few days to one year, this can disturb the adjustment process between the two-horizon market risk premium $\phi_{M,t}$ and its equilibrium value $\phi^*_{M,t}$. This is why the measurement equation (20) has been augmented by changes in the 1- and 12-month T-Bills rates. Only the latter appeared to be significant and the variable representing the changes in 1-month maturity T-Bills rate has therefore been dropped.[20] Insofar as an increase in ${}_{12}r_t$ increases the opportunity cost of maintaining the 3 or 6-month investment, one can expect that this rise fosters a higher level of $\phi_{M,t}$ to compensate this psychological disadvantage. The magnitude of this effect is represented by the coefficient δ associated with $\Delta_{12}r_t = {}_{12}r_t - {}_{12}r_{t-1}$.

4.1 Estimating the state-space model

Table 2 provides in the second column the estimates of our state-space model represented by the measurement equation (20) (augmented by $\Delta_{12}r_t$) and by the state equation (21) over our sample period. We accounted for the overlapping bias resulting from the difference between our 3-month expectation horizon and the monthly frequency of observations by introducing a second order (horizon time-span minus 1) moving average (MA) specification for the residuals (Hansen and Hodrick, 1980). The coefficients a_q of the Δz_{t-q} terms were found to be insignificant and were removed from the final adjustment, while the

[20] Because changes in 1- and 12 month T-Bills rates are significantly correlated (R=0.70), it is not surprising that only one of the two maturities is found to be significant.

coefficients of the two MA terms were found to be significant, implying that our estimates would have been biased if the overlapping problem was not accounted for.

Table 2. Kalman filter estimation results

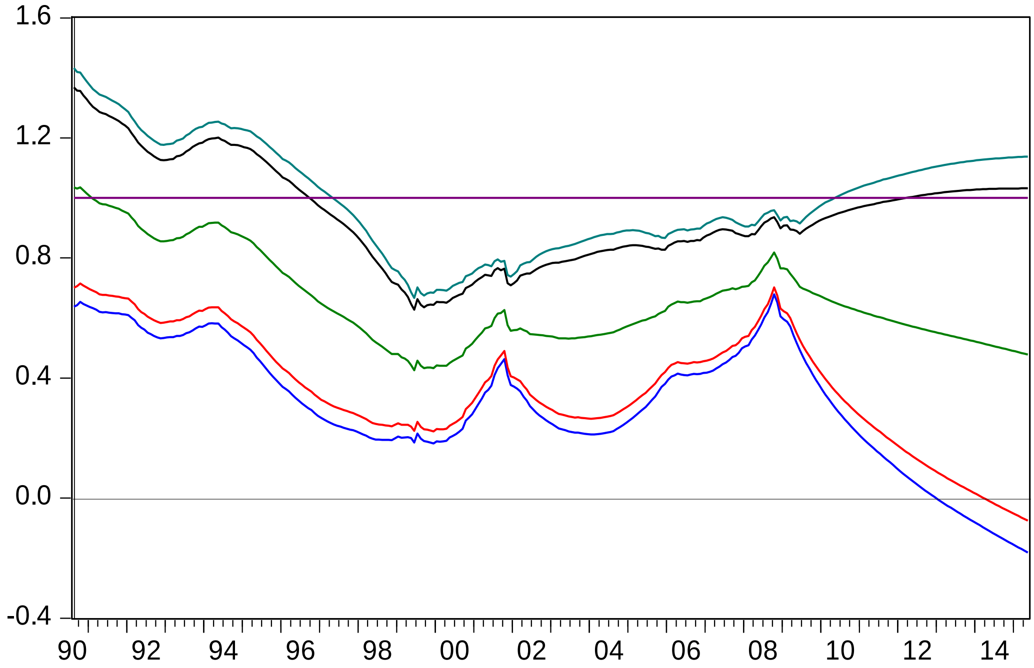
Sample	Aug. 1990 – May 2015 (*T*=298)	Aug. 1990 – Dec. 2008 (*T*=221)
Signal equation		
$\rho_3 N_o$	19.31*** (2.65)	19.49*** (3.42)
α	1.52*** (0.50)	1.58** (0.63)
δ	0.41*** (0.03)	0.41*** (0.03)
γ	0.66*** (0.04)	0.67*** (0.05)
b_1	0.86*** (0.18)	0.78** (0.34)
b_2	0.37*** (0.18)	0.35 (0.35)
MA(-1)	-0.24*** (0.07)	-0.25*** (0.08)
MA(-2)	0.16*** (0.06)	0.16** (0.07)
k_M	-4.18*** (0.06)	-3.91*** (0.08)
State equation		
λ_o	-	-
θ	0.99*** (0.005)	0.99*** (0.004)
k_S	-6.10*** (1.01)	-6.91*** (1.54)
$\overline{R}^2$	0.71	0.70
$\overline{R}_D^2$	0.57	0.56
Q(4)*	10.06	4.73
hH(h)*	35.01	46.20
AIC	-1.28	-1.02
SC	-1.27	-1.00
HQC	-1.28	-1.01
L	192.03	113.43

