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Ludovic A. Julien[1] & Gagnie P. Yebarth[2]

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This paper explores the possibility that a taxation mechanism always implements a Pareto-optimal allocation in bilateral exchange when the market participants behave strategically and noncooperatively. To this end, we reconsider the taxation mechanism, namely the endowment taxation with transfers, implemented in the strategic bilateral exchange models by Gabszewicz and Grazzini (JPET, 1999). In this framework of strategic bilateral exchange, we consider a general class of smooth utility functions, and we determine the conditions under which the taxation mechanism is Pareto-optimal, i.e., whether there exists an equilibrium tax such that endowment taxation with transfers *always* implements a Pareto-optimal allocation. Furthermore, we explain why this taxation mechanism could implement a Pareto-optimal allocation.

Key Words: Cournot-Nash equilibrium, Pareto-optimality, taxation
Subject Classification: C72, D41, H21

1. INTRODUCTION

Taxation mechanisms are generally implemented either for redistributive purposes (Mirrlees 1971; Tuomala 2016), or to correct market failures caused by negative externalities (Sandmo 1975; Kopczuk 2003), asymmetric information (Biais et al., 2023), or imperfectly competitive behavior (Aumann and Kurz 1977; Guesnerie and Laffont 1978; Myles 1989; Collie 2019). Taxation mechanisms that fulfill the dual role of levying and redistributing resources can be implemented in exchange economies in which agents behave strategically and non-cooperatively. From this perspective, Gabszewicz and Grazzini (1999) consider the effectiveness and the welfare implications of a taxation mechanism in strategic exchange named *endowment taxation with transfers*. A tax is levied on the initial endowments of agents before exchange takes place. After exchange, there is a redistribution of the tax product among the agents. This taxation mechanism implements a Pareto-optimal allocation when the preferences of agents are represented by linear, Cobb-Douglas, and CES utility functions. Elegbede et al. (2022) extend this result by considering general CES utility functions, and show that this taxation mechanism can implement a Pareto-optimal allocation when commodities are perfect complements or perfect substitutes. The main objective of this paper is to address the following question: to what extent does such a taxation mechanism *always* implement a Pareto-optimal allocation when all participants to exchange behave strategically and non-cooperatively?

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This question has empirical resonance as well as a wide range of applications in the globalized market economy, where large agents such as companies, retailers, or traders, who possess or use some scarce resources, behave strategically and non-cooperatively. From this perspective, theoretical models with empirical evidence have also considered the importance of interactions between markets with agents who behave strategically (Haufler 2008). As an illustration, taxation with redistribution is justified on the grounds that large, specialized firms located in different countries can affect competition on the world market (Colo-Martinez et al. 2007; Head and Spencer 2017). A further illustration is the presence of negative externalities due to pollution and caused by agents who exert their market power in strategic bilateral trade (Nkuiya and Plantinga 2021; Julien et al. 2023). These applications have a common feature: markets include large agents who manipulate relative prices to their own advantage so that the allocation resulting from market transactions is not Pareto-optimal. Thus, the problem is to determine to what extent a taxation mechanism based on two instruments, i.e., a tax scheme and a redistribution scheme through transfers, can restore Pareto-optimality.

The motivations associated with the main objective of this paper are twofold. First, we will study the robustness of the optimality result of Gabszewicz and Grazzini (1999). These authors consider three models with strategic trade in which the preferences are represented by linear, Cobb-Douglas and CES utility functions. The taxation mechanism implements a Pareto-optimal allocation in the Cobb-Douglas and CES exchange models.[3] Elegbede et al. (2022) extend this result to CES utility functions with non constant shares in consumption, and show that this taxation mechanism implements a first-best allocation when commodities are perfect complements or perfect substitutes. In all the above exchange economies, the preferences of traders are represented by homogeneous (then homothetic) utility functions. Thus, in view of this first motivation, these results will lead us to reframe the question addressed previously as follows: is there a wide class of smooth utility functions for which endowment taxation with transfers *always* implements a first-best allocation? Then, this will lead us to specify the properties of utility functions, that could be neither homogeneous nor homothetic, that guarantee that the allocation resulting from the market equilibrium and from these transfers is Pareto-optimal.

Second, we will determine the reason(s) why such a taxation mechanism would always implement a Pareto-optimal allocation. Indeed, we will determine whether the optimality of the tax mechanism will depend on the assumptions about the fundamentals such as endowments and preferences and/or on some other assumption. It turns out that beyond the technical assumptions made on the utility functions that will guarantee the existence of a market equilibrium with(out) taxation, a second kind of assumptions will concern the fundamentals of the exchange economy.

[3] The linear bilateral oligopoly automatically implements the competitive allocation when exchange takes place. Besides, Gabszewicz and Grazzini (2001) consider an exchange economy in which two goods are initially held by a finite number of inside agents, the traders, and an outside agent who owns nothing. The traders behave strategically and the outside agent does not participate in the exchange. They notably consider two kinds of taxes with transfer to the outside agent: *ad valorem taxation* which consists of taxing the supplies of traders when exchange takes place, and *endowment taxation* with consists of taxing the endowments of traders before exchange takes place. By assuming that the preferences are represented by the same Cobb-Douglas utility function, they show that the two kinds of fiscal policies can only reach a second-best. Indeed, without transfers among insiders, such fiscal policies are not sufficiently powerful to neutralize the market power of strategic traders. Elegbede et al. (2022) generalize these results to the case of CES utility functions.

These hypotheses will lead us to explore possible sources of heterogeneity. This heterogeneity will not directly apply to the structure of the game. The game associated with the exchange economy will be a quantity setting game *a la* Cournot with complete and imperfect information. The heterogeneity in question will stem more essentially from the fundamentals of the exchange economy, namely the number of agents per sector, the structure of initial endowments, and the utility functions. We will refer to the notion of *symmetric market* for an exchange economy that will satisfy the requisite hypotheses. Thus, in view of this second motivation, we are led to reframe the original question as follows: why this taxation mechanism would *always* implement a Pareto-optimal allocation? In this way, we will explore the assumptions on the fundamentals of the exchange economy that explain why the taxation mechanism implements a Pareto-optimal allocation.

To determine whether a taxation mechanism that plays this dual role can always implement a Pareto-optimal allocation in markets where all agents behave strategically and non-cooperatively, we consider a class of non-cooperative strategic exchange models. This class of models studies the strategic interactions in the framework of an exchange economy in which two commodities are initially held by a finite number of traders (Dubey and Shubik, 1978). The strategic behavior of traders is introduced by embedding the finite exchange economy within a noncooperative simultaneous move game in which the players are the traders, the strategies are their supplies, and the payoffs are the utility levels they achieve at equilibrium, i.e., at a Cournot-Nash equilibrium (CNE thereafter). Insofar as two commodities are exchanged, and each type of trader is initially endowed with only one commodity, this class of noncooperative strategic bilateral exchange models is akin to a bilateral oligopoly framework introduced by Gabszewicz and Michel (1997), and developed by Bloch and Ghosal (1997), Dickson and Hartley (2008), Amir and Bloch (2009), Busetto et al. (2020), among others.[4] This framework offers a natural starting point for studying the distortions generated by strategic behaviors in interrelated markets and the public policies to be implemented to possibly restore Pareto-optimality.

Within this framework of bilateral oligopoly, we consider that the preferences of traders are represented by strictly quasi-concave utility functions which satisfy some usual regularity technical assumptions.[5] First, we study the existence and uniqueness of an interior CNE without taxation. To show existence of a unique interior CNE without taxation, we adapt to our framework of strictly quasi-concave utility functions a proof made by Bloch and Ghosal (1997) to show existence of an interior CNE for strictly concave functions. The generalization of the model to strictly quasi-concave utility functions is not trivial: it requires taking into account situations where marginal utilities are increasing, which enlarges the number of

[4]In bilateral oligopoly all traders behave strategically using quantities as strategies. There is a trading post to which traders may offer a fraction of their endowment of the good to be exchanged for the other good, and which aggregates the strategic supplies of all traders and allocates the amounts traded to each trader in proportion of her supply. Ffor a survey, see Dickson and Tonin (2021). More broadly, the bilateral oligopoly model is a two commodity version of the strategic market game models (Shapley, 1976; Shapley and Shubik, 1977; Dubey and Shubik, 1978; Sahi and Yao, 1989; Amir et al., 1990). With two commodities and corner endowments, no distinction occurs between the prototypical models of strategic maket games.

[5]There are two class of bilateral oligopolies studied in Gabszewicz and Grazzini (1999) which do not implement automatically the competitive allocation, and in which the taxation mechanim is implemented: the Cobb-Douglas and CES bilateral oligopolies. The Cobb-Douglas and CES utility functions are strictly quasi-concave in the interior of the consumption set.

cases to be studied in order to prove the existence of a unique interior CNE without taxation. Second, we study the non-optimality property of the CNE without taxation. Third, we study the implementation of the taxation mechanism, namely, *endowment taxation with transfers*, introduced by Gabszewicz and Grazzini (1999). This taxation mechanism is based on two tools: a taxation scheme and a redistribution scheme through transfers. This kind of tax and transfers are not linked to the individual characteristics such as preferences, but merely to the commodities themselves: tax is imposed on traders in the same manner for the same commodity, and transfers are performed in such a way each trader receives the same amount of the commodity with which s/he is not initially endowed with, and which is proportional to the supply of the commodity s/he sends to the market for trade. This taxation mechanism works as follows: tax are levied before exchange takes place and transfers to traders are implemented after exchange has occurred. As a result, the strategic market game is modified in two ways: taxation affects strategic sets in such a way that new strategic sets are strictly included in old ones, and payoffs are altered by transfers. Thus, like in Gabszewicz and Grazzini (1999), the problem is to determine to what extent it is possible to manipulate the strategic possibilities of traders in such a way to lead the CNE allocation of the game with taxation and transfers to coincide with the competitive equilibrium allocation. Hence, we wonder whether such a taxation mechanism *always* implements a Pareto-optimal allocation when traders still behave strategically and noncooperatively.

The main result of the paper is a theorem which states the Pareto-optimality of a CNE of the game with taxation. Thus, based on the existence and uniqueness of a CNE without taxation, we show that there exists an interior CNE with taxation such that the taxation mechanism *always* implements a Pareto-optimal allocation in the game with taxation. This result is reminiscent of the second welfare theorem in general equilibrium analysis but with strategic trade. Additionally, a corollary of the theorem shows the uniqueness of the optimal tax, that is, there does not exist another endowment tax with transfers such that the overall-allocation resulting from the interior CNE of the game with taxation and transfers is Pareto-optimal. To put in a nutshell, our result extends the conditions under which a fiscal policy with transfers implements a Pareto-optimal allocation in strategic bilateral trade. On the one hand, we generalize the results of Gabszewicz and Grazzini (1999) to a broader class of utility functions, and thereby to a larger class of bilateral oligopoly models. On the other hand, our results also echo Elegbede et al. (2022) who show that when the preferences of traders are represented by CES utility functions with non unitary shares on consumption, the fiscal policies with transfers implement a first-best allocation in two limit cases, i.e., only when commodities are either perfect complements or perfect substitutes.

These results are based on two kinds of assumptions. The first kind are technical assumptions concerning utility functions. These technical assumptions are made to study the existence and uniqueness of an interior non-cooperative equilibrium in the strategic market game. Furthermore, in order to generalize the results of Gabszewicz and Grazzini (1999) and Elegbede et al. (2022), we do not require utility functions to be homogeneous or even homothetic. We only assume that the utility functions be smooth, strictly monotonic, and strictly quasi-concave. Thus, we also provide a core example with a class utility of functions that are neither homogeneous nor homothetic, namely the quasi-linear utility functions, allowing us to explore the effectiveness and the welfare implications of the taxation mechanism in quasi-linear bilateral oligopolies. Besides, we discuss the model to determine the reason why the tax mechanism is Pareto-optimal. To this end, we relax the assumption of a symmetric market mentioned earlier. Thus, we use examples to explore sources of heterogeneity. The objective is to test the robustness of our results when the market is no longer symmetric.

The remainder of the paper is organized as follows. In Section 2 we study the basic model, namely the existence, the uniqueness and the non-optimality of an interior Cournot-Nash equilibrium. Section 3 is devoted to the implementation of the taxation mechanism, and its effects and implications on welfare. Section 4 provides a core example with non-homogeneous and non-homothetic utility functions. In Section 5 we discuss the model. In Section 6 we conclude.

2. THE MODEL

This section is devoted to the presentation of the strategic market game we are considering. First, we describe the model, i.e., the exchange economy and the game associated with it. Second, we study the main properties of the noncooperative equilibrium of the game. In particular, the equilibrium analysis will focus on market inefficiencies at work as a prerequisite for the study of a taxation mechanism that can be implemented to restore Pareto optimality.

2.1. The strategic market game

Consider an exchange economy, namely $\mathcal{E}$. The space of commodities is $\mathbb{R}^2_+$. The two divisible homogeneous commodities are labeled X and Y, with unit prices p_X and p_Y, where $p_Y = 1$ (commodity Y is the numeraire). The set $\mathcal{T}$ of traders is purely atomic, i.e., it contains only atoms. Traders are of two types, namely 1 and 2, with $\mathcal{T} = T_1 \cup T_2$, and $T_1 \cap T_2 = \varnothing$, where T_1 and T_2 are the subsets of type 1 and type 2 traders respectively. We assume $2 \leqslant \#T_1 < \infty$ and $2 \leqslant \#T_2 < \infty$, where "#" denotes the cardinality of the set. Traders are indexed by i.

There are fixed initial endowments whose distribution satisfies the following assumption.

ASSUMPTION 1. $\forall i \in T_1$ $\mathbf{w}_i = (\alpha, 0)$, $1 \leqslant \alpha < \infty$, and $\forall i \in T_2$ $\mathbf{w}_i = (0, \beta)$, $1 \leqslant \beta < \infty$.

The preferences of type 1-traders are represented by a utility function $u_1 : \mathbb{R}^2_+ \to \mathbb{R}$, $(x, y) \mapsto u_1(x, y)$. Likewise, for type-2 traders, let $u_2 : \mathbb{R}^2_+ \to \mathbb{R}$, $(x, y) \mapsto u_2(x, y)$.

Among the class of possible exchange economies such as $\mathcal{E}$, one will be of particular interest. This leads us to define the notion of a *symmetric market.*

DEFINITION 1 (symmetric market). The market is said to be *symmetric* between both types of traders whenever the following three conditions are satisfied:

1. $\#T_1 = \#T_2$;
2. $\alpha = \beta$;
3. $\forall (x, y) \in \mathbb{R}^2_+$ $u_1(x, y) = u_2(y, x)$.

If at least one of the conditions in definition 1 is not met, the market will be said to be *asymmetric.*[6]

[6] To avoid messy repetitions, and with a slight abuse of notations, we will sometimes use the notation $u(x, y)$ when the distinction does not matter.

Let $(p_X, 1)$ be a price vector. A competitive equilibrium (CE thereafter) of $\mathcal{E}$ is a price vector $(p_X^*, 1)$ and an allocation $\mathcal{A}^* = (x_i^*, y_i^*)_{i \in \mathcal{T}}$ such that $\sum_{i \in \mathcal{T}} x_i^* = \alpha(\#T_1)$ and $\sum_{i \in \mathcal{T}} y_i^* = \beta(\#T_2)$, and, $\forall i \in T_1$ (x_i^*, y_i^*) solves $\max_{(x_i, y_i)} u_1(x_i, y_i)$ s.t. $p_X^* x_i + y_i \leqslant \alpha p_X^*$, and $\forall i \in T_2$ (x_i^*, y_i^*) solves $\max_{(x_i, y_i)} u_2(x_i, y_i)$ s.t. $p_X^* x_i + y_i \leqslant \beta$.

To $\mathcal{E}$, we associate the Dubey-Shubik (1978) strategic market game $\mathbf{\Gamma}$. The game $\mathbf{\Gamma}$ is a simultaneous move quantity setting game in which information is complete but imperfect. Insofar as two goods are exchanged, and each type of trader is initially endowed with only one commodity, this strategic market game is akin to a bilateral oligopoly model such as those introduced by Gabszewicz and Michel (1997). The traders behave strategically and can offer only a fraction of the commodity they initially hold. Let q_i (resp. b_i) be the supply of commodity X (resp. Y) by trader $i \in T_1$ (resp. $i \in T_2$). It represents the amount of commodity X (resp. Y) that trader $i \in T_1$ (resp. $i \in T_2$) offers in exchange for commodity Y (resp. X). Therefore, the strategy sets are given by:

$$\mathcal{Q}_i = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant \alpha\},\ i \in T_1; \tag{1}$$

$$\mathcal{B}_i = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant \beta\},\ i \in T_2. \tag{2}$$

A market price mechanism aggregates the strategic supplies of all traders and allocates the amounts traded to each trader. Indeed, given a $(\#T_1 + \#T_2)$-tuple of supply strategies $(\mathbf{q}; \mathbf{b})$, with $(\mathbf{q}; \mathbf{b}) \in \prod_{i \in T_1} \mathcal{Q}_i \times \prod_{i \in T_2} \mathcal{B}_i$, the relative price p_X, which satisfies the market clearing condition $\sum_{k \in T_1} q_k = \frac{1}{p_X} \sum_{k \in T_2} b_k$, may be written:

$$p_X(\mathbf{q}; \mathbf{b}) = \begin{cases} \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k}, \text{ if } \sum_{k \in T_2} b_k > 0 \text{ and } \sum_{k \in T_1} q_k > 0; \\ \\ 0, \text{ if } \sum_{k \in T_2} b_k = 0 \text{ or } \sum_{k \in T_1} q_k = 0. \end{cases} \tag{3}$$

Then, from (1), trader $i \in T_1$ consumes the amount $\alpha_i - q_i$ of commodity X, and obtains in exchange for q_i a quantity of good Y equal to $p_X q_i$ (recall $p_Y = 1$), so she ends up with the bundle of commodities $(x_i, y_i) = (\alpha_i - q_i, p_X q_i)$. Likewise, from (2), trader $i \in T_2$ consumes the amount $\beta_i - b_i$ of commodity Y, and obtains in exchange for b_i a quantity of good X equal to $\frac{1}{p_X} b_i$, so she ends up with the bundle of commodities $(x_i, y_i) = (\frac{1}{p_X} b_i, \beta_i - b_i)$. Therefore, the final allocation that results from trade may be written:

$$(x_i(\mathbf{q}; \mathbf{b}), y_i(\mathbf{q}; \mathbf{b})) = \begin{cases} \left(\alpha - q_i\ , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i\right), \text{ if } p_X > 0, \\ \\ (\alpha, 0), \text{ if } p_X = 0 \end{cases},\ i \in T_1; \tag{4}$$

$$(x_i(\mathbf{q}; \mathbf{b}), y_i(\mathbf{q}; \mathbf{b})) = \begin{cases} \left(\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i, \beta - b_i\right), \text{ if } p_X > 0, \\ \\ (0, \beta), \text{ if } p_X = 0 \end{cases},\ i \in T_2. \tag{5}$$

Then, the payoffs in $\mathbf{\Gamma}$ may be defined as $\pi_i : \prod_{i \in T_1} \mathcal{Q}_i \times \prod_{i \in T_2} \mathcal{B}_i \rightarrow \mathbb{R}$, $i \in T_1 \cup T_2$, with

$$\pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}) = u_1 \left(\alpha - q_i \ , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i \right), \, i \in T_1; \tag{6}$$

$$\pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}) = u_2 \left(\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i, \beta - b_i \right), \, i \in T_2, \tag{7}$$

where $\mathbf{q}_{-i} = (q_1, ..., q_{i-1}, q_{i+1}, ..., q_{\#T_1})$ is a $(\#T_1 - 1)$-tuple of supply strategies of type 1-traders, and $\mathbf{b}_{-i} = (b_{\#T_1+1}, ..., b_{\#T_1+1-i}, b_{\#T_1+1+i}, ..., b_{\#T_1+\#T_2})$ is a $(\#T_2-1)$-tuple of supply strategies of type 2-traders.