Notes - Numbers in parentheses are the standard deviations. The estimated state-space model is given by the system made by the measurement equation (20) and the state equation (21). The sample period in column 2 is the full sample, the one in column 3 is the sub-sample excluding the zero-lower bound of short term interest rates. The constant $\rho_3 N_o$ was first assessed at the value of a grid search corresponding to the lowest information criteria and re-estimated while the other parameters were set to their estimated values to compute its standard-deviation. The estimated intercept of the state equation being insignificantly different from zero, final estimates are obtained by setting $\lambda_o = 0$. To ensure positivity, the unconditional variances of ζ_t and η_t are estimated as exp(k_M) and exp(k_S), respectively. R_D^2 is a measure of goodness of fit which compares to a random walk with drift (Harvey, 1992). AIC, SC and HQC stand for Akaike, Schwarz and Hannan and Quinn information criteria.. The asymptotic critical values of the *Q**-statistic follow a χ^2 with $\sqrt{298} - 10 = 7$ d.o.f. at the full sample and $\sqrt{221} - 10 = 5$ d.o.f. at the subsample and are (12.0, 14.1, 18.5) and (9.2, 11.1, 15.1) at the (10%, 5%, 1%) levels of significance, respectively. The asymptotic critical values of the *hH** statistic follow a χ^2 with 298/3=99 at the full sample and 221/3=74 d.o.f. at the subsample and are (118, 124, 136) and (85.5, 90.5, 100) at the (10%, 5%, 1%) levels of significance, respectively.

The error correction mechanism in the measurement equation is validated, the significant error correction parameter $\gamma = 0.66$ indicating an average period of adjustment of the observed premium to its required value close to 2 months. The change in the 12-month T-Bill rate appeared significant with a positive sign consistent with the expected one (coefficient δ). The positive estimated value of the ratio α indicates an amount of the 3-month Bill that is more than 1.5 times the amount of the 6-month Bill. The estimated constant $\varphi\ {}_3N_o$ is positive, which is in accordance with the expected sign. The estimates $b_1 = 0.86$ and $b_2 = 0.37$ imply that the expected covariance (19) is determined by a weighted average of actual and past observed covariances with decreasing weights, which is rather intuitive. We now discuss the inference of the state variable λ_t (Eq.(21)). The intercept λ_o was removed in the final estimation since it failed to be significant. The slope θ of the lagged value is not significantly different from 1 at the 5% level, suggesting that λ_t seemingly follows a random walk. The time-pattern of the estimated values of λ_t lies between 0 and 1 (Figure 2), which means that, at any time, both horizons play a role in the determination of the risk premium. Moreover, the limit values 0 and 1 stand outside the 95% confidence interval during almost all the period for the former and most of the period for the latter, implying that generally they cannot be statistically accepted. This innovative result contradicts the invariant hypothesis of the literature according to which $\lambda_t = 1 \forall t$ and consequently that the short term rate is the riskless rate. Further to our result that the weight is different from 1, we find that it significantly varies over time. This is seen from the evidence that, on the one hand, no constant value can unchangingly be included in the confidence intervals along the sample period. On the other hand, consistently with its non-stationary pattern, λ_t follows a slight downward trend with swings around a mean of 0.65, which reflects a dominance of the 3-month horizon vis-à-vis the 6-month horizon by two-thirds / one third on average. While the market was dominated by the 3-month horizon in the beginning of the period and at the end of 2000s, the 3- and 6-month horizons seem to be balanced between late 1990's and early 2000s and tend towards a balanced distribution since the 2008 financial crisis. The post-2008 period is characterized by a collapse in the short term US interest rates to near-zero values, giving rise to a situation of liquidity trap and the implementation of unconventional monetary policies by the Federal Reserve. To check whether this zero-lower bound of short term interest rates have distorted the estimates over the period, we also performed the Kalman filter estimation over the sub-period excluding the zero-lower bound (i.e., over the sub-period

08/1990-12/2008). The results are provided in column 3, Table 2. A Wald test of equality between estimates from this sub-period and those from the full sample has shown that the null of equality very strongly failed to be rejected at the 5% level, indicating that no significant bias has resulted from the zero-lower bound.

Figure 2. Degree of preference for the 3-month horizon (state variable λ_t)



Note - The central line represents the smoothed estimated values of the state variable λ_t; the outer and inner intervals are calculated as $\lambda_t \pm 1.96\, SD_t$ and $\lambda_t \pm 1.645\ SD_t$ corresponding to the 95% and 90% confidence bounds, respectively (SD_t: conditional standard deviation of λ_t).

The statistical properties of the standardized smoothed residuals of the measurement equation can be examined by performing appropriate Ljung-Box autocorrelation Q^* test and heteroskedasticity hH^* test developed by Harvey (1992). The Q^* test is a modified version of the standard Portmanteau test, where the underlying Ljung-Box statistic follows a χ^2 distribution with a $\sqrt{T} - m + 1$ degrees of freedom, where T is the sample size and m the number of parameters. According to our test statistics values (with 4 lags), the null of no residual autocorrelation fails to be rejected at all levels of significance. The hH^* test compares the sum of squared smoothed standard residuals between two sub-periods defined as the one-third and the two-thirds of the sample period. The asymptotic distribution of the statistic is a χ^2 with $T/3$ d.o.f. According to this test, we can conclude that the null of no heteroskedasticity is not rejected for all levels of significance. Overall, these results suggest that innovations are well-behaved and that the conditions of application underlying the Kalman filter modelling are satisfied (see Stock and Watson (1998), Durbin and Koopman (2001).

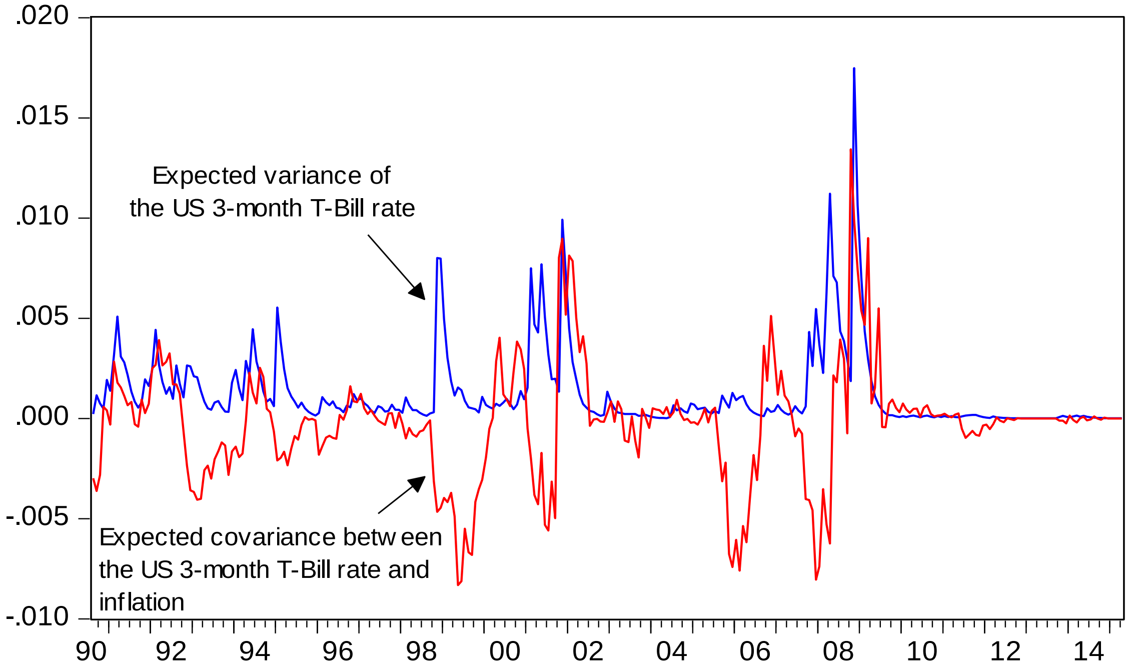
Furthermore, the non-stationarity and autocorrelation features of λ_t suggest that some underlying factors may be detected. We then carried out an empirical analysis consisting in regressing λ_t on observable variables. To ensure that our estimates are robust to heteroscedasticity and autocorrelation, we used the Newey-West method and found that more than 50% of the variance of λ_t can be represented by the expected rate of inflation and the expected growth in the real GDP, both for the next calendar year, denoted $E_t \pi_{y+1}$ and $E_t g_{y+1}$, respectively (the data were provided by Consensus Economics). Other variables tested and which appeared to be insignificant whether lagged or not were the uncertainty about future inflation (measured as the expected volatility estimated using a GARCH model), the degree of heterogeneity in interest rates expectations (standard errors of the 3- and 12-month ahead expected values of the three-month rate), the observed inflation rate, the observed real GDP growth rate and the 10-year maturity Treasury bonds yield (or its change). We finally obtain the following regression (*t* values in brackets) :