A noncooperative equilibrium is a Cournot-Nash equilibrium (CNE henceforth) of the bilateral oligopoly game Γ. More formally, we define a CNE as follows.

DEFINITION 2 (CNE). A Cournot-Nash equilibrium of Γ is given by a $(\#T_1 + \#T_2)$-tuple of strategies $(\tilde{q}_1, ..., \tilde{q}_{\#T_1}; \tilde{b}_{\#T_1+1}, ..., \tilde{b}_{\#T_1+\#T_2}) \in \prod_{i \in T_1} \mathcal{Q}_i \times \prod_{i \in T_2} \mathcal{B}_i$, such that:
$\forall i \in T_1 \ \pi_i(\tilde{q}_i, \tilde{\mathbf{q}}_{-i}; \tilde{\mathbf{b}}) \geqslant \pi_i(q_i, \tilde{\mathbf{q}}_{-i}; \tilde{\mathbf{b}})$, for all $q_i \in \mathcal{Q}_i$;
$\forall i \in T_2 \ \pi_i(\tilde{\mathbf{q}}; \tilde{b}_i, \tilde{\mathbf{b}}_{-i}) \geqslant \pi_i(\tilde{\mathbf{q}}; b_i, \tilde{\mathbf{b}}_{-i})$, for all $b_i \in \mathcal{B}_i$.

A CNE is said to be type-symmetric when $\tilde{q}_i = \tilde{q}$, for each $i \in T_1$, and $\tilde{b}_i = \tilde{b}$, for each $i \in T_2$. Moreover, a CNE is said to be symmetric when $\tilde{q} = \tilde{b}$. Finally, a type-symmetric CNE, which is symmetric, is said to be symmetric type-symmetric.

The $(\#T_1 + \#T_2)$-tuple of supply strategies $(\tilde{q}_1, ..., \tilde{q}_{\#T_1}; \tilde{b}_{\#T_1+1}, ..., \tilde{b}_{\#T_1+\#T_2}) = (0, ..., 0; 0, ..., 0)$, with $\pi_i(0, ..., 0; 0, ..., 0) = u_i(\alpha \ , 0)$, $i \in T_1$, and $\pi_i(0, ..., 0; 0, ..., 0) = u_i(0, \beta)$, $i \in T_2$, is always a Nash equilibrium of $\mathbf{\Gamma}$, i.e., the so-called *trivial equilibrium* (Cordella and Gabszewicz, 1998; Busetto and Codognato, 2006). In what follows, we will make some assumptions on the utility functions to characterize the interior noncooperative equilibria.

2.2. Noncooperative equilibrium without taxation

We now turn to the characterization of a CNE of Γ. To this end, we make two kinds of assumptions on the utility functions. First, the utility functions satisfy the following set of regularity assumptions, which we designate as Assumption 2.

ASSUMPTION 2.
2a. $u(x, y) \in \mathcal{C}^2(\mathbb{R}^2_{++}, \mathbb{R})$;
2b. $\forall (x, y) \in \mathbb{R}^2_{++} \left(\frac{\partial u(x,y)}{\partial x}, \frac{\partial u(x,y)}{\partial y} \right) >> (0, 0)$;
2c. $\forall (x, y) \in \mathbb{R}^2_{++} \ (-1)^2 \left| \mathcal{H}^2_u \right| > 0$, where $\mathcal{H}^2_u = \begin{bmatrix} 0 & \frac{\partial u(x,y)}{\partial x} & \frac{\partial u(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial x} & \frac{\partial^2 u(x,y)}{\partial x^2} & \frac{\partial^2 u(x,y)}{\partial x \partial y} \\ \frac{\partial u(x,y)}{\partial y} & \frac{\partial^2 u(x,y)}{\partial y \partial x} & \frac{\partial^2 u(x,y)}{\partial y^2} \end{bmatrix}$.

Assumptions 2a and 2b says that the utility functions are twice-continuously differentiable and strictly monotonic, and 2c, together with 2b, that they are strictly quasi-concave in the interior of the commodity space.

Moreover, as we focus on interior noncooperative equilibrium, we also make, as in Bloch and Ghosal (1997) who showed existence of an interior CNE with strictly concave utility functions, the following assumptions, which we designate as Assumption 3.

ASSUMPTION 3.

3a. $\lim_{x\to 0}\left(-\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)<0$ and $\lim_{x\to\alpha}\left(-\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)>0$;

3b. $\forall(x,y)\in\mathbb{R}^2_+\ \frac{\partial^2 u}{\partial x\partial y}\geqslant 0$;

3c. $2\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x\partial y}>0$.

Assumption 3 constitutes a set of sufficient conditions for the existence and uniqueness of an interior equilibrium.[7] More specifically, the boundary conditions of Assumption 3a ensure that the traders are always willing to trade a fraction of their initial endowment, which is needed for an interior equilibrium. Assumption 3b, which is related to complementarity in consumption, is a sufficient condition for the best responses of traders to be well defined. Assumption 3c guarantees that the best response of a trader of one type is increasing with the offers of traders of the other type.[8]

REMARK 1. Under Assumptions 1-2-3a, the exchange economy $\mathcal{E}$ has a competitive equilibrium, which is interior. Moreover, when the market is symmetric, with $\alpha=\beta=1$, the competitive supplies, namely q_i^* and b_i^*, are such that $q_i^*=q^*$, for $i\in T_1$, and $b_i^*=b^*$, for $j\in T_2$, with $0<q^*,b^*<1$. Additionnally, we have $q^*=b^*$. Then, the equilibrium relative price is $p_X^*=1$, and the competitive allocations may be written $(x_i^*,y_i^*)=(1-q^*,q^*)$, for $i\in T_1$, and $(x_i^*,y_i^*)=(q^*,1-q^*)$, for $i\in T_2$.[9]

The next proposition relates to the existence and uniqueness of a strategic equilibrium in Γ.

PROPOSITION 1. *Let Assumptions 1-3 be satisfied. Assume the market is symmetric, with* $\#T_1=\#T_2=n$ *and* $\alpha=\beta=1$. *Then, the game* Γ *has a unique symmetric Cournot-Nash equilibrium, which is interior and type-symmetric, and which is given by the vector of supplies* $(\tilde{q},\tilde{b})$, *where* $\tilde{q}$ *and* $\tilde{b}$ *are implicitly defined by the following system of equations*:

$$\begin{cases} -\frac{\partial u_1}{\partial x_i}(1-\tilde{q}\ ,\tilde{b})\ +\frac{\partial u_1}{\partial y_i}(1-\tilde{q}\ ,\tilde{b})\frac{n-1}{n}=0,\ i\in T_1; \\ \\ -\frac{\partial u_2}{\partial y_i}(1-\tilde{b}\ ,\tilde{q})+\frac{\partial u_2}{\partial x_i}(1-\tilde{b}\ ,\tilde{q})\frac{n-1}{n}=0,\ i\in T_2. \end{cases} \tag{8}$$

PROOF. See Appendix A.

The existence of an interior CNE in bilateral market with only atoms has been studied under different assumptions made on the utility functions. Cordella and Gabszewicz (1998) studied the case of linear utility functions, and provided a condition on the marginal rate of substitution of each trader at the endowment

[7] Assumptions 3a and 3c are stated for traders of type 1. The corresponding ones for traders of type 2 are given by $\lim_{y\to 0}(x\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y})<0$, $\lim_{y\to\beta}(x\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y})>0$, and $2\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}>0$.

[8] Assumption 3c mimics Novshek's (1985) sufficient condition 3 in his Theorem 3 (p. 90) which guarantees that any firm best response decreases with the actions of the others (see Bloch and Ghosal, 1997).

[9] Indeed, the two market excess demand functions for commodity X and Y, i.e., $\xi_X(p_X,1)=\sum_{i\in T_1\cup T_2}x_i(p_X,1)-n$, and $\xi_Y(p_X,1)=\sum_{i\in T_1\cup T_2}y_i(p_X,1)-n$, are (i) continuous in the interior of the commodity set from Assumptions 1-2-3a, (ii) homogeneous of degree zero in (p_X,p_Y), (iii) bounded by Assumption 1, and (iv) they satisfy Walras' law. Then, there exist a competitive equilibrium (for the two-good case, see Arrow, 1962). Moreover, as the market is symmetric and under Assumption 3a, the competitive equilibrium is interior and symmetric, so the market clearing relative price is $p_X^*=1$, and the type 1 trader's allocation is (x^*,y^*) and the type 2 trader's allocation is (y^*,x^*). Note that when the market is not symmetric, i.e., $u_1(x,y)\neq u_2(y,x)$, multiple competitive equilibria are possible (Toda and Walsh, 2017).

point to generate trade. Bloch and Ferrer (2001) showed existence of an interior type-symmetric CNE by assuming that the utility functions are strictly increasing, strictly concave, and satisfy the boundary conditions that the marginal utilities tend to infinity when the amount of the commodity consumed tends to zero. Amir and Bloch (2009) showed existence of an interior type-symmetric CNE under the assumptions of twice-differentiable, strictly increasing, strongly quasi-concave utility functions satisfying the same boundary conditions as in Bloch and Ferrer (2001). Dickson and Hartley (2008) proved existence of an interior CNE by assuming that the preferences are continuously differentiable, binormal and satisfy a weakened axiom of gross substitutability.[10] Here, the generalization of the model to strictly quasi-concave utility functions is not trivial for two reasons. It requires to consider the case of increasing marginal utilities, which enlarges the number of cases to be studied in order to prove the existence of a unique interior CNE without taxation. Moreover, this makes it possible to generalize the results of Gabszewicz and Grazzini (1999).

Finally, let us consider the welfare property of the CNE. To this end, for each $i \in T_1 \cup T_2$, let $MRS^i(x_i, y_i) = \frac{\partial u/\partial x_i}{\partial u/\partial y_i} \mid_{(x_i,y_i)}$, be agent i's marginal rate of substitution between commodities X and Y at (x_i, y_i), which is well-defined by Assumption 2a. We can state the following proposition.

PROPOSITION 2. *The interior symmetric type-symmetric Cournot-Nash equilibrium of* Γ *is not Pareto-optimal.*

PROOF. See Appendix B.

The marginal rates of substitution differ across traders as $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n-1}{n}$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n}{n-1}$, for $i \in T_2$. As commodities are substitutable, the traders can manipulate the relative price to their own advantage by supplying less than the competitive supplies. At the CNE, the relative price is equal to the marginal rate of substitution, which is like a marginal cost, times a markup factor given by the quantity $\frac{n}{n-1}$ (resp. $\frac{n-1}{n}$) for trader of type 1 (resp. 2) which measures the market power of traders of type 1 (resp. 2).[11]

We now wonder whether there exists some taxation mechanism that will be sufficiently powerful to eliminate the distortions caused by the strategic behavior of traders, and thereby to restore Pareto-optimality.

3. NONCOOPERATIVE EQUILIBRIUM WITH TAXATION

Let us now consider the implementation of the taxation mechanism. A tax is levied before exchange takes place, and after exchange has taken place, the tax product is redistributed among the market participants (Gabszewicz and Grazzini, 1999). First, we introduce the taxation mechanism. Second, we consider the effects and the welfare implications of the implementation of the taxation mechanism.

[10] Busetto et al. (2020) used an atomless version of the boundary conditions given in Bloch and Ferrer (2001) to show existence of an interior CNE in markets with atoms and an atomless part.

[11] To see this, pick one $i \in T_1$. Let $u_1(1 - q_i, p_X q_i)$, where $p_X = \frac{B}{Q}$. Then, at the interior symmetric type-symmetric CNE, the first-order condition may be written as $-\frac{\partial u_1}{\partial x_i} + p_X(1 - \frac{1}{n})\frac{\partial u_1}{\partial y_i} = 0$, where $-\frac{1}{n} = -\frac{\tilde{q}_i}{Q} = \tilde{q}_i(\frac{\partial p_X}{\partial q_i}) \mid_{q_i = \tilde{q}_i}$, which leads to the required expression, i.e., $p_X = \frac{n}{n-1} \frac{\frac{\partial u_1}{\partial x_i}}{\frac{\partial u_1}{\partial y_i}}$. Moreover, $\lim_{n\to\infty} \frac{n}{n-1} = 1$: when the number of traders becomes large on both sides of the market the CNE converges to the unique interior competitive equilibrium.

3.1. Taxation mechanism

Let us now introduce the taxation mechanism in the model studied in Section 2. A tax is levied on the initial endowments of traders before exchange takes place. After strategic exchange has taken place, the tax product is redistributed among the traders. Thus, the original strategic market game Γ is modified in two ways: first, taxation reduces the set of admissible strategies as the strategy set of each trader is a now an interval strictly included in the older interval, and, second, transfers modify the payoffs of each trader insofar as the transfers she receives are added to the income she obtains from strategic trade. More formally, consider that, before exchange takes place, a tax t_X, with $0 < t_X < \alpha$, is levied on the endowments of commodity X, and a tax t_Y, with $0 < t_Y < \beta$, is levied on the endowments of commodity Y, generating respectively a total tax revenue equal to $\#T_1 t_X$ units of commodity X and $\#T_2 t_Y$ units of commodity Y. Therefore, this taxation mechanism generates a new stategic market game, namely $\mathbf{\Gamma}^t$. The strategy sets in $\mathbf{\Gamma}^t$ are given by:[12]

$$\mathcal{Q}_i(t_X) = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant \alpha - t_X\},\ i \in T_1; \tag{9}$$

$$\mathcal{B}_i(t_Y) = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant \beta - t_Y\},\ i \in T_2. \tag{10}$$

Furthermore, after exchange has taken place, the product of this tax is redistributed among traders as follows. Each trader is assigned a share of the total tax revenue on the commodity she does not initially own that is proportional to the supply of the commodity she is initially endowed with. Indeed, the transfer received by each trader increases with the amount of her supply. Formally, after trade has occurred at an $(\#T_1+\#T_2)$-tuple of strategies $(q_1, ..., q_{\#T_1}; b_{\#T_1+1}, ..., b_{\#T_1+\#T_2}) \in \prod_{i\in T_1} \mathcal{Q}_i(t_X) \times \prod_{i\in T_2} \mathcal{B}_i(t_Y)$, a share $s_X \equiv \frac{t_X}{\alpha - t_X - x^*}$ of the total tax product in commodity X is transferred to each trader $i \in T_2$, the denominator of which representing the solution q_i to $\alpha - t_X -\ q_i = x_i^* = x^*$, where x^* is the amount of commodity X corresponding to a competitive allocation of trader $i \in T_1$ at an interior competitive equilibrium without taxation. Likewise, let $s_Y \equiv \frac{t_Y}{\beta - t_Y - y^*}$ be the share of the total tax product of commodity Y transferred to each trader of type 1, the denominator of which representing the solution b_i to $\beta - t_Y -\ b_i = y_i^* = y^*$, where y^* is the amount of commodity Y corresponding to a competitive allocation of trader $i \in T_2$ at an interior competitive equilibrium without taxation. For each $(t_X, t_Y) \in (0, \alpha) \times (0, \beta)$, these transfers are feasible.[13]

Given an admissible strategy profile $(\mathbf{q}; \mathbf{b}) \in \prod_{i\in T_1} \mathcal{Q}_i(t_X) \times \prod_{i\in T_2} \mathcal{B}_i(t_Y)$, the relative price obtains as $p_X(\mathbf{q}; \mathbf{b}; t_X, t_Y) = \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k}$, and the resulting post tax allocation may be written:

[12] Given (t_X, t_Y), type 1 (resp. 2) trader i's equilibrium strategy in Γ^t will now be denoted by $\tilde{q}_i(t_X, t_Y)$ (resp. $\tilde{b}_i(t_X, t_Y)$). To save notations, the corresponding $2n$-tuple of strategies $(\tilde{q}_1(t_X, t_Y), ..., \tilde{q}_n(t_X, t_Y); \tilde{b}_{n+1}(t_X, t_Y), ..., \tilde{b}_{2n}(t_X, t_Y))$ is denoted by $(\tilde{\mathbf{q}}(t_X, t_Y); \tilde{\mathbf{b}}(t_X, t_Y))$, and $(\tilde{x}_i(t_X, t_Y), \tilde{y}_i(t_X, t_Y))_{i\in T_1\cup T_2}$ will denote the CNE allocation in Γ^t: no confusion will arise.

[13] To see this, consider (9) and (10), and let $0 < t_X < \alpha$ and $0 < t_Y < \beta$). From (9), we have $\forall i \in T_1\ q_i \leqslant \alpha - t_X$, which implies $\sum_{i\in T_1} q_i \leqslant \#T_1(\alpha - t_X)$, so $\sum_{i\in T_1} q_i + \#T_1 t_X \leqslant \#T_1\alpha$, while from (10), we have $\forall i \in T_2\ b_i \leqslant \beta - t_Y$, which implies $\sum_{i\in T_2} b_i \leqslant \#T_2(\beta - t_Y)$, so $\sum_{i\in T_2} b_i + \#T_2 t_Y \leqslant \#T_2\beta$.

$$(x_i(t_X,t_Y),y_i(t_X,t_Y)) = \begin{cases} \left(\alpha - t_X - q_i, \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + \frac{t_Y}{\beta - t_Y - y^*} q_i\right), \, i \in T_1; \\ \\ \left(\frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + \frac{t_X}{\alpha - t_X - x^*} b_i, \beta - t - b_i\right), \, i \in T_2. \end{cases} \tag{11}$$

Therefore, the payoffs in $\mathbf{\Gamma}^t$, i.e., $\pi_i : \prod_{i\in T_1} Q_i(t_X) \times \prod_{i\in T_2} \mathcal{B}_i(t_Y) \to \mathbb{R}$, for each $i \in T_1 \cup T_2$, may be written as follows:

$$\pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}; t_X, t_Y) = u_1\left(\alpha - t_X - q_i, \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + \frac{t_Y}{\beta - t_Y - y^*} q_i\right), \, i \in T_1; \tag{12}$$

$$\pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}; t_X, t_Y) = u_2\left(\frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + \frac{t_X}{\alpha - t_X - x^*} b_i, \beta - t_Y - b_i\right), \, i \in T_2, \tag{13}$$

Let us now determine whether the fiscal policy with transfers implement a first-best allocation.

3.2. Pareto-optimality of the taxation mechanism

We now address the following question: to what extent is it possible to manipulate the strategic possibilities of traders in such a way to lead the CNE of the game Γ^t to coïncide with the CE? Before stating the main result of the paper, in what follows, the quantity z^* will denote indifferently the amount of commodity X or Y corresponding to the competitive allocation of trader $i \in T_1 \cup T_2$ at an interior symmetric competitive equilibrium without taxation, i.e., $z^* \equiv \frac{\sum_{i\in T_1} x_i^*}{n} = \frac{\sum_{i\in T_2} y_i^*}{n}$. Moreover, t will denote an uniform tax for which $t = t_X = t_Y$. We are now able to state the following result which states that endowment taxation with transfers implements a Pareto-optimal allocation in bilateral oligopoly.

THEOREM 1. *Let Assumptions 1-3 be satisfied. Assume the market is symmetric, with $\#T_1 = \#T_2 = n$ and $\alpha = \beta = 1$. Let $z^* \in (0,1)$. Then, there exists an uniform tax on both commodities $\tilde{t} = \frac{1-z^*}{n+1}$ with transfers to traders $\tilde{s} = \frac{\tilde{t}}{1-\tilde{t}-z^*}$ such that the game Γ^t has a unique symmetric type-symmetric interior Cournot-Nash equilibrium given by the vector of supplies $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t})) = \left(\frac{(1-z^*)n}{n+1}, \frac{(1-z^*)n}{n+1}\right)$. Additionally, the allocation $\widetilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t}))_{i\in T_1\cup T_2}$ resulting from the Cournot-Nash equilibrium of the game Γ^t and from these transfers is Pareto-optimal.*

PROOF. See Appendix C.

Theorem 1 states that a taxation mechanism that levies a tax on traders' endowments and redistributes the product of the tax to the traders can implement a competitive equilibrium allocation as the unique symmetric type-symmetric interior CNE of the game with taxation Γ^t. Thus, this result is evocative of the second welfare theorem in general equilibrium analysis but with strategic trade insofar as it focuses on the redistributive purpose of the tax as well as on the optimality of the corresponding taxation mechanism when agents have market power.

The inefficiency of the equilibrium of the game without taxation is explained by the exercise of market power by traders whose strategic behavior results in the contraction of their supply. While the tax mechanism mitigates the effects of market power, its effectiveness depends on the redistribution scheme. As each trader is assigned a share of the total tax revenue on the commodity she does not initially own that is proportional to the supply of the commodity she owns, the transfer received by each trader increases with the amount of her supply. These transfers reshape each trader's payoff since they add to the consumption already obtained by strategic exchange. Thus, as the additional consumption which results from this transfer increases with supply, this redistribution creates an incentive to trade more, which increases the volume of trade.

Theorem 1 generalizes the results of Gabszewicz and Grazzini (1999, 2001) on the optimality of the endowment taxation mechanism with transfers to agents, which are obtained for specific utility functions. It also complements the results of Elegbede et al. (2022) who show that, when the preferences of traders are represented by CES utility functions with non unitary shares on consumption, the fiscal policies with transfers implement a first-best allocation only when commodities are perfect complements or perfect substitutes. Indeed, apart from the special case of linear utility functions for which the competitive allocation is implemented by the equilibrium with trade (see Gabszewicz and Grazzini 1999), our result covers, as special cases, bilateral oligopoly models with Cobb-Douglas and CES utility functions. It turns out that it should also hold when the preferences of traders are represented by non-homogeneous and non-homothetic utility functions as will be illustrated in Section 4. Thus, our result extends the set of exchange economies for which first-best optimal taxation is effective.

Correlatively, Theorem 1 has the following implication that there does not exist another tax on endowment with transfers which leads to a Pareto optimal allocation.

COROLLARY 1. *Let Assumptions 1-3 be satisfied. Assume the market is symmetric, with $\#T_1 = \#T_2 = n$ and $\alpha = \beta = 1$. The equilibrium tax with transfers to traders $\tilde{t}$ is unique, i.e., there does not exist another endowment tax with transfers to traders such that the overall-allocation resulting from the symmetric type-symmetric interior Cournot-Nash equilibrium of the game with taxation and transfers Γ^t is Pareto-optimal.*

PROOF. See Appendix D.

As a consequence, any other feasible endowment tax with transfers can only reach a second-best. Moreover, it can be shown that any endowment tax without transfer to traders does not lead to a Pareto-optimal allocation unless commodities are perfect complements or perfect substitutes (Elegbede et al. 2022). Finally, it can be shown that endowment taxation without transfer to traders can only reach a second-best allocation (Gabszewicz and Grazzini 2001; Elegbede et al. 2022). The reason which explains the sub-optimality of such taxation mechanisms stems from the fact that, without transfer to traders, the effects of the tax are not strong enough to eliminate the market distortions created by the strategic behavior of traders.

4. EXAMPLE: QUASI-LINEAR BILATERAL OLIGOPOLIES

Theorem 1 shows that the optimality of endowment taxation mechanism with transfers is consistent for a large class of utility functions. As a result, it should also hold when the preferences of traders are represented by non-homogeneous and non-homothetic utility functions which satisfy the technical Assumptions 2 and 3.[14] In Gabszewicz and Grazzini (1999), the Pareto optimality of endowment taxation with transfers is obtained for homothetic utulity functions, i.e., with Cobb-Douglas, linear, CES utility functions. In particular, we ask to what extent the fact that when: i. income demand elasticity is no longer unitary, ii. demand is no longer multiplicatively separable in price and income, and iii. the elasticity of substitution is no longer constant, may affect the effectiveness and optimality of the taxation mechanism. To illustrate this, assume that the preferences of traders are represented by *quasi-linear* utility functions.[15] Thus, the following example deserves two purposes. First, it extends the optimality result to *quasi-linear* bilateral oligopolies.[16] Second, it constitutes a core example to highligh the main features of the optimal taxation mechanism.

Consider the exchange economy $\mathcal{E}$ with the following specification. Assume the market is symmetric, with $\#T_1 = \#T_2 = n$, $\alpha = \beta = 1$, and utility functions:

$$u_i(x_i, y_i) = \begin{cases} \gamma v(x_i) + y_i, \text{ for } i \in T_1; \\ \\ x_i + \gamma v(y_i), \text{ for } i \in T_2, \end{cases} \tag{14}$$

where $0 < \gamma < 1$, with $v' > 0$, $v'' < 0$, and $\lim_{(x_i,y_i)\to(0,0)}(v'(x_i), v'(y_i)) = (\infty, \infty)$.[17]

The unique interior CE of $\mathcal{E}$ is given by $p_X^* = 1$, and $(x_i^*, y_i^*) = ((v')^{-1}(\frac{1}{\gamma}), 1 - (v')^{-1}(\frac{1}{\gamma}))$, for $i \in T_1$, and $(x_i^*, y_i^*) = (1 - (v')^{-1}(\frac{1}{\gamma}), (v')^{-1}(\frac{1}{\gamma}))$, for $i \in T_2$. The CE is Pareto-optimal as $MRS^i(x_i^*, y_i^*) = 1$, for $i \in T_1 \cup T_2$.

Consider now the computation of the CNE of the game Γ associated with $\mathcal{E}$. The strategy sets are given by $\mathcal{Q}_i = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant 1\}$, for $i \in T_1$, and $\mathcal{B}_i = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant 1\}$, for $i \in T_2$. The unique CNE of Γ, which is interior and symmetric type-symmetric, is given by $\tilde{q} = \tilde{b} = 1 - (v')^{-1}(\frac{1}{\gamma}\frac{n-1}{n})$, and $(\tilde{x}_i, \tilde{y}_i) = ((v')^{-1}(\frac{1}{\gamma}\frac{n-1}{n}), 1 - (v')^{-1}(\frac{1}{\gamma}\frac{n-1}{n}))$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = (1 - (v')^{-1}(\frac{1}{\gamma}\frac{n-1}{n}), (v')^{-1}(\frac{1}{\gamma}\frac{n-1}{n}))$, for $i \in T_2$. The CNE is not Pareto-optimal as $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n-1}{n}$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n}{n-1}$, for $i \in T_2$.

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[14] Remind that a utility function is homothetic if it is a monotone transformation of a homogeneous utility function. Every homogeneous utility function is homothetic, and, every continuous homothetic utility function is a monotonic transformation of a homogeneous utility function. Then, a utility function which is not homothetic is not homogeneous.

[15] On the properties of quasi-linear utility functions, see pages 164-169 in Varian (1992). The quasi-linear utility function is often used in both empirical and theoretical work. It is notably used in public economics to study optimal taxation under the assumption of perfect competition (Mirrlees 1971), for two of its properties, i.e., the absence of income effects for the commodity whose marginal utility is variable and the computation of consumers' surplus. For instance, by using a general oligopoly equilibrium model with production, Collie (2019) shows that tax effects can at best be Pareto-improving when consumers have identical quasi-linear preferences. See also Konishi et al. (1990).

[16] The same conclusion as the ones in this example can be reached with Stone-Geary utility functions, for instance with $u_i(x_i, y_i) = (x_i - \gamma)y_i$, for $i \in T_1$, and $u_i(x_i, y_i) = x_i(y_i - \gamma)$ for $i \in T_2$, where $0 < \gamma < x_i, y_i$.

[17] Special cases of $v(.)$ are $\ln(.)$ or $\sqrt{(.)}$.

Consider finally the taxation mechanism with transfers. Consider an uniform tax $t \in (0,1)$ is levied on endowments before exchange takes place.[18] After trade has occurred at an $2n$-tuple of strategies $(\mathbf{q};\mathbf{b}) \in [0,1-t]^n \times [0,1-t]^n$, a share $s(t) = \frac{t}{1-(v')^{-1}(\frac{1}{\gamma})-t}$, with $0 < t < 1-(v')^{-1}(\frac{1}{\gamma})$, of the total tax product in commodity X (resp. Y) is transferred to each agent $i \in T_2$ (resp. $i \in T_1$). This taxation mechanism generates a new strategic market game Γ^t. The strategy sets are now given by $Q_i(t) = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant 1-t\}$, for $i \in T_1$, and by $\mathcal{B}_i(t) = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant 1-t\}$, for $i \in T_2$. Indeed, given $(\mathbf{q};\mathbf{b}) \in [0,1-t]^n \times [0,1-t]^n$, a new relative price $p_X(\mathbf{q};\mathbf{b};t) = \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k}$ is determined, and the post-tax allocation of Γ^t at a vector of strategies $(\mathbf{q};\mathbf{b}) \in [0,1-t]^n \times [0,1-t]^n$ now obtains as

$$(x_i(t), y_i(t)) = \begin{cases} \left(1-t-q_i, \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} q_i\right), \, i \in T_1; \\ \\ \left(\frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} b_i, 1-t-b_i\right), \, i \in T_2. \end{cases} \quad (15)$$

Then, the payoffs of Γ^t are given by

$$\pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}; t) = \gamma v(1-t-q_i) + \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} q_i, \; i \in T_1; \quad (16)$$

$$\pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}; t) = \frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} b_i + \gamma v(1-t-b_i), \; i \in T_2. \quad (17)$$

Given $0 < t < 1-(v')^{-1}(\frac{1}{\beta})$, the CNE with endowment tax with transfers is the simultaneous solution to

$$\max_{q_i \in Q_i(t)} \gamma v(1-t-q_i) + \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + \frac{t}{1-t-(v')^{-1}(\frac{1}{\beta})} q_i, \; i \in T_1; \quad (18)$$

$$\max_{b_i \in \mathcal{B}_i(t)} \frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + \frac{t}{1-t-(v')^{-1}(\frac{1}{\beta})} b_i + \gamma v(1-t-b_i), \; i \in T_2. \quad (19)$$

The first-order sufficient conditions $\frac{\partial \pi_i}{\partial q_i} = 0$, for $i \in T_1$, and $\frac{\partial \pi_i}{\partial bi} = 0$, for $i \in T_2$, are given by:

$$-\gamma v'(1-t-q_i) + \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} \frac{\sum_{k\in T_1\setminus\{i\}} q_k}{\sum_{k\in T_1} q_k} + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} = 0, \; i \in T_1; \quad (20)$$

$$-\gamma v'(1-t-b_i) + \frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} \frac{\sum_{k\in T_2\setminus\{i\}} b_k}{\sum_{k\in T_2} b_k} + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} = 0, \; i \in T_2. \quad (21)$$

According to Theorem 1, we know, on the one hand, that traders of the same type must adopt the same strategy at equilibrium, i.e., $\tilde{q}_i(t) = \tilde{q}(t)$, for each $i \in T_1$,

[18] Since the market is symmetric, we assume from the outset that a uniform tax t is levied on the endowments of commodities X and Y.

and $\tilde{b}_i(t) = \tilde{b}(t)$, for each $i \in T_2$, and on the other hand, that as the market is symmetric, we have $\tilde{q}(t) = \tilde{b}(t)$. Therefore, the solution to these $2n$ problems is a unique symmetric CNE, which is interior and type-symmetric, so the first-order sufficient conditions which must be satisfied at this CNE may be written as:

$$-\gamma v'(1-t-q_i) + \frac{n-1}{n} + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} = 0,\ i \in T_1; \tag{22}$$

$$-\gamma v'(1-t-b_i) + \frac{n-1}{n} + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})} = 0,\ i \in T_2. \tag{23}$$

Given $0 < t < 1-(v')^{-1}(\frac{1}{\gamma})$, the unique interior and symmetric type-symmetric CNE of Γ^t is given by:

$$\tilde{q}(t) = \tilde{b}(t) = 1 - t - (v')^{-1}\left(\frac{1}{\gamma}\left(\frac{n-1}{n} + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})}\right)\right). \tag{24}$$

At the CNE of Γ^t, the post-tax allocations are given by:

$$(\tilde{x}_i(t), \tilde{y}_i(t)) = \begin{cases} \left((v')^{-1}(\eta(t)), \frac{1-(v')^{-1}(\frac{1}{\gamma})}{1-t-(v')^{-1}(\frac{1}{\gamma})}(1-t-(v')^{-1}(\eta(t)))\right),\ i \in T_1; \\ \left(\frac{1-(v')^{-1}(\frac{1}{\gamma})}{1-t-(v')^{-1}(\frac{1}{\gamma})}(1-t-(v')^{-1}(\eta(t))), (v')^{-1}(\eta(t))\right),\ i \in T_2, \end{cases} \tag{25}$$

where $\eta(t) \equiv \frac{1}{\gamma}\left(\frac{n-1}{n} + \frac{t}{1-t-(v')^{-1}(\frac{1}{\gamma})}\right)$, with corresponding payoffs:

$$\tilde{\pi}_i(t) = \frac{1}{\gamma}v(v')^{-1}(\eta(t)) + \frac{1-(v')^{-1}(\frac{1}{\gamma})}{1-t-(v')^{-1}(\frac{1}{\gamma})}(1-t-(v')^{-1}(\eta(t)),\ i \in T_1 \cup T_2. \tag{26}$$

We now wonder whether the endowment tax with transfers to traders can implement a Pareto-optimal allocation. If, at the CNE of the game Γ^t, the endowment tax with incentive transfers $\tilde{t}$ implements the competitive allocation without taxation as outcome, then, we have that

$$(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t})) = (x_i^*, y_i^*) = \begin{cases} \left((v')^{-1}\left(\frac{1}{\gamma}\right), 1-(v')^{-1}\left(\frac{1}{\gamma}\right)\right),\ i \in T_1; \\ \left(1-(v')^{-1}\left(\frac{1}{\gamma}\right), (v')^{-1}\left(\frac{1}{\gamma}\right)\right),\ i \in T_2. \end{cases} \tag{27}$$

By using (25), we can deduce:

$$\tilde{t} = \frac{1-(v')^{-1}\left(\frac{1}{\gamma}\right)}{n+1}. \tag{28}$$

By substituting the value of $\tilde{t}$ given by (28) into (24), we have that

$$\tilde{q}(\tilde{t}) = \tilde{b}(\tilde{t}) = \left(1-(v')^{-1}\left(\frac{1}{\gamma}\right)\right)\frac{n}{n+1}. \tag{29}$$

Then, by using (28) and (29), the CNE allocations of Γ^t given by (20) obtain as

$$(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t})) = \begin{cases} \left((v')^{-1}\left(\frac{1}{\gamma}\right), 1-(v')^{-1}\left(\frac{1}{\gamma}\right)\right), \ i \in T_1; \\ \\ \left(1-(v')^{-1}\left(\frac{1}{\gamma}\right), (v')^{-1}\left(\frac{1}{\gamma}\right)\right), \ i \in T_2, \end{cases} \tag{30}$$

which shows that the CNE outcomes at $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$ of Γ^t are the same as the CE outcomes without taxation. Indeed, the marginal rates of substitution of all traders are equal to one. We conclude that the overall-allocation is Pareto-optimal.

Finally, by virtue of Corollary 1, it can be shown that any other feasible endowment tax with transfers does not lead to a Pareto-optimal allocation. Indeed, consider a share $s(t)$, $0 < s(t) < 1$, of the total tax product of commodity X (resp. Y) is transferred to each agent $i \in T_2$ (resp. $i \in T_1$). The payoffs in Γ^t are given by $\pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}; t) = \gamma v(1-t-q_i) + \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + s(t) q_i$, for $i \in T_1$, and $\pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}; t) = \frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + s(t) b_i + \gamma v(1-t-b_i)$, for $i \in T_2$. Simple calculations show that the first-order sufficient conditions $\frac{\partial \pi_i}{\partial q_i} = 0$, for $i \in T_1$, and $\frac{\partial \pi_i}{\partial bi} = 0$, for $i \in T_2$, yield $\tilde{q}(t) = \tilde{b}(t) = 1 - t - (v')^{-1}\left(\frac{1}{\gamma}\left(\frac{n-1}{n} + s(t)\right)\right)$. Then, the allocations are given by $(\tilde{x}_i(t), \tilde{y}_i(t)) = \big((v')^{-1}(\kappa(t)), (1-t-(v')^{-1}(\kappa(t)))(1+s(t)\big)$, $i \in T_1$, and $(\tilde{x}_i(t), \tilde{y}_i(t)) = \big((v')^{-1}(\kappa(t)), (1-t-(v')^{-1}(\kappa(t)))(1+s(t)\big)$, $i \in T_2$, where $\kappa(t) \equiv \frac{1}{\gamma}\left(\frac{n-1}{n} + s(t)\right)$. Fix $\tilde{s} = s(\tilde{t})$, with $0 < \tilde{s} < 1$. Then, the marginal rates of substitution differ across traders as $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n-1}{n} + \tilde{s}$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n}{n-1} + \tilde{s}$, for $i \in T_2$, unless $\tilde{s} = \frac{1}{n}$, in which case $\tilde{t} = \frac{1-(v')^{-1}\left(\frac{1}{\gamma}\right)}{n+1}$.

5. DISCUSSION OF THE MODEL

Theorem 1 together with the core example of Section 4 show that endowment taxation with transfers can implement a first-best allocation in strategic bilateral trade even if the utility functions are neither homogeneous nor homothetic. To discuss the model, we will not consider the technical assumptions 2 and 3, which are sufficient conditions for the existence of a unique interior symmetric type-symmetric equilibrium. Instead, we will explore the consequences of introducing sources of heterogeneity into the strategic market game, so that the market is no longer symmetric. The objective is to test the robustness of Theorem 1 when the market is no longer symmetric. To this end, we relax the three assumptions which underly this symmetry: first, the equal number of traders on both sides of the market; second, the fact that traders have symmetric utility functions; and third, the fact that the amount of the endowment initially distributed to the traders is the same on both sides of the market. This leads to consider the following question: Does the taxation mechanism with transfers implement a Pareto-optimal allocation when the market is no longer symmetric, i.e., when the interior CNE of the game is type-symmetric but not symmetric type-symmetric? In all examples, we compute the CE, the CNE without taxation, and the CNE with taxation.