$$\lambda_t = \underset{(4.11)}{0.40} + \underset{(7.26)}{0.16} E_t \pi_{y+1} - \underset{(-2.50)}{0.05} E_t g_{y+1} + e_t \qquad R^2 = 0.558 \quad DW = 0.10$$

where e_t is the residual term. When expected inflation for the next year increases, to avoid negative real interest rates at longer horizons agents prefer to shorten their investment horizon, which implies a higher value of λ_t. Conversely, when the expected growth rate in real GDP increases, positive real interest rates are expected at longer horizons, hence encouraging investors to extend their time horizon, which implies a lower value of λ_t. May these outcomes result from arbitrary regressor selection, they enhance the credibility of the estimated dynamics of λ_t described by the state equation. However, the *DW* statistic shows that they are strongly auto-correlated, which clearly indicates a problem of missing factors. A more thorough theoretical investigation on the factors explaining the degree of preference for the short horizon should be carried on and more adequately tested directly at the stage of the estimation process of the structural model; however, this research is beyond the scope of this study.

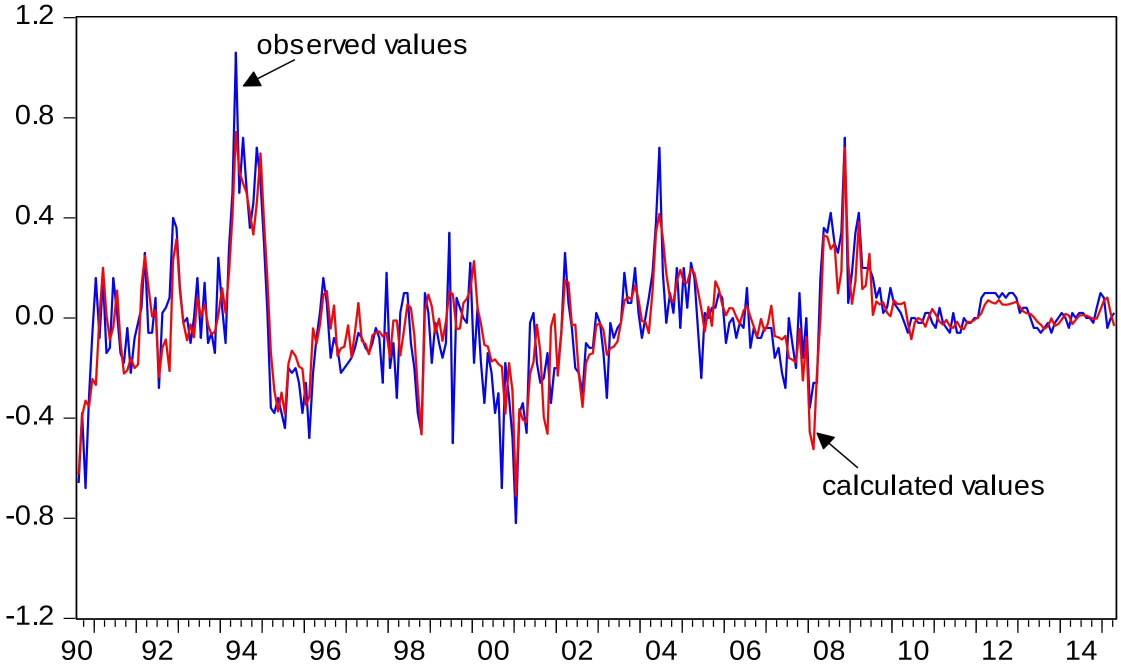
Figure 3 exhibits the estimated values of the expected covariance between the 3-month rate and inflation (Cov_t) and of the expected variance of the 3-month rate (V_t), which both are influential factors of the risk premium. It can be seen that these two factors fluctuate with the same order of magnitude. The weak correlation between V_t and Cov_t over the period shows that they are complementary in explaining the premium, except during the 2008 financial crisis where they move together with sharp fluctuations before they die out at the zero-lower bound of short term interest rates. Interestingly, these findings are consistent with Filardo and Hofmann (2014) who show that the forward guidance, one of the unconventional monetary policies conducted by the Federal Reserve at the zero-lower bound, led to lower the volatility of expectations about the future path of policy interest rates.