The following example considers a first kind of nonsymmetric market: the number of traders is not the same on both sides of the market. Therefore, we address the following problem: does the taxation mechanism with transfers implement a Pareto-optimal allocation when the number of traders differs between both sectors?

EXAMPLE 1. Consider the following specification for the exchange economy $\mathcal{E}$. Let $\#T_1 \neq \#T_2$, with $T_1 = \{1, ..., m\}$ and $T_2 = \{m+1, ..., m+n\}$; $\alpha = \beta = 1$; and, $u_i(x_i, y_i) = \frac{1}{2} \ln x_i + y_i$, for $i \in T_1$, and $u_i(x_i, y_i) = x_i + \frac{1}{2} \ln y_i$, for $i \in T_2$. The unique interior CE of $\mathcal{E}$ is given by $p_X^* = \frac{m+2n}{2m+n}$, and $(x_i^*, y_i^*) = (\frac{2m+n}{2(m+2n)}, \frac{3n}{2(2m+n)})$, for $i \in T_1$, and $(x_i^*, y_i^*) = (\frac{3m}{2(m+2n)}, \frac{m+2n}{2(2m+n)})$, for $i \in T_2$, which is Pareto-optimal as $MRS^i(x_i^*, y_i^*) = \frac{m+2n}{2m+n}$, for $i \in T_1 \cup T_2$. Consider now the game Γ associated with $\mathcal{E}$. The problems of traders may be written as $\max\limits_{q_i \in [0,1]} \{\frac{1}{2} \ln(1-q_i) + \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i\}$, for $i \in T_1$, and $\max\limits_{b_i \in [0,1]} \{\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i + \frac{1}{2} \ln(1-b_i)\}$, for $i \in T_2$. The unique interior type-symmetric CNE of Γ is given by the strategies $\tilde{q}_i = \frac{4}{m}(\frac{m-1}{m})(\frac{n-1}{n}) - \frac{1}{m}/(2\frac{n-1}{n}(\frac{1}{n} + \frac{2}{m}\frac{m-1}{m}))$, for $i \in T_1$, $\tilde{b}_i = \frac{4}{n}(\frac{m-1}{m})(\frac{n-1}{n}) - \frac{1}{n}/(2\frac{m-1}{m}(\frac{1}{m} + \frac{2}{n}\frac{n-1}{n}))$, for $i \in T_2$, the price $\tilde{p}_X = \frac{n-1}{n}(\frac{1}{n} + \frac{2}{m}\frac{m-1}{m})/(\frac{m-1}{m}(\frac{1}{m} + \frac{2}{n}\frac{n-1}{n}))$, and the allocations $(\tilde{x}_i, \tilde{y}_i) = (\frac{2}{n}\frac{n-1}{n} + \frac{1}{m}/(2\frac{n-1}{n}(\frac{1}{n} + \frac{2}{m}\frac{m-1}{m})), \frac{4}{m}(\frac{m-1}{m})(\frac{n-1}{n}) - \frac{1}{m}/(2\frac{m-1}{m}(\frac{1}{m} + \frac{2}{n}\frac{n-1}{n}))$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = (\frac{4}{n}(\frac{m-1}{m})(\frac{n-1}{n}) - \frac{1}{n}/(2\frac{n-1}{n}(\frac{1}{n} + \frac{2}{m}\frac{m-1}{m})), \frac{2}{m}\frac{m-1}{m} + \frac{1}{n}/(2\frac{m-1}{m}(\frac{1}{m} + \frac{2}{n}\frac{n-1}{n}))$, for $i \in T_2$. The CNE is not Pareto-optimal as $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n-1}{n}(\frac{1}{n} + \frac{2}{m}\frac{m-1}{m})/(\frac{2}{n}\frac{n-1}{n} + \frac{1}{m})$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{2}{m}\frac{m-1}{m-1} + \frac{1}{n}/(\frac{m-1}{m}(\frac{1}{m} + \frac{2}{n}\frac{n-1}{n})$, for $i \in T_2$. Consider finally the endowment taxation with transfers. Let $t_X \in (0,1)$ and $t_Y \in (0,1)$. The strategy sets in Γ^t are given by $Q_i = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant 1 - t_X\}$, for $i \in T_1$, and by $\mathcal{B}_i = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant 1 - t_Y\}$, for $i \in T_2$. After trade has occurred at an $(m+n)$-tuple of strategies $(\mathbf{q}; \mathbf{b}) \in [0, 1-t_X]^m \times [0, 1-t_Y]^n$, a share $s_X \equiv \frac{t_X}{1-t_X-\frac{2m+n}{2(m+2n)}} = \frac{2(m+2n)t_X}{3n-2(m+2n)t_X}$ of the total tax product in commodity X is transferred to each agent $i \in T_2$; correspondingly, a share $s_Y \equiv \frac{t_Y}{1-t_Y-\frac{m+2n}{2(2m+n)}} = \frac{2(2m+n)t_Y}{3m-2(2m+n)t_Y}$ of the total tax product in commodity Y is transferred to each agent $i \in T_1$. Then, the problems of traders may now be written as $\max\limits_{q_i \in [0,1-t_X]} \{\frac{1}{2} \ln(1 - t_X - q_i) + \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i + s_Y q_i\}$, for $i \in T_1$, and $\max\limits_{b_i \in [0,1-t_Y]} \{\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i + s_X b_i + \frac{1}{2} \ln(1 - t_Y - b_i)\}$, for $i \in T_2$. Any interior type-symmetric CNE of Γ^t is the solution to the $(m+n)$ equations $-\frac{1}{2}\frac{1}{1-t_X-q_i} + \frac{B}{Q}\frac{m-1}{m} + s_Y = 0$, $i \in T_1$, and $\frac{Q}{B}\frac{n-1}{n} + s_X - \frac{1}{2}\frac{1}{1-t_Y-b_i} = 0$, $i \in T_2$. Let $Q = aB$, with $a \neq 1$. Then, as at a type-symmetric CNE, we must have $s_X = \frac{1}{m}$ and $s_Y = \frac{1}{n}$, the preceding $(m+n)$ equations lead to $\frac{Q}{m} = 1 - t_X - \frac{1}{2(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n})}$, for $i \in T_1$, and $\frac{B}{n} = 1 - t_Y - \frac{1}{2(a\frac{n-1}{n} + \frac{1}{m})}$, for $i \in T_1$. The relative price $p_X = \frac{B}{Q}$ may be written as $p_X = \frac{m}{n}(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n})(2(1-t_Y)(a\frac{n-1}{n} + \frac{1}{m}) - 1)/(a\frac{n-1}{n} + \frac{1}{m})(2(1-t_X)(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n}) - 1)$. Then, the allocations are given by $(\tilde{x}_i, \tilde{y}_i) = (1/(2(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n})), \frac{m}{n}(2(1-t_Y)(a\frac{n-1}{n} + \frac{1}{m}) - 1)/(2(a\frac{n-1}{n} + \frac{1}{m})) + \frac{1}{n}(2(1-t_X)(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n}) - 1)/(2(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n})))$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = (\frac{n}{m}(2(1-t_X)(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n}) - 1)/(\frac{1}{a}\frac{m-1}{m} + \frac{1}{n}) + \frac{1}{m}(2(1-t_Y)(a\frac{n-1}{n} + \frac{1}{m}) - 1)/(2(a\frac{n-1}{n} + \frac{1}{m}))\ , 1/(2(a\frac{n-1}{n} + \frac{1}{m})1)$, for $i \in T_2$. Finally, we have $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{1}{a}\frac{m-1}{m} + \frac{1}{n}$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = a\frac{n-1}{n} + \frac{1}{m}$, for $i \in T_2$. These marginal rates of substitution are equal to each other if and only if $m = n$. Indeed, if $m = n$, then $a = 1$. Then, $MRS^i(\tilde{x}_i, \tilde{y}_i) = 1$, for $i \in T_1 \cup T_2$. Conversely, if $\frac{1}{a}\frac{m-1}{m} + \frac{1}{n} = a\frac{n-1}{n} + \frac{1}{m}$, then $m(n-1)a^2 + (n-m)a - (m-1)n = 0$. As at a symmetric type-symmetric CNE, we must have $a = 1$, then the preceding polynom reduces to $2(n-m) = 0$.□

The reason why the marginal rates of substitution are not equal in this case explains as follows. The subsidy received by each trader, i.e., $\tilde{s}_X = \frac{1}{m}$, for $i \in T_2$, and $\tilde{s}_Y = \frac{1}{n}$, for $i \in T_1$, is not sufficiently strong to compensate the difference of market power measured by their market shares. Indeed, when the market is symmetric, i.e., when $m = n$, so $t_X = t_Y = t$, the unique symmetric type-symmetric interior CNE of Γ^t is given by the strategies $\tilde{q} = \tilde{b} = 1 - t - \frac{(1-2t)n}{2(n-1)+4t}$, the relative price $\tilde{p}_X = 1$, and the allocations $(\tilde{x}_i, \tilde{y}_i) = \Big(\frac{(1-2t)n}{2(n-1)+4t}, (\frac{1}{1-2t})(1 - t - \frac{(1-2t)n}{2(n-1)+4t})\Big)$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = \Big((\frac{1}{1-2t})(1 - t - \frac{(1-2t)n}{2(n-1)+4t}), \frac{(1-2t)n}{2(n-1)+4t}\Big)$, $i \in T_2$. It turns out that there exists an endowment tax with transfers $\tilde{t} \in (0, 1)$ such that the overall-allocation resulting from the unique interior CNE of the game Γ^t is Pareto-optimal. As the CE allocation is given by $(x^*_i, y^*_i) = (\frac{1}{2}, \frac{1}{2})$, for $i \in T_1 \cup T_2$, we deduce $\tilde{t} = \frac{1}{2(n+1)}$, and $\tilde{s} = \frac{1}{n}$. Then, $MRS^i(\tilde{x}_i, \tilde{y}_i) = 1$, for $i \in T_1 \cup T_2$.

We now turn to the case for which the utility functions of traders no longer satisfy $u_1(x, y) = u_2(y, x)$. Therefore, we address the following problem: does the taxation mechanism with transfer implement a Pareto-optimal allocation when the preferences are heterogeneous?

EXAMPLE 2. Consider the following specification for the exchange economy $\mathcal{E}$. Let $\#T_1 = \#T_2 = n$, $n \geqslant 2$; $\alpha = \beta = 1$; and, $u_i(x_i, y_i) = x_i + y_i$, for $i \in T_1$, and $u_i(x_i, y_i) = x_i y_i$, for $i \in T_2$. The unique interior CE of $\mathcal{E}$ is given by $p^*_X = 1$, and $(x^*_i, y^*_i) = (\frac{1}{2}, \frac{1}{2})$, for $i \in T_1 \cup T_2$, which is Pareto-optimal as $MRS^i(x^*_i, y^*_i) = 1$, for $i \in T_1 \cup T_2$. Consider now the game Γ associated with $\mathcal{E}$. The problems of traders may be written $\max_{q_i \in [0,1]} \{1 - q_i + \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i\}$, for $i \in T_1$, and $\max_{b_i \in [0,1]} \{\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i(1 - b_i)\}$, for $i \in T_2$. The unique interior type-symmetric CNE of Γ is given by the strategies $\tilde{q}_i = \frac{(n-1)^2}{n^2(2n-1)}$, for $i \in T_1$, $\tilde{b}_i = \frac{n-1}{2n-1}$, for $i \in T_2$, the price $\tilde{p}_X = \frac{n}{n-1}$, and the allocations $(\tilde{x}_i, \tilde{y}_i) = (1 - \frac{(n-1)^2}{n^2(2n-1)}, \frac{1}{n}\frac{n-1}{2n-1})$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = (\frac{(n-1)^2}{n(2n-1)}, \frac{n}{2n-1})$, for $i \in T_2$. The CNE is not Pareto-optimal as $MRS^i(\tilde{x}_i, \tilde{y}_i) = 1$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = (\frac{n}{n-1})^2$, for $i \in T_2$. Consider finally the endowment taxation with transfers. Let $t_X \in (0, 1)$ and $t_Y \in (0, 1)$. The strategy sets in Γ^t are given by $Q_i = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant 1 - t_X\}$, for $i \in T_1$, and by $\mathcal{B}_i = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant 1 - t_Y\}$, for $i \in T_2$. After trade has occurred at an $2n$-tuple of strategies $(\mathbf{q}; \mathbf{b}) \in [0, 1 - t_X]^n \times [0, 1 - t_Y]^n$, a share $s_X = \frac{2t_X}{1-2t_X}$ of the total tax product in commodity X is transferred to each agent $i \in T_2$; correspondingly, a share $s_Y = \frac{2t_Y}{1-2t_Y}$ of the total tax product in commodity Y is transferred to each agent $i \in T_1$. Then, the problems of traders may be written as $\max_{q_i \in [0,1-t_X]} \{1 - t_X - q_i + \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i + \frac{2t_Y}{1-2t_Y} q_i\}$, for $i \in T_1$, and $\max_{b_i \in [0,1-t_Y]} \{(\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i + \frac{2t_X}{1-2t_X} b_i)(1 - t_Y - b_i)\}$, for $i \in T_2$. The unique CNE of Γ^t, which is interior and type-symmetric, is given by the strategies $\tilde{q}_i = \frac{1-4t_Y}{1-2t_Y}\frac{n-1}{n}(1 - t_Y)[\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})^2 + \frac{2t_X}{1-2t_X}]/[\frac{4t_X}{1-2t_X} + \frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})(\frac{2n-1}{n})]$, for $i \in T_1$, and $\tilde{b}_i = (1 - t_Y)[\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})^2 + \frac{2t_X}{1-2t_X}]/[\frac{4t_X}{1-2t_X} + \frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})(\frac{2n-1}{n})]$, for $i \in T_2$, the relative price $\tilde{p}_X = \frac{n}{n-1}\frac{1-2t_Y}{1-4t_Y}$, and the allocations $(\tilde{x}_i, \tilde{y}_i) = (1 - t_X - \frac{1-4t_Y}{1-2t_Y}\frac{n-1}{n}(1-t_Y)[\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})^2 + \frac{2t_X}{1-2t_X}]/[\frac{4t_X}{1-2t_X} + \frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})(\frac{2n-1}{n})], \frac{1-4t_Y}{1-2t_Y}\frac{n-1}{n}(1 - t_Y)[\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})^2 + \frac{2t_X}{1-2t_X}](\frac{n}{n-1}\frac{1-2t_Y}{1-4t_Y} + \frac{1}{n})/[\frac{4t_X}{1-2t_X} + \frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})(\frac{2n-1}{n})])$, for $i \in T_1$, and $(\tilde{x}_i,\tilde{y}_i)=((1-t_Y)[\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})^2+\frac{2t_X}{1-2t_X}]\frac{(1-4t_Y)n+2t_Y}{n(1-2t_Y)}/[\frac{4t_X}{1-2t_X}+\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})(\frac{2n-1}{n})]$ $,(1-t_Y)[\frac{1-4t_Y}{1-2t_Y}\frac{n-1}{n}+\frac{2t_X}{1-2t_X}]/[\frac{4t_X}{1-2t_X}+\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})(\frac{2n-1}{n})])$, for $i\in T_2$. Finally, we have $MRS^i(\tilde{x}_i,\tilde{y}_i)=1$, for $i\in T_1$, and $MRS^i(\tilde{x}_i,\tilde{y}_i)=(\frac{1-4t_Y}{1-2t_Y}\frac{n-1}{n}+\frac{2t_X}{1-2t_X})/\{[\frac{1-4t_Y}{1-2t_Y}(\frac{n-1}{n})^2+\frac{2t_X}{1-2t_X}]\frac{(1-4t_Y)n+2t_Y}{n(1-2t_Y)}\}\neq 1$, for $i\in T_2$.□

The reason why the marginal rates of substitution are not equal in this case explains as follows. Here, the preferences of traders are heterogeneous: commodities are perfectly (resp. imperfectly) substitutable for traders of type 1 (resp. type 2). As $\mathbf{w}_i=(1,0)$, for $i\in T_1$, and $\mathbf{w}_i=(0,1)$, for $i\in T_2$, and $\#T_1=\#T_2=n$, at the interior type-symmetric CNE, each traders has the same market power, measured by her market share, which is equal to $\frac{1}{n}$. Moreover, as the CE is such that traders receive the same allocation, i.e., $(x_i^*,y_i^*)=(\frac{1}{2},\frac{1}{2})$, for each $i\in T_1\cup T_2$, the traders receive the same subsidy, i.e., $\tilde{s}_X=\frac{1}{n}$, for $i\in T_2$, and $\tilde{s}_Y=\frac{1}{n}$, for $i\in T_1$. But such a subsidy is not sufficiently strong to compensate the distortions caused by the strategic behavior of traders when the preferences are heterogeneous.

The next two examples consider a third kind of heterogeneity which is associated with a nonsymmetric market, the one for which the traders do not the receive the same amount of endowments. Therefore, we address the following problem: does the taxation mechanism with transfers implement a Pareto-optimal allocation when the amount of endowment differs between both types of traders?

EXAMPLE 3. Consider the following specification for the exchange economy $\mathcal{E}$. Let $\#T_1=\#T_2=n$; $\mathbf{w}_i=(\alpha,0)$, for $i\in T_1$, and $\mathbf{w}_i=(0,\beta)$; and, $u_i(x_i,y_i)=\frac{1}{2}\ln x_i+y_i$, for $i\in T_1$, and $u_i(x_i,y_i)=x_i+\frac{1}{2}\ln y_i$, for $i\in T_2$. The unique interior CE is given by $p_X^*=\frac{1+2\beta}{1+2\alpha}$, and $(x_i^*,y_i^*)=(\frac{1+2\alpha}{2(1+2\beta)},\frac{4\alpha\beta-1}{2(1+2\alpha)})$, for $i\in T_1$, and $(x_i^*,y_i^*)=(\frac{4\alpha\beta-1}{2(1+2\beta)},\frac{1+2\beta}{2(1+2\alpha)})$, for $i\in T_2$, which is Pareto-optimal as $MRS^i(x_i^*,y_i^*)=\frac{1+2\beta}{1+2\alpha}$, for $i\in T_1\cup T_2$. Consider now the game Γ associated with $\mathcal{E}$. The problems of traders may be written as $\max\limits_{q_i\in[0,1]}\{\frac{1}{2}\ln(\alpha-q_i)+\frac{\sum_{k\in T_2}b_k}{\sum_{k\in T_1}q_k}q_i\}$, for $i\in T_1$, and $\max\limits_{b_i\in[0,1]}\{\frac{\sum_{k\in T_1}q_k}{\sum_{k\in T_2}b_k}b_i+\frac{1}{2}\ln(\beta-b_i)\}$, for $i\in T_2$. The unique interior type-symmetric CNE of Γ is given by $\tilde{q}_i=[4\alpha\beta(\frac{n-1}{n})^2-1)]/[2\frac{n-1}{n}(1+2\beta\frac{n-1}{n})]$, for $i\in T_1$, $\tilde{b}_i=(4\alpha\beta(\frac{n-1}{n})^2-1)/(2(1+2\alpha\frac{n-1}{n}))$, for $i\in T_2$, $\tilde{p}_X=\frac{1+2\beta\frac{n-1}{n}}{1+2\alpha\frac{n-1}{n}}$, and $(\tilde{x}_i,\tilde{y}_i)=((1+2\alpha\frac{n-1}{n})/(2\frac{n-1}{n}(1+2\beta\frac{n-1}{n})),(4\alpha\beta(\frac{n-1}{n})^2-1)/(2\frac{n-1}{n}(1+2\alpha\frac{n-1}{n})))$, for $i\in T_1$, and $(\tilde{x}_i,\tilde{y}_i)=((4\alpha\beta(\frac{n-1}{n})^2-1)/(2\frac{n-1}{n}(1+2\beta\frac{n-1}{n}))\,,(1+2\beta\frac{n-1}{n})/(2\frac{n-1}{n}(1+2\alpha\frac{n-1}{n})))$, for $i\in T_2$. The CNE is not Pareto-optimal as $MRS^i(\tilde{x}_i,\tilde{y}_i)=(\frac{n-1}{n}(1+2\beta\frac{n-1}{n}))/(1+2\alpha\frac{n-1}{n})$, for $i\in T_1$, and $MRS^i(\tilde{x}_i,\tilde{y}_i)=(1+2\beta\frac{n-1}{n})/(\frac{n-1}{n}(1+2\alpha\frac{n-1}{n}))$, for $i\in T_2$. Consider finally the endowment taxation with transfers. Let $t_X\in(0,\alpha)$ and $t_Y\in(0,\beta)$. The strategy sets in Γ^t are given by $Q_i=\{q_i\in\mathbb{R}\mid 0\leqslant q_i\leqslant\alpha-t_X\}$, for $i\in T_1$, and by $\mathcal{B}_i=\{b_i\in\mathbb{R}\mid 0\leqslant b_i\leqslant\beta-t_Y\}$, for $i\in T_2$. After trade has occurred at an $2n$-tuple of strategies $(\mathbf{q};\mathbf{b})\in[0,\alpha-t_X]^n\times[0,\beta-t_Y]^n$, a share $s_X=\frac{2(1+2\beta)t_X}{4\alpha\beta-1-2(1+2\beta)t_X}$ of the total tax product in commodity X is transferred to each agent $i\in T_2$; correspondingly, a share $s_Y=\frac{2(1+2\alpha)t_Y}{4\alpha\beta-1-2(1+2\alpha)t_Y}$ of the total tax product in commodity Y is transferred to each agent $i\in T_1$. Then, the problems of traders are $\max\limits_{q_i\in[0,1-t_X]}\{\frac{1}{2}\ln(\alpha-t_X-q_i)+\frac{\sum_{k\in T_2}b_k}{\sum_{k\in T_1}q_k}q_i+s_Yq_i\}$, for $i\in T_1$, and $\max\limits_{b_i\in[0,1-t_Y]}\{\frac{\sum_{k\in T_1}q_k}{\sum_{k\in T_2}b_k}b_i+s_Xb_i+\frac{1}{2}\ln(\beta-t_Y-b_i)\}$, for $i\in T_2$. Any CNE of Γ^t, which is interior and type-symmetric, is the solution to the $2n$ equations $\frac{1}{2}\frac{1}{\alpha-t_X-q_i} - \frac{B}{Q}\frac{n-1}{n} - s_Y = 0$, $i \in T_1$, and $\frac{1}{2}\frac{1}{\beta-t_Y-b_i} - \frac{Q}{B}\frac{n-1}{n} - s_X = 0$, $i \in T_2$, where $\frac{B}{Q} = \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k}$. But, at an interior type-symmetric CNE, $\tilde{s}_X = \tilde{s}_Y = \frac{1}{n}$, so we must have $\tilde{t}_X = \frac{4\alpha\beta-1}{2(1+2\beta)(n+1)}$ and $\tilde{t}_Y = \frac{4\alpha\beta-1}{2(1+2\alpha)(n+1)}$. Assume the taxes $\tilde{t}_X = \frac{4\alpha\beta-1}{2(1+2\beta)(n+1)}$ and $\tilde{t}_Y = \frac{4\alpha\beta-1}{2(1+2\alpha)(n+1)}$, and the transfers $\tilde{s}_X = \tilde{s}_Y = \frac{1}{n}$, are such that the allocation $\widetilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}_X, \tilde{t}_Y), \tilde{y}_i(\tilde{t}_X, \tilde{t}_Y))_{i\in T_1\cup T_2}$ resulting from the CNE of the game Γ^t and from these transfers is Pareto-optimal. The n equilibrium conditions $-\frac{1}{2}\frac{1}{\alpha-\tilde{t}_X-\tilde{q}_i} + \left(\frac{\tilde{B}}{\tilde{Q}}\right)\frac{n-1}{n} + s_Y = 0$ yield $\left(\frac{\tilde{Q}}{\tilde{B}}\right) = \frac{(1+2\alpha)(n-1)}{(1+2\beta)n-(1+2\alpha)}$, so the remaining n conditions $\left(\frac{\tilde{Q}}{\tilde{B}}\right)\frac{n-1}{n} + s_X - \frac{1}{2}\frac{1}{\beta-\tilde{t}_Y-\tilde{b}_i} = 0$ are such that $\frac{1+2\alpha}{1+2\beta} = \frac{(1+2\alpha)(n-2)+(1+2\beta)}{(1+2\beta)n-(1+2\alpha)}$. A contradiction when $\alpha \neq \beta$. Indeed, the allocation $\widetilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}_X, \tilde{t}_Y), \tilde{y}_i(\tilde{t}_X, \tilde{t}_Y))_{i\in T_1\cup T_2}$ resulting from the CNE of the game Γ^t and from these transfers is Pareto-optimal if and only if $\alpha = \beta$. Indeed, if $\alpha = \beta$, we have $\left(\frac{\tilde{Q}}{\tilde{B}}\right) = 1$ and $\tilde{t}_X = \tilde{t}_Y = \frac{2\alpha-1}{2(n+1)}$. The $2n$ equilibrium conditions may be written $-\frac{1}{2}\frac{1}{\alpha-\tilde{t}_X-\tilde{q}_i} + 1 = 0$ and $1 - \frac{1}{2}\frac{1}{\beta-\tilde{t}_Y-\tilde{b}_i} = 0$, and they lead to the strategies $\tilde{q}_i = \frac{(2\alpha-1)n}{2(n+1)}$, for $i \in T_1$, $\tilde{b}_i = \frac{(2\alpha-1)n}{2(n+1)}$, for $i \in T_2$, and the allocations $(\tilde{x}_i, \tilde{y}_i) = (\frac{1}{2}, \frac{2\alpha-1}{2})$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = (\frac{2\alpha-1}{2}, \frac{1}{2})$, for $i \in T_2$. The CNE is Pareto-optimal as $MRS^i(\frac{1}{2}, \frac{2\alpha-1}{2}) = 1$, for $i \in T_1$, and $MRS^i(\frac{2\alpha-1}{2}, \frac{1}{2}) = 1$, for $i \in T_2$. Conversely, $\left(\frac{\tilde{Q}}{\tilde{B}}\right) = \frac{(1+2\alpha)(n-1)}{(1+2\beta)n-(1+2\alpha)} = 1$ and $\frac{(1+2\alpha)(n-2)+(1+2\beta)}{(1+2\beta)n-(1+2\alpha)} = \frac{1+2\alpha}{1+2\beta}$ only if $\alpha = \beta$.□

Nevertheless, even if the amounts of endowments are not equally spread among the participants on both sides of the markets, the taxation mechanism can lead to a Pareto-optimal allocation. Thus, the following example illustrates that the fact that the amount of the initial allocations of commodity X and commodity Y distributed to the traders is identical, i.e., $\mathbf{w}_i = (\alpha, 0)$, for $i \in T_1$, and $\mathbf{w}_i = (0, \alpha)$, for $i \in T_2$, is not a necessary condition for the optimality of the tax-and-transfer mechanism.

EXAMPLE 4. Consider the following specification for the exchange economy $\mathcal{E}$. Let $\#T_1 = \#T_2 = n$; $\mathbf{w}_i = (\alpha, 0)$, for $i \in T_1$, and $\mathbf{w}_i = (0, \beta)$, for $i \in T_2$; and, $u_i(x_i, y_i) = x_i + \frac{1}{2}\ln y_i$, for $i \in T_1$, and $u_i(x_i, y_i) = \frac{1}{2}\ln x_i + y_i$, for $i \in T_2$. The unique interior CE is given by $p^*_X = 1$, and $(x^*_i, y^*_i) = (\alpha - \frac{1}{2}, \frac{1}{2})$, for $i \in T_1$, and $(x^*_i, y^*_i) = (\frac{1}{2}, \beta - \frac{1}{2})$, for $i \in T_2$, which is Pareto-optimal as $MRS^i(x^*_i, y^*_i) = 1$, for $i \in T_1 \cup T_2$. Consider now the game Γ associated with $\mathcal{E}$. The problems of traders may be written as $\max_{q_i\in[0,1]} \{(\alpha - q_i) + \frac{1}{2}\ln(\frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i)\}$, for $i \in T_1$, and $\max_{b_i\in[0,1]} \{\frac{1}{2}\ln(\frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i) + (\beta - b_i)\}$, for $i \in T_2$. The unique interior type-symmetric CNE of Γ is given by $\tilde{q}_i = \frac{n-1}{2n}$, for $i \in T_1$, $\tilde{b}_i = \frac{n-1}{2n}$, for $i \in T_2$, $\tilde{p}_X = 1$, and $(\tilde{x}_i, \tilde{y}_i) = (\frac{(2\alpha-1)n+1}{2n}, \frac{n-1}{2n})$, for $i \in T_1$, and $(\tilde{x}_i, \tilde{y}_i) = (\frac{n-1}{2n}, \frac{(2\beta-1)n+1}{2n})$, for $i \in T_2$. The CNE is not Pareto-optimal as $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n-1}{n}$, for $i \in T_1$, and $MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n}{n-1}$, for $i \in T_2$. Consider finally the endowment taxation with transfers. Let $t_X \in (0, \alpha)$ and $t_Y \in (0, \beta)$. The strategy sets in Γ^t are given by $Q_i = \{q_i \in \mathbb{R} \mid 0 \leqslant q_i \leqslant \alpha - t_X\}$, for $i \in T_1$, and by $\mathcal{B}_i = \{b_i \in \mathbb{R} \mid 0 \leqslant b_i \leqslant \beta - t_Y\}$, for $i \in T_2$. After trade has occurred at an $2n$-tuple of strategies $(\mathbf{q}; \mathbf{b}) \in [0, \alpha - t_X]^n \times [0, \beta - t_Y]^n$, a share $s_X = \frac{2t_X}{1-2t_X}$ of the total tax product in commodity X is transferred to each agent $i \in T_2$; correspondingly, a share $s_Y = \frac{2t_Y}{1-2t_Y}$ of the total tax product in commodity Y is transferred to each agent $i \in T_1$. Then, the problems of traders may be written as $\max_{q_i \in [0,1-t_X]} \{(\alpha - t_X - q_i) + \frac{1}{2}\ln(\frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + s_Y q_i)\}$, for $i \in T_1$, and $\max_{b_i \in [0,1-t_Y]} \{\frac{1}{2}\ln(\frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + s_X b_i) + (\beta - t_Y - b_i)\}$, for $i \in T_2$. Any CNE of Γ^t, which is interior and type-symmetric, is the solution to the $2n$ equations $-1 + \frac{1}{2}\frac{\frac{B}{Q}\frac{n-1}{n} + s_Y}{\frac{B}{Q}q_i + s_Y q_i} = 0$, $i \in T_1$, and $\frac{1}{2}\frac{\frac{Q}{B}\frac{n-1}{n} + s_X}{\frac{Q}{B}b_i + s_X b_i} - 1 = 0$, $i \in T_2$, where $\frac{B}{Q} = \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k}$. But, at an interior type-symmetric CNE, $\tilde{s}_X = \tilde{s}_Y = \frac{1}{n}$. Assume $B = aQ$, $a \neq 1$. The preceding $2n$ equilibrium conditions lead to $Q = \frac{n}{2}\frac{a(n-1)+1}{an+1}$ and $Q = \frac{n}{2a}\frac{n-1+a}{n+a}$, which are consistent if and only if $a = 1$. Then, $\tilde{q}_i(t_X, t_Y) = \frac{1}{2}(1-2t_Y)\frac{n-1}{n} + t_Y$, for $i \in T_1$, and $\tilde{b}_i(t_X, t_Y) = \frac{1}{2}(1-2t_X)\frac{n-1}{n} + t_X$, for $i \in T_2$. As $Q = B$, from $\alpha - t_X - \tilde{q}_i(t_X, t_Y) = \alpha - \frac{1}{2}$ or $\beta - t_Y - \tilde{b}_i(t_X, t_Y) = \beta - \frac{1}{2}$, we get $\tilde{t}_X = \tilde{t}_Y = \frac{1}{2(n+1)}$. Finally, we wonder whether, the tax $\tilde{t}_X = \tilde{t}_Y = \frac{1}{2(n+1)}$ and transfers $\tilde{s}_X = \tilde{s}_Y = \frac{1}{n}$, are such that the allocation $\widetilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}_X, \tilde{t}_Y), \tilde{y}_i(\tilde{t}_X, \tilde{t}_Y))_{i\in T_1\cup T_2}$ resulting from the CNE of the game Γ^t and from these transfers is Pareto-optimal. The strategies are $\tilde{q}_i = \frac{n}{2(n+1)}$, for $i \in T_1$, and $\tilde{b}_i = \frac{n}{2(n+1)}$, for $i \in T_2$, and the allocations are $(\tilde{x}_i, \tilde{y}_i) = (\alpha - \frac{1}{2}, \frac{1}{2})$, for $i \in T_1$, and $(x_i^*, y_i^*) = (\frac{1}{2}, \beta - \frac{1}{2})$, for $i \in T_2$. The CNE is Pareto-optimal as $MRS^i(\alpha - \frac{1}{2}, \frac{1}{2}) = 1$, for $i \in T_1$, and $MRS^i(\frac{1}{2}, \beta - \frac{1}{2}) = 1$, for $i \in T_2$.□

The salient difference between the last two examples may be grasped as follows. In Example 3 the taxation mechanism leads to a Pareto-optimal allocation when the interior CNE is symmetric type-symmetric, i.e., when $\alpha = \beta$. In Example 4 the taxation mechanism leads to a Pareto-optimal allocation when the interior CNE is symmetric type-symmetric, which is nonetheless consistent with $\alpha \neq \beta$. In Example 3, the commodity initially held by any trader enters in the nonlinear part of her utility function: it is as though commodity X (resp. Y) were a "necessity" for traders of type 1 (resp. 2). For trader $i \in T_1$ (resp. $i \in T_2$) her consumption of commodity X (resp. Y) as well as her equilibrium supply depend on the amount of her endowment α (resp. β). Moreover, for a value of income just above $\alpha > \frac{1}{2}$, with $\tilde{q}_i(1 + \tilde{s}_Y) = \frac{(2\alpha-1)n}{2(n+1)}(1 + \frac{1}{n}) = \frac{1}{2}$ (resp. $\tilde{b}_i(1 + \tilde{s}_Y) = \frac{(2\alpha-1)n}{2(n+1)}(1 + \frac{1}{n}) = \frac{1}{2}$), all the income is used to purchase commodity Y (resp. X). By constrat, in Example 4, as the commodity initially held by any trader enters in the linear part of her utility function: it is as though the purchased commodity Y (resp. X) were a "necessity" for traders of type 1 (resp. 2). As the equilibrium supply does not depend on the amount of the endowment, the amount of commodity purchased is independent of her endowment, i.e., $\tilde{q}_i(1 + \tilde{s}_Y) = \frac{n}{2(n+1)}(1 + \frac{1}{n}) = \frac{1}{2}$ (resp. $\tilde{b}_i(1 + \tilde{s}_X) = \frac{n}{2(n+1)}(1 + \frac{1}{n}) = \frac{1}{2}$).

The preceding four examples illustrate that if a uniform tax with transfers cannot implement a first-best allocation, then the market is not symmetric (the converse is false as Example 4 illustrates). Moreover, as in all examples the interior CNE is type-symmetric, the subsidy transferred to each trader corresponds to her market share, i.e., $\frac{1}{\#T_h}$, $h = 1, 2$. This means that heterogeneity, whether it stems from differences in preferences, the amounts of initial allocations or the number of agents on each side of the market, can result in the implementation of the tax-and-transfer mechanism being insufficient to wipe out the distortions caused by strategic behavior. These examples complement the results of Gabszewicz and Grazzini (1999) who show that endowment taxation with transfers to traders leads to a Pareto-optimal allocation when the bilateral market is symmetric under the following three assumptions: 1. symmetric homogeneous (and then homothetic) utility functions, 2. unit corner endowments, and, 3. a same number of traders on both sides of the market.

6. CONCLUSION

In this paper we investigated the implementation of a taxation mechanism in a noncooperative model of strategic bilateral trade. First, we showed existence, uniqueness, and non-optimality of a symmetric type symmetric interior CNE without taxation for a large class of smooth utility functions. Then, we considered the implementation of a taxation mechanism with supply subsidy, namely endowment taxation with transfers to traders. We showed existence of a unique symmetric type-symmetric interior CNE for which there was an uniform equilibrium tax such that the taxation mechanism *always* implemented a Pareto-optimal allocation when all agents behaved strategically and non-cooperatively in trade.

Our main result was evocative of the second welfare theorem in general equilibrium analysis but with strategic exchange. It generalized the results of Gabszewicz and Grazzini (1999) on the optimality of the endowment taxation with transfers to traders, which were obtained for specific utility functions. Indeed, the Pareto-optimality of this taxation mechanism was robust for a large class of smooth utility functions. Our result also complemented Elegbede et al. (2022) who showed that when the preferences of traders were represented by CES utility functions with non unitary shares on consumption, the taxation with transfers implemented a first-best allocation only when commodities were perfect complements or perfect substitutes. Moreover, a core example - *the quasi-linear bilateral oligopoly* - illustrated that the optimality of the endowment taxation mechanism also holds when the preferences of traders were represented by non-homogeneous and non-homothetic utility functions. Indeed, by enlarging the set of utility functions for which the taxation mechanism was Pareto-optimal, our result extended the class of exchange economies for which first-best optimal taxation was effective. Beyond this result on robustness, we highlighted the reason why such a taxation mechanism led to a Pareto-optimal allocation. Thus, the reason stemmed fron the fact that the market was symmetric, i.e., the number of traders, the amounts of endowments, and the utility functions on each side of the market were identical.

This has led us to discuss the assumption of a symmetric market. Thus, to explore sources of heterogeneity other than that associated with the distribution of (corner) endowments, we have relaxed the assumption of a symmetric market. The objective was to test the robustness of Theorem 1 when the market was no longer symmetric. To this end, we relaxed the three assumptions which underlied this symmetry: first, the equal number of traders on both sides of the market; second, the symmetric utility functions; and third, the distribution of a same amount of initial endowment on both sides of the market. Moreover, if an uniform tax with transfers could no longer implement a first-best allocation, then the market was not symmetric, i.e., when the interior CNE of the game was type-symmetric but not symmetric. But as Example 4 illustrated the converse was false: if the supply of a trader did not depend upon the amount of her endowment, the tax-and-transfer mechanism could be sufficient to wipe out the distortions caused by strategic behavior.