Figure 3 : Expected variance of the US 3-month interest rate and expected covariance between interest rate and inflation



We now examine the ability of our state-space model to describe the risk premium dynamics. Figure 4 compares the "observed" values of the market *ex-ante* risk premium to the fitted values from the measurement equation (20): the major fluctuations are well reproduced and especially no systematic lags between observed and fitted values can be reported. We further checked the relevance of this fit using Harvey's (1992) modified coefficient of determination R_D^2 that assesses the goodness of the fit with respect to a simple random walk plus drift process.[21] The R_D^2 value indicates that the residual variance of the signal equation is 0.43 times the one of the random walk model, which implies that our model strongly outperforms the random walk. It should also be noted that, when the market risk premium model (17) is estimated assuming that the weight in the required risk premium (15) is constant, all of the information criteria *AIC*, *SC* and *HQC* take much higher values than those obtained assuming a time-varying weight. This suggests that our model with time-varying weight outperforms the one with constant weight.

Figure 4. Observed and fitted values of the two-horizon ex-ante risk premium



4 Conclusion

This article aims at revisiting the standard model of the risk premium, which is a key component of the term structure of interest rates. We propose a new approach relaxing the commonly accepted joint hypothesis that the riskless debt is the one-period maturity claim, while the risky debt is the n-period maturity claim (n>1). This joint hypothesis is indeed valid

[21] Harvey's (1992) goodness of fit measure is given by $R_D^2 = 1 - SSR \,/ \sum_{t=2}^{T} (\Delta y_t - \overline{\Delta y})^2$ where y_t and *SSR* are the dependent variable and the sum of the squared residuals of the measurement equation, respectively. A negative R_D^2 would imply that the estimated model is beaten by a simple random walk plus drift.

only when the investor wishes to make a one-period investment. When the investor prefers to undertake a *n*-period investment, the riskless rate is the rate on a (zero-coupon) claim whose duration coincides with the investor's horizon.

We consider a representative agent investing both at the 1-period and 2-period horizons and comparing for each horizon the *ex-ante* market premium with the required value so as to choose between a riskless investment and a risky investment. On theoretical grounds, we find that both the 1- and 2-period risk premia offered by the market can take positive or negative values but are strictly of opposite signs at any time. At the demand side, we calculate the required values of the 1- and 2-period premia as solutions of a portfolio choice model, according to which the representative agent maximizes the expected real value of their future wealth. Contrary to the premium offered, the required premia are not symmetrical although they can also take positive or negative values. Because the representative investor considers at any time both horizons, at the equilibrium the observed *ex-ante* premium offered by the market is equal to the weighted average of the 1- and 2-period required premia, where the time-varying weight associated with the 1-period (resp. 2-periods) required premium measures the degree of preference of the market for the 1-period (resp. 2-period) horizon. Moreover, due to market frictions, we assume that the market premium adjusts towards its required value gradually. Our model is thus a more general approach than the standard hypothesis according to which the riskless debt is exclusively the short maturity claim. While our model can reduce to the standard model as a special case, it allows to measure the share of the market for which the riskless rate is the short rate and the share for which the riskless rate is the long rate.

By setting one period equal to three months, we use 3-month ahead expected values of the US 3-month Treasury bill rate provided by Consensus Economics surveys and the 3- and 6-month Bill rates to estimate our 3- and 6-month horizon risk premium model over the period November 1989 – May 2015. With these data, we found that the risk premium explains about half of the variance of the spread, which reflects the importance of modeling the risk premium to explain the term structure of interest rates. We estimate our hybrid model of the risk premium with time varying weights using the Kalman filter methodology. Our results strongly invalidate the restrictive hypothesis of the literature that the riskless (risky) rate is always given by the short (long) rate and support the evidence that, at any time, the market refers to both maturities to define the riskless and risky rates with a 3-month maturity weight varying around its mean of 0.65 over the sample period. Overall, our hybrid two-horizon model fits well the *ex-ante* market risk premium, which suggests that, to describe adequately risky strategies, the market preferences for both short and long horizons must be taken into account when modeling the term structure of interest rates.

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