Some possible extensions, which are left for future research, could consist of investigating the effectiveness and the optimality of taxation mechanisms in more complex environments, by considering either exchange economies with more than two commodities or the possibility of asymmetric strategic behavior. For instance, in this latter case, asymmetric behavior could be based on the fact that traders (possibly of the same type) are heterogeneous to the extent that either their utility functions differ or that traders do not have the same market power due to their initial endowment or their size. It could open the way to study taxation mechanism in the context of mixed markets, i.e., in markets on which large traders, the atoms, compete with small traders, the atomless part. This heterogeneity could also be reflected in the fact that agents consider not only their income but also the income of their relative market position with respect to the distribution of endowments (Bruce and Peng 2018). It would also be interesting to focus on the mechanism design with small perturbations of the game that would increase efficiency (Zhang, 2024).

7. APPENDIX

7.1. Appendix A: proof of Proposition 1

The logic of the proof is as follows. First, we assume the existence of an interior CNE, and we show it must be a symmetric type-symmetric CNE, i.e., it satisfies Equations (8). Then, we show the symmetric type-symmetric CNE must be unique and globally stable. Finally, we show existence of a SNE. To this end, we adapt to our framework of strictly quasi-concave utility functions the proof given in Bloch and Ghosal (1997) for concave utility functions. Assume the market is symmetric, with $\#T_1 = \#T_2 = n$ and $\alpha = \beta = 1$. Consider an interior CNE, i.e., a $2n$-tuple of supplies $(\tilde{q}_1, ..., \tilde{q}_n; \tilde{b}_{n+1}, ..., \tilde{b}_{2n}) \in [0,1]^n \times [0,1]^n$ for which $\sum_{k \in T_1} \tilde{q}_k > 0$ and $\sum_{k \in T_2} \tilde{b}_k > 0$.

The $2n$ problems of traders may be written:

$$\begin{cases} \max\limits_{q_i \in [0,1]} \pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}) = u_1 \left(1 - q_i \ , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i \right), \ i \in T_1 \\ \\ \max\limits_{b_i \in [0,1]} \pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}) = u_2 \left(\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i, 1 - b_i \right), \ i \in T_2. \end{cases} \tag{A1}$$

The $2n$ first-order derivatives $\frac{\partial \pi_i}{\partial q_i}$, for $i \in T_1$, and $\frac{\partial \pi_i}{\partial b_i}$, for $i \in T_2$, are given by:

$$\begin{cases} \frac{\partial \pi_i}{\partial q_i} = -\frac{\partial u_1}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} \frac{\sum_{k \in T_1 \setminus \{i\}} q_k}{\sum_{k \in T_1} q_k}, \ i \in T_1 \\ \\ \frac{\partial \pi_i}{\partial b_i} = \frac{\partial u_2}{\partial x_i} \frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} \frac{\sum_{k \in T_2 \setminus \{i\}} b_k}{\sum_{k \in T_2} b_k} - \frac{\partial u_2}{\partial y_i}, \ i \in T_2. \end{cases} \tag{A2}$$

As from Assumption 3a, we have $\lim_{q_i \to 1} \frac{\partial \pi_i}{\partial q_i} < 0$ and $\lim_{q_i \to 0} \frac{\partial \pi_i}{\partial q_i} > 0$, for $i \in T_1$. Likewise, we have $\lim_{b_i \to 1} \frac{\partial \pi_i}{\partial b_i} < 0$ and $\lim_{b_i \to 0} \frac{\partial \pi_i}{\partial b_i} > 0$, for $i \in T_2$. Then, at an interior equilibrium, the $2n$ first-order necessary conditions may be written:

$$
\begin{cases}
-\frac{\partial u_1}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} \frac{\sum_{k \in T_1 \setminus \{i\}} q_k}{\sum_{k \in T_1} q_k} = 0, \, i \in T_1 \\
\\
\frac{\partial u_2}{\partial x_i} \frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} \frac{\sum_{k \in T_2 \setminus \{i\}} b_k}{\sum_{k \in T_2} b_k} - \frac{\partial u_2}{\partial y_i} = 0, \, i \in T_2.
\end{cases}
\tag{A3}
$$

Furthermore, the $2n$ second-order derivatives $\frac{\partial^2 \pi_i}{\partial q_i^2}$, for $i \in T_1$, and $\frac{\partial^2 \pi_i}{\partial b_i^2}$, for $i \in T_2$, may be written:

$$
\begin{cases}
\frac{\partial^2 \pi_i}{\partial q_i^2} = -\frac{1}{(\frac{\partial u_1}{\partial y_i})^2} \left| \mathcal{H}^2_{u_1}(x_i, y_i) \right| - 2 p_X \frac{\sum_{k \in T_1 \setminus \{i\}} q_k}{\left(\sum_{k \in T_1} q_k\right)^2} \frac{\partial u_1}{\partial y_i}, \, i \in T_1 \\
\\
\frac{\partial^2 \pi_i}{\partial b_i^2} = -\frac{1}{(\frac{\partial u_2}{\partial x_i})^2} \left| \mathcal{H}^2_{u_2}(x_i, y_i) \right| - 2 \frac{1}{p_X} \frac{\sum_{k \in T_2 \setminus \{i\}} b_k}{\left(\sum_{k \in T_2} b_k\right)^2} \frac{\partial u_2}{\partial x_i}, \, i \in T_2.
\end{cases}
\tag{A4}
$$

where $\left| \mathcal{H}^2_{u_1}(x_i, y_i) \right| = -\frac{\partial^2 u_1}{\partial x_i^2} + 2A \frac{\partial^2 u_1}{\partial x_i \partial y_i} - A^2 \frac{\partial^2 u_1}{\partial y_i^2}$, with $A \equiv \frac{\sum_{k \in T_2} b_k \sum_{k \in T_1 \setminus \{i\}} q_k}{\left(q_i + \sum_{k \in T_1 \setminus \{i\}} q_k\right)^2}$, is the determinant of the bordered Hessian matrix of the utility function u_1 evaluated at the point $(x_i(\mathbf{q}; \mathbf{b}), y_i(\mathbf{q}; \mathbf{b}))$, where $(\mathbf{q}; \mathbf{b})$ is a stationary point, i.e., a solution to (A3). A similar expression holds for $\left| \mathcal{H}^2_{u_2}(x_i, y_i) \right|$, $i \in T_2$. From Assumption 2b, for each $i \in T_1$, we have that $\frac{\partial u_i}{\partial y_i} > 0$. Moreover, from Assumption 2c, we have $\left| \mathcal{H}^2_{u_1}(x_i, y_i) \right| > 0$, for $i \in T_1$. Likewise, for $i \in T_2$, we have that $\frac{\partial u_2}{\partial x_i} > 0$ and $\left| \mathcal{H}^2_{u_2}(x_i, y_i) \right| > 0$. Then, we have

$$
\begin{cases}
\frac{\partial^2 \pi_i}{\partial q_i^2} < 0, \, i \in T_1 \\
\\
\frac{\partial^2 \pi_i}{\partial b_i^2} < 0, \, i \in T_2,
\end{cases}
\tag{A5}
$$

so the maximization problem of each trader is strictly concave in her own strategy. In addition, as from Assumption 3a, we have $\lim_{q_i \to 1} \frac{\partial \pi_i}{\partial q_i} < 0$ and $\lim_{q_i \to 0} \frac{\partial \pi_i}{\partial q_i} > 0$, $i \in T_1$, and $\lim_{b_i \to 1} \frac{\partial \pi_i}{\partial b_i} < 0$ and $\lim_{b_j \to 0} \frac{\partial \pi_i}{\partial b_i} > 0$, $i \in T_2$, then, the maximization problem of each trader has a unique interior solution $q_i \in (0, 1)$, $i \in T_1$, and $b_i \in (0, 1)$, $i \in T_2$, which is given by the $2n$ sufficient first-order conditions (A3).

Next, we show that the strategies of traders of type 1 are type-symmetric, i.e., $q_i = q$, for $i \in T_1$ (a similar argument holds for $i \in T_2$, with $b_i = b$). To this end, pick $\{i, j\} \in T_1$, with $i \neq j$, and assume that $q_i \neq q_j$, with $q_i > q_j$. Then, at an equilibrium, by using (A3), we must have:

$$
\begin{cases}
\frac{\frac{\partial u_1}{\partial x_i}\left(1 - q_i \, , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1 \setminus \{i\}} q_k + q_i} q_i\right)}{\frac{\partial u_1}{\partial y_i}\left(1 - q_i \, , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1 \setminus \{i\}} q_k + q_i} q_i\right)} = \frac{\sum_{k \in T_2} b_k \sum_{k \in T_1 \setminus \{i\}} q_k}{\left(\sum_{k \in T_1 \setminus \{i\}} q_k + q_i\right)^2} \\
\\
\frac{\frac{\partial u_1}{\partial x_j}\left(1 - q_j \, , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1 \setminus \{j\}} q_k + q_j} q_j\right)}{\frac{\partial u_1}{\partial y_j}\left(1 - q_j \, , \frac{\sum_{j \in T_2} b_j}{\sum_{k \in T_1 \setminus \{i'\}} q_k + q_j} q_j\right)} = \frac{\sum_{k \in T_2} b_k \sum_{k \in T_1 \setminus \{j\}} q_k}{\left(\sum_{k \in T_1 \setminus \{j\}} q_k + q_j\right)^2}
\end{cases}
\tag{A6}
$$

Define the function $f : \mathcal{Q} \to [0, \infty]$, $q \mapsto f(q)$, with $f(q) = \frac{\frac{\partial u_1}{\partial x}\left(1 - q \, , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q\right)}{\frac{\partial u_1}{\partial y}\left(1 - q \, , \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q\right)}$.

As $f'(q) = \frac{1}{\partial u_1 / \partial y} \left| \mathcal{H}^2_{u_1}(x, y) \right| > 0$ by 2b-2c, it follows that $f(q_i) > f(q_j)$. But, from

(A6), we have that $1 < \frac{f(q_i)}{f(q_j)} = \frac{\sum_{k \in T_1 \setminus \{i\}} q_k}{\sum_{k \in T_1 \setminus \{j\}} q_k} < 1$, a contradiction. Then, the CNE is type-symmetric, and we will denote it by $(\tilde{q}, \tilde{b})$. Therefore, as $\frac{B}{Q} q = \frac{B}{n} = b$ and $\frac{Q}{B} b = \frac{Q}{n} = q$, the CNE is characterized by the solution $(\tilde{q}, \tilde{b})$ to the system of $2n$ equations:

$$\begin{cases} -\frac{\partial u_1}{\partial x_i}(1-q\ ,b) + \frac{\partial u_1}{\partial y_i}(1-q\ ,b)\frac{b}{q}\frac{n-1}{n} = 0,\ i \in T_1 \\ \\ \frac{\partial u_2}{\partial x_i}(q\ ,1-b)\frac{q}{b}\frac{n-1}{n} - \frac{\partial u_2}{\partial y_i}(q, 1-b) = 0,\ i \in T_2. \end{cases} \tag{A7}$$

Next, we show the CNE is symmetric, i.e., $q = b$. To this end, define the total supplies on the two sides of the market as $Q \equiv nq$ and $B \equiv nb$, so the equilibrium conditions (A7) may be written:

$$\begin{cases} -\frac{\partial u_1}{\partial x_i}\left(1-\frac{Q}{n}\ ,\frac{B}{n}\right) + \frac{\partial u_1}{\partial y_i}\left(1-\frac{Q}{n}\ ,\frac{B}{n}\right)\frac{B}{Q}\frac{n-1}{n} = 0,\ i \in T_1 \\ \\ \frac{\partial u_2}{\partial x_i}\left(\frac{Q}{n}\ ,1-\frac{B}{n}\right)\frac{Q}{B}\frac{n-1}{n} - \frac{\partial u_2}{\partial y_i}\left(\frac{Q}{n}\ ,1-\frac{B}{n}\right) = 0,\ i \in T_2, \end{cases} \tag{A8}$$

Assume $Q \neq B$. Let $Q = aB$, with $a > 0$ and $a \neq 1$. Then, we have:

$$\begin{cases} -\frac{\partial u_1}{\partial x_i}\left(1-\frac{aB}{n}\ ,\frac{B}{n}\right) + \frac{\partial u_1}{\partial y_i}\left(1-\frac{aB}{n}\ ,\frac{B}{n}\right)\frac{1}{a}\frac{n-1}{n} = 0,\ i \in T_1 \\ \\ \frac{\partial u_2}{\partial x_i}\left(\frac{aB}{n}\ ,1-\frac{B}{n}\right)a\frac{n-1}{n} - \frac{\partial u_2}{\partial y_i}\left(\frac{aB}{n}\ ,1-\frac{B}{n}\right) = 0,\ i \in T_2, \end{cases} \tag{A9}$$

As the market is symmetric, i.e., $\forall (x,y) \in \mathbb{R}_+^2\ u_1(x,y) = u_2(y,x)$, then, $\frac{\partial u_1}{\partial x_i} = \frac{\partial u_2}{\partial y_i}$ and $\frac{\partial u_1}{\partial y_i} = \frac{\partial u_2}{\partial x_i}$, so we must have

$$\begin{cases} -\frac{\partial u_1}{\partial x_i}\left(1-\frac{aB}{n}\ ,\frac{B}{n}\right) + \frac{\partial u_1}{\partial y_i}\left(1-\frac{aB}{n}\ ,\frac{B}{n}\right)a\frac{n-1}{n} = 0,\ i \in T_1 \\ \\ \frac{\partial u_2}{\partial x_i}\left(\frac{aB}{n}\ ,1-\frac{B}{n}\right)\frac{1}{a}\frac{n-1}{n} - \frac{\partial u_2}{\partial y_i}\left(\frac{aB}{n}\ ,1-\frac{B}{n}\right) = 0,\ i \in T_2. \end{cases} \tag{A10}$$

But (A10) is consistent with (A9) if and only if $a = 1$, i.e., $Q = B$. Then, $\tilde{Q} = \tilde{B}$, so any interior type-symmetric CNE is symmetric.

Let us now consider the equilibrium conditions (A7) following the change of variables $Q \equiv nq$ and $B \equiv nb$, which may now be written:

$$\begin{cases} g(Q,B) = -\frac{\partial u_1}{\partial x_i}\left(1-\frac{Q}{n},\frac{B}{n}\right) + \frac{n-1}{n}\frac{B}{Q}\frac{\partial u_1}{\partial y_i}\left(1-\frac{Q}{n},\frac{B}{n}\right) = 0 \\ \\ h(Q,B) = -\frac{\partial u_2}{\partial y_i}\left(\frac{Q}{n},1-\frac{B}{n}\right) + \frac{n-1}{n}\frac{Q}{B}\frac{\partial u_2}{\partial x_i}\left(\frac{Q}{n},1-\frac{B}{n}\right) = 0. \end{cases} \tag{A11}$$

We show that the system of equations (A11) has a unique solution $(\tilde{Q}, \tilde{B})$. To this end, we first show that if (A11) has a solution, then it is unique and globally stable. To show this, we will prove that, from (A11), we can define implicitly the best responses of traders, and that these best responses have strictly positive slopes less than one. First, let us note that:

$$\frac{\partial g}{\partial Q} = -\frac{1}{n}\left(-\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}\right) \tag{A12}$$

and

$$\frac{\partial h}{\partial B} = -\frac{1}{n}\left(-\frac{\partial^2 u_1}{\partial y_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{B}\frac{\partial u_2}{\partial x_i}\right). \tag{A13}$$

We show that $\frac{\partial g}{\partial Q} < 0$ (a similar reasoning will hold for the case $\frac{\partial h}{\partial B} < 0$). It leads to show that:

$$-\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i} > 0. \tag{A14}$$

Obviously, from Assumptions 2b and 3b, we have $\frac{\partial g}{\partial Q} < 0$ when $\frac{\partial^2 u_1}{\partial x_i^2} \leqslant 0$. Consider the case $\frac{\partial^2 u_1}{\partial x_i^2} > 0$. Multiplying each term of the inequality in Assumption 3c, i.e., here $2\frac{\partial u_1}{\partial y_i} + y_i \frac{\partial^2 u_1}{\partial y_i^2} - \frac{\partial^2 u_1}{\partial x_i \partial y_i} > 0$, by $\frac{n-1}{B} > 0$ yields the inequality:

$$2\frac{n-1}{B}\frac{\partial u_1}{\partial y_i} + \frac{n-1}{B} y_i \frac{\partial^2 u_1}{\partial y_i^2} - \frac{n-1}{B}\frac{\partial^2 u_1}{\partial x_i \partial y_i} > 0. \tag{A15}$$

Assume by way of contradiction that (A14) does not hold, i.e., $-\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i} \leqslant 0$. With $\frac{\partial^2 u_1}{\partial x_i^2} > 0$ and Assumption 3b, we have that $\frac{n-1}{Q}\frac{\partial u_1}{\partial y_i} \leqslant \frac{\partial^2 u_1}{\partial x_i^2} - \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i}$. But then, as $Q = B$ and $B = nb$, with $y = b$, from (A15), we deduce:

$$2\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2} - \frac{n-1}{n}(2 + \frac{1}{b})\frac{\partial^2 u_1}{\partial x_i \partial y_i} > 0. \tag{A16}$$

Then, we deduce $\frac{\partial^2 u_1}{\partial x_i^2} > \chi$, with $\chi \equiv \frac{n-1}{2n}(2 + \frac{1}{b})\frac{\partial^2 u_1}{\partial x_i \partial y_i} - \frac{n-1}{2n}\frac{\partial^2 u_1}{\partial y_i^2}$. In addition, from Assumption 2c, we have $\frac{\partial^2 u_1}{\partial x_i^2} < \delta$, with $\delta \equiv 2\frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} - (\frac{n-1}{n})^2 \frac{\partial^2 u_1}{\partial y_i^2}$. But as $b \in (0,1)$, $(2 + \frac{1}{b}) > 2$ and $\frac{n-1}{n} < 1$, we have $\chi > \delta$, with $\frac{\partial^2 u_1}{\partial x_i^2} > \chi$ and $\frac{\partial^2 u_1}{\partial x_i^2} < \delta$, a contradiction. Then, (A14) holds.

Then, from the above, by the implicit function theorem, the equation $g(Q, B) = 0$ (resp. $h(Q, B) = 0$) in (A11) defines implicitly the continuously differentiable best response function of the traders of type 1 (resp. 2), namely $Q = \sigma(B)$ (resp. $B = \phi(Q)$). Then, we have $g(\sigma(B), B) \equiv 0$ and $h(Q, \phi(Q)) \equiv 0$. By implicit differentiation, and, as from Assumption 3c we also have $\frac{\partial g}{\partial B} > 0$ and $\frac{\partial h}{\partial Q} > 0$, we deduce:

$$\begin{cases} \frac{d\sigma}{dB} \equiv -\frac{\frac{\partial g}{\partial B}}{\frac{\partial g}{\partial Q}} = \frac{-\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}}{-\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}} > 0 \\ \\ \frac{d\phi}{dQ} \equiv -\frac{\frac{\partial h}{\partial Q}}{\frac{\partial h}{\partial B}} = \frac{-\frac{\partial^2 u_2}{\partial x_i \partial y_i} + \frac{n-1}{n}\frac{\partial^2 u_2}{\partial x_i^2} + \frac{n-1}{B}\frac{\partial u_2}{\partial x_i}}{-\frac{\partial^2 u_2}{\partial y_i^2} + \frac{n-1}{n}\frac{\partial^2 u_2}{\partial x_i \partial y_i} + \frac{n-1}{B}\frac{\partial u_2}{\partial x_i}} > 0, \end{cases} \tag{A17}$$

so the best response functions are strictly increasing. Moreover, define, as in Bloch and Ghosal (1997), the elasticities of the best response functions as $\frac{d\sigma}{dB}\frac{B}{Q}$ and $\frac{d\phi}{dQ}\frac{Q}{B}$.

We now show that the CNE, if it exists, must be unique and globally stable, i.e., $\frac{d\sigma}{dB}\frac{B}{Q} < 1$ and $\frac{d\phi}{dQ}\frac{Q}{B} < 1$. Consider the best response $\sigma(B)$ (a similar argument holds for $\phi(B)$). At a symmetric type symmetric interior CNE, we have $Q = B$, so the elasticity of $\sigma(B)$ is given by $\frac{d\sigma}{dB}\frac{B}{Q} = \frac{d\sigma}{dB}$. Let $R = \frac{-\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}}{-\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}}$. As $\frac{\partial^2 u_1}{\partial x_i \partial y_i} \geqslant 0$, the position of R with respect to 1 depends critically on both the signs and the relative magnitudes of $\frac{\partial^2 u_1}{\partial x_i^2}$ and $\frac{\partial^2 u_1}{\partial y_i^2}$. As the case $\frac{\partial^2 u_1}{\partial x_i^2} = 0$ and $\frac{\partial^2 u_1}{\partial y_i^2} = 0$ is precluded from Assumption 2c, there are here height cases to examine, i.e., 6 cases involved by $\frac{\partial^2 u_1}{\partial x_i^2} \lesseqqgtr 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} \lessgtr 0$, and 2 cases involved by $\frac{\partial^2 u_1}{\partial x_i^2} \lessgtr 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} = 0$.

1. The case for which $\frac{\partial^2 u_1}{\partial x_i^2} < 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} < 0$ (strict concavity) is immediate (see Bloch and Ghosal, 1997, p. 381).

2. When $\frac{\partial^2 u_1}{\partial x_i^2} < 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} = 0$, we immediately get $R < 1$.

3. Consider the two cases $\frac{\partial^2 u_1}{\partial x_i^2} \leqslant 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} > 0$. Assume $\frac{\partial^2 u_1}{\partial x_i^2} = 0$. From Assumption 2c, we must have $\frac{\partial^2 u_1}{\partial x_i \partial y_i} > 0$. Assume $R < 1$. Then, we have $\frac{\partial^2 u_1}{\partial y_i^2} < \frac{2n-1}{n-1}\frac{\partial^2 u_1}{\partial x_i \partial y_i}$. From Assumption 2c, we must have $\frac{\partial^2 u_1}{\partial y_i^2} < 2\frac{\partial u_1/\partial y_i}{\partial u_1/\partial x_i} = 2\frac{n}{n-1}$. Then, $\frac{2n-1}{n-1}\frac{\partial^2 u_1}{\partial x_i \partial y_i} < 2\frac{n}{n-1}\frac{\partial^2 u_1}{\partial x_i \partial y_i}$, which is true as $\frac{2n-1}{n-1} < 2\frac{n}{n-1}$. Assume $\frac{\partial^2 u_1}{\partial x_i^2} < 0$ and $R > 1$. We deduce $\frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2} - \frac{2n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{\partial^2 u_1}{\partial x_i^2} > 0$. As $\frac{\partial u_1/\partial x_i}{\partial u_1/\partial y_i} = \frac{n-1}{n}$, by algebraic manpulations, we obtain $\left|\mathcal{H}_u^2\right| > \frac{1}{n}(\frac{\partial u_1}{\partial y_i})^2(\frac{\partial^2 u_1}{\partial x_i^2} - \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i})$, where $\left|\mathcal{H}_u^2\right| = (\frac{\partial u_1}{\partial x_i})^2\frac{\partial^2 u_1}{\partial y_i^2} - 2\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial y_i}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + (\frac{\partial u_1}{\partial y_i})^2\frac{\partial^2 u_1}{\partial x_i^2}$. Then, $\left|\mathcal{H}_u^2\right| < 0$, a contradiction by virtue of Assumption 2c. Then, we have $R < 1$.

4. Consider the four cases, i.e., $\frac{\partial^2 u_1}{\partial x_i^2} \geqslant 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} < 0$, $\frac{\partial^2 u_1}{\partial x_i^2} > 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} = 0$, and $\frac{\partial^2 u_1}{\partial x_i^2} > 0$ with $\frac{\partial^2 u_1}{\partial y_i^2} > 0$. To this end, let $S = \frac{-\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}}{-\frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q}\frac{\partial u_1}{\partial y_i}} > R$. Assume $S \geqslant 1$. It leads to the inequality $\frac{n-1}{n}(\frac{\partial^2 u_1}{\partial x_i^2} + \frac{\partial^2 u_1}{\partial y_i^2}) \geqslant \frac{2n-1}{n}\frac{\partial^2 u_1}{\partial x_i \partial y_i}$. From Assumption 2c, we get $\frac{\partial^2 u_1}{\partial x_i \partial y_i} > \frac{1}{2}\frac{n}{n-1}\frac{\partial^2 u_1}{\partial x_i^2} + \frac{1}{2}\frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2}$, so we deduce $\frac{1}{2}C < \frac{\partial^2 u_1}{\partial x_i \partial y_i} \leqslant \frac{n-1}{2n-1}D$, where $C \equiv \frac{n}{n-1}\frac{\partial^2 u_1}{\partial x_i^2} + \frac{n-1}{n}\frac{\partial^2 u_1}{\partial y_i^2}$ and $D \equiv \frac{\partial^2 u_1}{\partial x_i^2} + \frac{\partial^2 u_1}{\partial y_i^2}$. But then, we must have $C < D$, which leads to $\frac{3n-2}{2(n-1)(2n-1)}\frac{\partial^2 u_1}{\partial x_i^2} - \frac{n-1}{2n(2n-1)}\frac{\partial^2 u_1}{\partial y_i^2} < 0$ (*). First, $\frac{\partial^2 u_1}{\partial x_i^2} \geqslant 0$ and $\frac{\partial^2 u_1}{\partial y_i^2} < 0$ are inconsistent with (*). Second, $\frac{\partial^2 u_1}{\partial x_i^2} > 0$ and $\frac{\partial^2 u_1}{\partial y_i^2} = 0$ are inconsistent with (*). Third, consider $\frac{\partial^2 u_1}{\partial x_i^2} > 0$ and $\frac{\partial^2 u_1}{\partial y_i^2} > 0$, with $\frac{\partial^2 u_1}{\partial x_i \partial y_i} > 0$ (otherwise u_1 is strictly quasi-convex). From (*) we have $\frac{\partial^2 u_1}{\partial y_i^2} > \frac{n(3n-2)}{(n-1)^2}\frac{\partial^2 u_1}{\partial x_i^2}$ and, by symmetry of u, we also have $\frac{\partial^2 u_2}{\partial y_i^2} > \frac{(n-1)^2}{n(3n-2)}\frac{\partial^2 u_2}{\partial x_i^2}$. As, at a symmetric type-symmetric CNE, we have $\frac{\partial^2 u_1}{\partial y_i^2} = \frac{\partial^2 u_2}{\partial x_i^2}$ and $\frac{\partial^2 u_1}{\partial x_i^2} = \frac{\partial^2 u_2}{\partial y_i^2}$, then $\frac{\partial^2 u_1}{\partial x_i^2} > \frac{n(3n-2)}{(n-1)^2}\frac{\partial^2 u_1}{\partial y_i^2}$ and (*) both hold, a contradiction. In all four cases we have that $\frac{d\sigma}{dB} \leqslant S$, with $S < 1$. Then, we deduce:

$$\begin{cases} \frac{d\sigma}{dB}\frac{B}{Q} = \frac{d\sigma}{dB} < 1 \\ \\ \frac{d\phi}{dQ}\frac{Q}{B} = \frac{d\phi}{dQ} < 1. \end{cases} \tag{A18}$$

Therefore, from (A18), if a symmetric type-symmetric CNE exists, then, it is unique and globally stable.

Finally, to show existence of a unique interior CNE, define the continuously differentiable function $\Upsilon : [0, n] \to \mathbb{R}$, $Q \mapsto \Upsilon(Q)$, with

$$\Upsilon(Q) = Q - \sigma[\phi(Q)]. \tag{A19}$$

Therefore, $(\tilde{Q}, \tilde{B})$, where $\tilde{B} = \phi(\tilde{Q})$, is an equilibrium if and only if $\Upsilon(\tilde{Q}) = 0$. Let us first determine the sign of Υ at the two boundary points 0 and n, i.e., we want to show that $\lim_{Q \to 0} \Upsilon(Q) < 0$ and $\lim_{Q \to n} \Upsilon(Q) > 0$.

Consider the case $\lim_{Q \to 0} \Upsilon(Q) < 0$. To show this, write (A19) as $\frac{\Upsilon(Q)}{Q} = 1 - \frac{\sigma[\phi(Q)]}{Q}$, and let $\frac{\sigma[\phi(Q)]}{Q} = \frac{\sigma[\phi(Q)]}{\phi(Q)} \frac{\phi(Q)}{Q}$. As, on the one hand, we have that $\lim_{Q \to 0} \phi(Q) = 0$, and, on other hand, as from (A18) we have that $\frac{d\phi}{dQ} < 1$, and from Assumption 3a, we have that $\lim_{Q \to 0} \frac{\phi(Q)}{Q} = +\infty$ and $\lim_{B \to 0} \frac{\sigma(B)}{B} = +\infty$, then we deduce $\lim_{Q \to 0} \frac{\sigma[\phi(Q)]}{\phi(Q)} = +\infty$. Then, (A18) is such that $\lim_{Q \to 0} \Upsilon(Q) < 0$.

Consider now the case $\lim_{Q \to n} \Upsilon(Q) > 0$. To show this, we prove that for all $B \in (0, n)$, we have $\sigma(B) < n$. Assume that there exists some $B \in (0, n)$ such that $\sigma(B) = n$. This implies that $\lim_{Q \to n}(-\frac{\partial u_1}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \frac{B}{Q} \frac{n-1}{n}) > 0$, contradicting the assumption that, for each $i \in T_1$, $\lim_{q_i \to 1}(-\frac{\partial u_1}{\partial x_i} + \frac{\partial u_1}{\partial y_i} y_i) > 0$. Then, we have $\lim_{Q \to n} \Upsilon(Q) > 0$.

Second, we show that Υ is a strictly increasing function on $(0, n)$. By differentiating (A19) with respect to Q, we have that:

$$\frac{d\Upsilon(Q)}{dQ} = 1 - \frac{d\sigma}{dB} \frac{d\phi}{dQ} < 1. \tag{A20}$$

As from (A18) we have $\frac{d\sigma}{dB} < 1$ and $\frac{d\phi}{dQ} < 1$, then $\frac{d\Upsilon(Q)}{dQ} > 0$, for all $Q \in (0, n)$. Then, we can conlude that:

$$\exists!\ \tilde{Q} \in (0, n) \mid \Upsilon(\tilde{Q}) = 0. \tag{A21}$$

Therefore, as $B = \phi(Q)$, with $\frac{d\phi}{dQ} > 0$, there is a unique $\tilde{B} \in (0, n)$. Then, the game Γ has a unique interior CNE $(\tilde{q}, \tilde{b})$.■

7.2. Appendix B: proof of Proposition 2

To show that the CNE is not Pareto-optimal, consider trader i's marginal rate of substitution at the interior CNE. From (8), we have:

$$\begin{cases} \frac{\tilde{b}}{\tilde{q}} = \frac{n}{n-1} MRS^i(\tilde{x}_i, \tilde{y}_i), \text{ for } i \in T_1; \\ \\ \frac{\tilde{q}}{\tilde{b}} = \frac{n}{n-1} MRS^i(\tilde{x}_i, \tilde{y}_i), \text{ for } i \in T_2. \end{cases} \tag{B1}$$

As, from (3), we have $\tilde{p}_X = \frac{\sum_{i \in T_2} \tilde{b}_i}{\sum_{i \in T_1} \tilde{q}_i}$, and the SNE is type-symmetric, i.e., $\tilde{q}_i = \tilde{q}$, for each $i \in T_1$,and $\tilde{b}_j = \tilde{b}$, for each $i \in T_2$, then $\tilde{p}_X = \frac{n\tilde{b}_i}{n\tilde{q}_i} = \frac{\tilde{b}}{\tilde{q}}$. Then, we have:

$$\begin{cases} \tilde{p}_X = \frac{n}{n-1} MRS^i(\tilde{x}_i, \tilde{y}_i), \text{ for } i \in T_1; \\ \\ \frac{1}{\tilde{p}_X} = \frac{n}{n-1} MRS^i(\tilde{x}_i, \tilde{y}_i), \text{ for } i \in T_2. \end{cases} \tag{B2}$$

In addition, as the interior CNE is symmetric, i.e., $\tilde{q} = \tilde{b}$, we deduce $\tilde{p}_X = 1$. Then, we have:

$$\begin{cases} MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n-1}{n}, \text{ for } i \in T_1; \\ \\ MRS^i(\tilde{x}_i, \tilde{y}_i) = \frac{n}{n-1}, \text{ for } i \in T_2, \end{cases} \tag{B3}$$

so, the marginal rates of substitution differ across traders of both types. Therefore, the CNE allocation is not Pareto-optimal.■

7.3. Appendix C: proof of Theorem 1

The logic of the proof is as follows. First, we show that, given $t_X \in (0,1)$ and $t_Y \in (0,1)$, there exists a unique symmetric CNE, which is interior and type-symmetric. Second, we determine the equilibrium tax $\tilde{t}$, i.e, $\tilde{t} = \frac{1-z^*}{n+1}$, with $0 < z^* < 1$. Third, we show that, for this tax, the corresponding CNE post-tax allocation $\widetilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t}))_{i \in T_1 \cup T_2}$ of Γ^t is Pareto-optimal.

First, we show there exists a unique interior symmetric type-symmetric CNE. Assume the market is symmetric, with $\#T_1 = \#T_2 = n$ and $\alpha = \beta = 1$. Let $(\tilde{q}_1, ..., \tilde{q}_n; \tilde{b}_{n+1}, ..., \tilde{b}_{2n}) \in [0, 1-t_X]^n \times [0, 1-t_Y]^n$, $0 < t_X < 1$ and $0 < t_Y < 1$, be a nontrivial equilibrium for which $\sum_{i \in T_1} \tilde{q}_i > 0$ and $\sum_{i \in T_2} \tilde{b}_i > 0$. Assumption 3a may now be written as $\lim_{x \to 0} \left(-\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y \right) < 0$ and $\lim_{x \to 1-t_X} \left(-\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y \right) > 0$.

Given $0 < t_X < 1$ and $0 < t_Y < 1$, the $2n$ maximization problems of traders may be written:

$$\begin{cases} \max\limits_{q_i \in [0, 1-t_X]} \pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}; t_X, t_Y) = u_1 \left(1 - t_X - q_i, \frac{\sum_{k \in T_2} b_k}{\sum_{k \in T_1} q_k} q_i + \frac{t_Y}{1-t_Y-y^*} q_i \right), \, i \in T_1 \\ \\ \max\limits_{b_i \in [0, 1-t_Y]} \pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}; t_X, t_Y) = u_2 \left(\frac{\sum_{k \in T_1} q_k}{\sum_{k \in T_2} b_k} b_i + \frac{t_X}{1-t_X-x^*} b_i, 1 - t_Y - b_i \right), \, i \in T_2. \end{cases} \tag{C1}$$

As Assumptions 2-3 still hold, using the same reasoning as in Appendix A, the CNE is interior and type-symmetric. Then, we have $Q(t_X, t_Y) \equiv nq(t_X, t_Y)$ and $B(t_X, t_Y) \equiv nb(t_X, t_Y)$. Then, the CNE is given by the solution $(\tilde{q}(t_X, t_Y), \tilde{b}(t_X, t_Y))$ to the system of $2n$ first-order sufficient conditions:

$$\begin{cases} -\frac{\partial u_1(1-t_X-q, b+\frac{t_Y}{1-t_Y-y^*}q)}{\partial x_i} + \left(\frac{b}{q} \frac{n-1}{n} + \frac{t_Y}{1-t_Y-y^*} \right) \frac{\partial u_1(1-t_X-q, b+\frac{t_Y}{1-t_Y-y^*}q)}{\partial y_i} = 0 \\ \\ -\frac{\partial u_2(q+\frac{t_X}{1-t_X-x^*}b, 1-t_Y-b)}{\partial y_i} + \left(\frac{q}{b} \frac{n-1}{n} + \frac{t_X}{1-t_X-x^*} \right) \frac{\partial u_2(q+\frac{t_X}{1-t_X-x^*}b, 1-t_Y-b)}{\partial x_i} = 0. \end{cases} \tag{C2}$$

Next, as the market is symmetric, with $u_1(x,y) = u_2(y,x)$, then, we can show, like in Appendix A, that we must have $Q(t_X, t_Y) = B(t_X, t_Y)$. Then, $\tilde{Q}(t_X, t_Y) = \tilde{B}(t_X, t_Y)$. Moreover, at an interior symmetric competitive equilibrium without taxation, we have $\frac{\sum_{i \in T_1} x_i^*}{n} = \frac{\sum_{i \in T_2} y_i^*}{n}$, so we must have $t_X = t_Y = t$. Then, as the market is symmetric, the equilibrium of Γ^t is a symmetric CNE, which is interior and type-symmetric, so (C2) may be written:

$$\begin{cases} -\frac{\partial u_1(1-t-q,b+\frac{t}{1-t-y^*}q)}{\partial x_i} + \left(\frac{n-1}{n} + \frac{t}{1-t-y^*}\right)\frac{\partial u_1(1-t-q,b+\frac{t}{1-t-y^*}q)}{\partial y_i} = 0 \\ -\frac{\partial u_2(q+\frac{t}{1-t-x^*}b,1-t-b)}{\partial y_i} + \left(\frac{n-1}{n} + \frac{t}{1-t-x^*}\right)\frac{\partial u_2(q+\frac{t}{1-t-x^*}b,1-t-b)}{\partial x_i} = 0. \end{cases} \tag{C3}$$

Let us now consider the equilibrium conditions (C3), following the change of variables $Q(t) \equiv nq(t)$ and $B(t) \equiv nb(t)$, which may be written:

$$\begin{cases} k(Q(t),B(t)) = -\frac{\partial u_1\left(1-\frac{Q(t)}{n},\frac{B(t)}{n}\right)}{\partial x_i} + \left(\frac{B(t)}{Q(t)}\frac{n-1}{n} + \frac{t}{1-t-y^*}\right)\frac{\partial u_1\left(1-\frac{Q(t)}{n},\frac{B(t)}{n}\right)}{\partial y_i} = 0 \\ l(Q(t),B(t)) = -\frac{\partial u_2\left(\frac{Q(t)}{n},1-\frac{B(t)}{n}\right)}{\partial y_i} + \left(\frac{Q(t)}{B(t)}\frac{n-1}{n} + \frac{t}{1-t-x^*}\right)\frac{\partial u_2\left(\frac{Q(t)}{n},1-\frac{B(t)}{n}\right)}{\partial x_i} = 0. \end{cases} \tag{C4}$$

Using a similar approach to that used for proving Proposition 1, we now show that the system of equations (C4) has a unique interior solution $(\tilde{Q}(t), \tilde{B}(t))$. First, as $Q(t) = B(t)$, and by using Assumption 2c and Assumptions 3b-3c, we deduce:

$$\frac{\partial k(t)}{\partial Q(t)} = -\frac{1}{n}[-\frac{\partial^2 u_1}{\partial x_i^2} + (\frac{n-1}{n} + \frac{t}{1-t-y^*})\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q(t)}\frac{\partial u_1}{\partial y_i}] < 0 \tag{C5}$$

and

$$\frac{\partial l(t)}{\partial B(t)} = -\frac{1}{n}[-\frac{\partial^2 u_1}{\partial x_i^2} + (\frac{n-1}{n} + \frac{t}{1-t-x^*})\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{B(t)}\frac{\partial u_1}{\partial y_i}] < 0. \tag{C6}$$

Then, the equation $k(Q(t), B(t)) = 0$ (resp. $l(Q(t), B(t)) = 0$) in (C4) defines implicitly the best response function of the traders of type 1 (resp. 2), namely $Q(t) = \varphi(B(t))$ (resp. $B(t) = \psi(Q(t))$). Then, we have $g(\varphi(B(t)), B(t)) \equiv 0$ and $l(Q, \psi(Q(t))) \equiv 0$. By implicit differentiation, and, as from Assumption 3c we also have $\frac{\partial k(t)}{\partial B(t)} > 0$ and $\frac{\partial l(t)}{\partial Q(t)} > 0$, we deduce:

$$\begin{cases} \frac{d\varphi(B(t))}{dB(t)} \equiv -\frac{\frac{\partial k(t)}{\partial B(t)}}{\frac{\partial k(t)}{\partial Q(t)}} = \frac{-\frac{\partial^2 u_1}{\partial x_i^2} + (\frac{n-1}{n} + \frac{t}{1-t-y^*})\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q(t)}\frac{\partial u_1}{\partial y_i}}{-\frac{\partial^2 u_1}{\partial x_i^2} + (\frac{n-1}{n} + \frac{t}{1-t-y^*})\frac{\partial^2 u_1}{\partial x_i \partial y_i} + \frac{n-1}{Q(t)}\frac{\partial u_1}{\partial y_i}} > 0 \\ \frac{d\psi(Q(t))}{dQ(t)} \equiv -\frac{\frac{\partial l(t)}{\partial Q(t)}}{\frac{\partial l(t)}{\partial B(t)}} = \frac{-\frac{\partial^2 u_2}{\partial x_i \partial y_i} + (\frac{n-1}{n} + \frac{t}{1-t-x^*})\frac{\partial^2 u_2}{\partial x_i^2} + \frac{n-1}{B(t)}\frac{\partial u_2}{\partial y_i}}{-\frac{\partial^2 u_2}{\partial y_i^2} + (\frac{n-1}{n} + \frac{t}{1-t-x^*})\frac{\partial^2 u_2}{\partial x_i \partial y_i} + \frac{n-1}{B(t)}\frac{\partial u_2}{\partial x_i}} > 0, \end{cases} \tag{C7}$$

so the best response functions are strictly increasing. Moreover, using a similar approach to that used for Proposition 1, we can deduce:

$$\begin{cases} \frac{d\varphi(B(t))}{dB(t)}\frac{B(t)}{Q(t)} = \frac{d\varphi(B(t))}{dB(t)} < 1 \\ \frac{d\psi(Q(t))}{dQ(t)}\frac{Q(t)}{B(t)} = \frac{d\psi(Q(t))}{dQ(t)} < 1. \end{cases} \tag{C8}$$

Therefore, from (C8), given $t \in (0, 1)$, if a CNE with taxation exists, then, it is unique and globally stable.

To show existence of a unique interior CNE with taxation, define the continuously differentiable function $\Psi : [0, n(1-t)] \to \mathbb{R}$, $Q(t) \mapsto \Psi(Q(t))$, with

$$\Psi(Q(t)) = Q(t) - \varphi[\psi(Q(t))]. \tag{C9}$$

Thus, given $t \in (0,1)$, $(\tilde{Q}(t), \tilde{B}(t))$, where $\tilde{B}(t) = \psi(\tilde{Q}(t))$, is an equilibrium if and only if $\Psi(\tilde{Q}(t)) = 0$. Let us first determine the sign of Ψ at the two boundary points 0 and $n(1-t)$.

Consider the case $\lim_{Q(t)\to 0} \Psi(Q(t)) < 0$. To this end, write (C9) ,as $\frac{\Psi(Q(t))}{Q(t)} = 1 - \frac{\varphi[\psi(Q(t))]}{Q(t)}$, and let $\frac{\varphi[\psi(Q(t))]}{Q(t)} = \frac{\varphi[\psi(Q(t))]}{\psi(Q(t))}\frac{\psi(Q(t))}{Q(t)}$. As, on the one hand, we have $\lim_{Q(t)\to 0} \psi(Q(t)) = 0$, and, on other hand, as from (C9) we have that $\frac{d\psi(Q(t))}{dQ(t)} < 1$, and from Assumption 3a, we have that $\lim_{Q(t)\to 0} \frac{\psi(Q(t))}{Q(t)} = +\infty$ and $\lim_{B(t)\to 0} \frac{\varphi(B(t))}{B(t)} = +\infty$, then we deduce $\lim_{Q(t)\to 0} \frac{\varphi[\psi(Q(t))]}{Q(t)} = +\infty$. Then, (C9) is such that $\lim_{Q(t)\to 0} \Psi(Q(t)) < 0$.

Consider now the case $\lim_{Q(t)\to n(1-t)} \Psi(Q(t)) > 0$. To this end, we show that for all $B \in (0, n(1-t))$, we have $\varphi(B(t)) < n(1-t)$. Assume that there exists some $B(t) \in (0, n(1-t))$ such that $\varphi(B(t)) = n(1-t)$. This implies that $\lim_{Q(t)\to n(1-t)}[-\frac{\partial u_1}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(\frac{B}{Q}\frac{n-1}{n} + \frac{t}{1-t-y^*})] > 0$, contradicting the assumption that, for each $i \in T_1$, $\lim_{q_i(t)\to 1-t}(-\frac{\partial u_1}{\partial x_i} + \frac{\partial u_1}{\partial y_i} y) > 0$. Then, $\lim_{Q(t)\to n(1-t)} \Psi(Q(t)) > 0$.

Next, we show that Ψ is a strictly increasing function on $(0, n(1-t))$. By differentiating (C9) with respect to $Q(t)$, we deduce:

$$\frac{d\Psi(Q(t))}{dQ(t)} = 1 - \frac{d\varphi(B(t))}{dB(t)}\frac{d\psi(Q(t))}{dQ(t)}. \tag{C10}$$

As from (C8) we have $\frac{d\varphi(B(t))}{dB(t)} < 1$ and $\frac{d\psi(Q(t))}{dQ(t)} < 1$, then $\frac{d\Psi(Q(t))}{dQ(t)} > 0$, for all $Q(t) \in (0, n(1-t))$. Then, we can conlude that:

$$\exists! \tilde{Q}(t) \in (0, n(1-t)) \mid \Psi(\tilde{Q}(t)) = 0. \tag{C11}$$

Therefore, as $B(t) = \varphi(Q(t))$, with $\frac{d\varphi}{dQ} > 0$, there is a unique $\tilde{B}(t) \in (0, n(1-t))$. Then, the game Γ^t has a unique symmetric Cournot-Nash equilibrium $(\tilde{q}(t), \tilde{b}(t))$, which is interior and type-symmetric.

Second, we determine the equilibrium tax $\tilde{t}$. If, at the symmetric type-symmetric interior CNE of the game Γ^t, the endowment tax t with transfers implements the allocation $(x_i^*, y_i^*)_{i\in T_1\cup T_2}$ corresponding to the interior symmetric CE without taxation, then, the allocation $(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t}))_{i\in T_1\cup T_2}$ corresponding to the symmetric type-symmetric interior CNE of the game Γ^t is such that:

$$(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t})) = (x_i^*, y_i^*),\ i \in T_1 \cup T_2, \tag{C12}$$

which may be written:

$$\left(1 - \tilde{t} - \tilde{q}(\tilde{t}), \tilde{q}(\tilde{t}) + \frac{\tilde{t}}{1-\tilde{t}-y^*}\tilde{q}(\tilde{t})\right) = (x_i^*, y_i^*),\ i \in T_1; \tag{C13}$$

$$\left(\tilde{b}(\tilde{t}) + \frac{\tilde{t}}{1-\tilde{t}-x^*}\tilde{b}(\tilde{t}), 1 - \tilde{\tau} - \tilde{b}(\tilde{t})\right) = (x_i^*, y_i^*),\ i \in T_2, \tag{C14}$$

with $x^* = \frac{\sum_{i\in T_1} x_i^*}{n}$ and $y^* = \frac{\sum_{i\in T_2} y_i^*}{n}$. As, at a symmetric type-symmetric interior CNE, $\tilde{q}(t) = \tilde{b}(t)$, and, each trader $i \in T_2$ receives the same share $\frac{t}{1-t-x^*} = \frac{t}{1-t-z^*}$

of commodity X, while each trader $i \in T_1$ receives the same share $\frac{t}{1-t-y^*} = \frac{t}{1-t-z^*}$ of commodity Y, then, we have that $\frac{\tilde{t}}{1-\tilde{t}-y^*} = \frac{\tilde{t}}{1-\tilde{t}-x^*} = \frac{t}{1-t-z^*} = \frac{1}{n}$. Then, from (C13) and (C14), and by using the fact that $x^* + y^* = 1$ at a symmetric type-symmetric interior CE (for which $p_X^* = 1$), we deduce the value of $\tilde{t}$, which is given by:

$$\tilde{t} = \frac{1 - z^*}{n+1}, \tag{C15}$$

where $z^* \equiv x^* = y^*$, with $0 < z^* < 1$.

In the remainder of the proof, we show the Pareto-optimality of the post-tax allocation associated with the equilibrium tax $\tilde{t}$. To this end, we prove that, for the equilibrium tax $\tilde{t} = \frac{1-z^*}{n+1}$, the equilibrium supplies of Γ^t given by $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$ implement a Pareto-optimal allocation. By substituting the value of $\tilde{t}$ given by (C15) into (C3), at the symmetric type-symmetric interior CNE of Γ^t the $2n$ first-order (sufficient) conditions may now be written:

$$\begin{cases} -\frac{\partial u_1(1-\tilde{t}-\tilde{q}(\tilde{t}), \frac{n+1}{n}\tilde{q}(\tilde{t}))}{\partial x_i} + \left(\frac{n-1}{n} + \frac{1}{n}\right) \frac{\partial u_1(1-\tilde{t}-\tilde{q}(\tilde{t}), \frac{n+1}{n}\tilde{q}(\tilde{t}))}{\partial y_i} = 0, \; i \in T_1 \\ -\frac{\partial u_2(\frac{n+1}{n}\tilde{b}(t), 1-\tilde{t}-\tilde{b}(\tilde{t}))}{\partial y_i} + \left(\frac{n-1}{n} + \frac{1}{n}\right) \frac{\partial u_2(\frac{n+1}{n}\tilde{b}(t), 1-\tilde{t}-\tilde{b}(\tilde{t}))}{\partial x_i} = 0, \; i \in T_2. \end{cases} \tag{C16}$$

The only interior solution $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$ to (C16) is such that:

$$\begin{cases} \forall i \in T_1 \;\; \frac{\frac{\partial u_1}{\partial x_i}}{\frac{\partial u_1}{\partial y_i}} \Big|_{(1-\tilde{t}-\tilde{q}(\tilde{t}), \tilde{q}(\tilde{t}) + \frac{1}{n}\tilde{q}(\tilde{t}))} = 1 \\ \forall i \in T_2 \;\; \frac{\frac{\partial u_2}{\partial x_i}}{\frac{\partial u_2}{\partial y_i}} \Big|_{(\tilde{b}(\tilde{t}) + \frac{1}{n}\tilde{b}(\tilde{t}), 1-\tilde{t}-\tilde{b}(\tilde{t}))} = 1, \end{cases} \tag{C17}$$

which implies that the marginal rates of substitution of all traders are equal at the symmetric type-symmetric interior CNE given by $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$. Indeed, the allocation $\tilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t}))_{i \in T_1 \cup T_2}$ is Pareto-optimal.

Finally, we determine $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$, and we show the allocation $(\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t}))$ corresponds to the competitive allocation. By substituting the value of $\tilde{t}$ given by (C15) into (C13)-(C14), and using the fact that $z^* \equiv \frac{\sum_{i \in T_1} x_i^*}{n} = \frac{\sum_{i \in T_2} y_i^*}{n}$, we have that:

$$\tilde{q}(\tilde{t}) = \tilde{b}(\tilde{t}) = \frac{n}{n+1}(1 - z^*). \tag{C18}$$

Then, we deduce:

$$(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t})) = \begin{cases} (z^*, 1 - z^*), \; i \in T_1; \\ (1 - z^*, z^*), \; i \in T_2. \end{cases} \tag{C19}$$

From (C19) and as, for each trader, we have $x^* + y^* = 1$, then, for each $i \in T_1$, the allocation is $(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t})) = (z^*, 1 - z^*) = (1 - q^*, q^*) = (x_i^*, y_i^*)$, while, for each $i \in T_2$, the allocation is $(\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t})) = (1 - z^*, z^*) = (q^*, 1 - q^*) = (x_i^*, y_i^*)$. But, then, as the allocation $\mathcal{A}^* = (x_i^*, y_i^*)_{i\in T_1\cup T_2}$ is Pareto-optimal, we have that:

$$\begin{cases} \forall i \in T_1 \ \frac{\frac{\partial u_1}{\partial x_i}}{\frac{\partial u_1}{\partial y_i}} \mid_{(z^*,1-z^*)} = \frac{\frac{\partial u_1}{\partial x_i}}{\frac{\partial u_1}{\partial y_i}} \mid_{(1-\tilde{t}-\tilde{q}(\tilde{t}),\frac{n+1}{n}\tilde{q}(\tilde{t}))} = 1 \\ \\ \forall i \in T_2 \ \frac{\frac{\partial u_2}{\partial x_i}}{\frac{\partial u_2}{\partial y_i}} \mid_{(1-z^*,z^*)} = \frac{\frac{\partial u_2}{\partial x_i}}{\frac{\partial u_2}{\partial y_i}} \mid_{(\frac{n+1}{n}\tilde{b}(\tilde{t}),1-\tilde{t}-\tilde{b}(\tilde{t}))} = 1. \end{cases} \tag{C18}$$

■

7.4. Appendix D: proof of Corollary 1

Assume the market is symmetric, with $\#T_1 = \#T_2 = n$ and $\alpha = \beta = 1$. Since the market is symmetric, we know from the outset (Theorem 1) that a uniform tax $t \in (0,1)$ is levied on the endowments of commodities X and Y before exchange takes place, generating a total tax revenue equal to nt units of each commodity. We have to show that the tax t is unique.

The tax t leads to the same strategy sets as the ones given by (9) and (10) with $\alpha = \beta = 1$. Furthermore, after exchange has taken place, each trader is assigned a share of the total tax revenue on the commodity she does not initially own that is proportional to the supply of the commodity she owns. Formally, after trade has occurred at an $2n$-tuple of strategies $(q_1, ..., q_n; b_{n+1}, ..., b_{2n}) \in [0, 1-t]^n \times [0, 1-t]^n$, a share $s(t) \equiv \frac{t}{1-t-y^*}$ of the total tax product in commodity X is transferred to each trader $i \in T_2$. Likewise, a share $s(t) \equiv \frac{t}{1-t-x^*}$, with $0 < s(t) < 1$, of the total tax product of commodity Y is transferred to each trader $i \in T_1$. For each $t \in (0,1)$, these transfers are feasible.

Let $(\tilde{q}_1, ..., \tilde{q}_n; \tilde{b}_{n+1}, ..., \tilde{b}_{2n}) \in [0, 1-t]^n \times [0, 1-t]^n$, $0 < t < 1$, be a nontrivial equilibrium for which $\sum_{i\in T_1} \tilde{q}_i(t) > 0$ and $\sum_{i\in T_2} \tilde{b}_i(t) > 0$. Given $0 < t < 1$, the $2n$ maximization problems of traders may be written:

$$\begin{cases} \max\limits_{q_i\in[0,1-t]} \pi_i(q_i, \mathbf{q}_{-i}; \mathbf{b}; t) = u_1\left(1 - t - q_i, \frac{\sum_{k\in T_2} b_k}{\sum_{k\in T_1} q_k} q_i + s(t)q_i\right), \ i \in T_1 \\ \\ \max\limits_{b_i\in[0,1-t]} \pi_i(\mathbf{q}; b_i, \mathbf{b}_{-i}; t) = u_2\left(\frac{\sum_{k\in T_1} q_k}{\sum_{k\in T_2} b_k} b_i + s(t)b_i, 1 - t - b_i\right), \ i \in T_2. \end{cases} \tag{D1}$$

As Assumptions 2-3 still hold, given $0 < t < 1$, there is a unique symmetric CNE, which is interior and type-symmetric. The solution $(\tilde{q}(t), \tilde{b}(t))$ is such that $\tilde{q}(t) = \tilde{b}(t)$, and is characterized by the system of $2n$ first-order sufficient conditions:

$$\begin{cases} -\frac{\partial u_1(1-q(t),b(t)+s(t)q(t))}{\partial x_i} + \left(\frac{n-1}{n} + s(t)\right) \frac{\partial u_1(1-q(t),b(t)+s(t)q(t))}{\partial y_i} = 0 \\ \\ -\frac{\partial u_2(q(t)+s(t)b(t),1-b(t))}{\partial y_i} + \left(\frac{n-1}{n} + s(t)\right) \frac{\partial u_2(q(t)+s(t)b(t),1-b(t))}{\partial x_i} = 0. \end{cases} \tag{D2}$$

But, if $\tilde{t}$, with $0 < \tilde{t} < 1$, is imposed on each trader i's endowment, and the share associated with the transfer to each trader is $\tilde{s} = s(\tilde{t})$, then the only solution to (D2) given by $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$ is such that:

$$\begin{cases} \forall i \in T_1 \quad \frac{\frac{\partial u_1}{\partial x_i}}{\frac{\partial u_1}{\partial y_i}} \Big|_{(1-\tilde{q}(\tilde{t}),\tilde{q}(\tilde{t})+s(\tilde{t})\tilde{q}(\tilde{t}))} = \frac{n(1+\tilde{s})-1}{n} \\ \\ \forall i \in T_2 \quad \frac{\frac{\partial u_2}{\partial x_i}}{\frac{\partial u_2}{\partial y_i}} \Big|_{(\tilde{b}(\tilde{t})+s(\tilde{t})\tilde{b}(\tilde{t}),1-\tilde{b}(\tilde{t}))} = \frac{n}{n(1+\tilde{s})-1}. \end{cases} \quad \text{(D3)}$$

Then, the marginal rates of substitution differ across all traders at the symmetric type-symmetric interior CNE given by $(\tilde{q}(\tilde{t}), \tilde{b}(\tilde{t}))$ if and only if $\tilde{s} \neq \frac{1}{n}$. Then, when $\tilde{s} \neq \frac{1}{n}$, the allocation $\widetilde{\mathcal{A}} = (\tilde{x}_i(\tilde{t}), \tilde{y}_i(\tilde{t}))_{i \in T_1 \cup T_2}$ is not Pareto-optimal.■

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