Characterization of Domains Admitting Strategy-Proof and Non-Dictatorial Social Choice Functions with Infinite Sets of Alternatives

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Abstract

We show, with an example, that the theorem on the characterization of the domains admitting strategy-proof and non-dictatorial social choice functions by Kalai and Muller (1977) does not hold when the set of alternatives is infinite. We consider two ways of overcoming this problem. The first identifies a set of domains admitting strategy-proof and non-dictatorial social choice functions when the set of alternatives is infinite. The second defines a class of social choice functions for which the theorem is true with both finite and infinite sets of alternatives.

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1 Introduction

Kalai and Muller (1977) contains the first published characterization of both the domains of preferences admitting non-dictatorial social welfare func-

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tions and those admitting strategy-proof and non-dictatorial social choice functions.[1] They showed that these domains coincide. However, these authors did not say anything about whether their theorems hold for any set of alternatives, finite or infinite.

On the contrary, Arrow (1963) explicitly addressed the issue concerning the cardinality of the set of alternatives when considering the domain restrictions which would have made it possible to circumvent his impossibility theorem. In particular, he redefined the notion of single-peaked preferences in order to cover "the general case of any number of alternatives" (see p. 76).

In the same vein, we reconsider here Kalai and Muller's characterization results to analyze if they hold for any number of alternatives. While we verify that the theorems concerning social welfare functions are true for both finite and infinite sets of alternatives, we provide an example showing that the theorem concerning social choice functions does not hold when the set of alternatives in infinite.

We consider two ways of overcoming this problem. The first identifies a set of domains admitting strategy-proof and non-dictatorial social choice functions when the set of alternatives is infinite. The second defines a class of social choice functions for which the theorem is true with both finite and infinite sets of alternatives.

2 Notation and definitions

Let I be any initial finite segment of the natural numbers with at least two elements and let $|I|$, the cardinality of I, be denoted by n. Elements of I are called individuals.

Let A be a set such that $|A| \geq 3$. Elements of A are called alternatives.

Let $\mathcal{A}$ be the set of all the non-empty subsets of A, called feasible sets.

Let $\mathcal{P}$ be the set of all the complete, transitive, and antisymmetric binary relations on A, called preference orderings.

Given a preference ordering $P \in \mathcal{P}$, let P^{-1} denote a preference ordering such that, for all $x, y \in A$, xPy if and only if $yP^{-1}x$.

Let Ω denote a nonempty subset of $\mathcal{P}$. Elements of Ω are called admissible preference orderings.

[1] Maskin (1976) proposed a similar characterization in his unpublished Ph.D. thesis.

Let $\overline{\Omega}$ denote the set of preference orderings such that, for each $P \in \overline{\Omega}$, there exists an alternative $x \in A$ such that xPy, for all $y \in A$.

Let $\underline{\Omega}$ denote the set of preference orderings such that, for each $P \in \underline{\Omega}$, there exists an alternative $x \in A$ such that yPx, for all $y \in A$.

Given a feasible set $X \in \mathcal{A}$, two admissible preference orderings $P, P' \in \Omega$ are said to agree on X whenever, for all $x, y \in X$, xPy if and only if $xP'y$.

Let Ω^n denote the n-fold cartesian product of Ω. Elements of Ω^n are called preference profiles.

Given $X \in \mathcal{A}$, two preference profiles $\mathbf{P}, \mathbf{P'} \in \Omega^n$ are said to agree on X if, for all $i \in I$, P_i and P'_i agree on X.

Given $\mathbf{P} \in \Omega^n$ and $P'_i \in \Omega$, $\mathbf{P} \setminus P'_i$ denotes the preference profile $(P_1, \ldots, P_{i-1}, P'_i, P_{i+1}, \ldots, P_n)$.

A social welfare function (SWF) on Ω is a function $w : \Omega^n \to \mathcal{P}$.

w is Pareto Optimal (PO) if, for all $\mathbf{P} \in \Omega^n$ and for all $x, y \in A$, xP_iy, for all $i \in I$, implies $xw(\mathbf{P})y$.

w is Independent of Irrelevant Alternatives (IIA) if, for all $\mathbf{P}, \mathbf{P'} \in \Omega^n$ and for all $x, y \in A$, $\mathbf{P}, \mathbf{P'}$ agree on $\{x, y\}$ implies $w(\mathbf{P})$ and $w(\mathbf{P'})$ agree on $\{x, y\}$.

w is dictatorial if there exists an individual $d \in I$ such that, for all $\mathbf{P} \in \Omega^n$ and for all $x, y \in A$, xP_dy implies $xw(\mathbf{P})y$. w is Non-Dictatorial (ND) if it is not dictatorial.

A social choice function (SCF) on Ω is a function $f : \Omega^n \times \mathcal{A} \to A$ such that, for all $\mathbf{P} \in \Omega^n$ and for all $X \in \mathcal{A}$, $f(\mathbf{P}, X) \in X$.[2]

f is Pareto Optimal (PO) if, for all $\mathbf{P} \in \Omega^n$, for all $X \in \mathcal{A}$, and for all $x, y \in X$, xP_iy, for all $i \in I$, implies $f(\mathbf{P}, X) \neq y$.

f is Independent of Non-Optimal Alternatives (INOA) if, for all $\mathbf{P} \in \Omega^n$ and for all $X, Y \in \mathcal{A}$ such that $Y \subseteq X$, $f(\mathbf{P}, X) \in Y$ implies $f(\mathbf{P}, X) = f(\mathbf{P}, Y)$.[3]

f is manipulable by an individual $i \in I$ at $\mathbf{P} \in \Omega^n$ and $X \in \mathcal{A}$ via $P'_i \in \Omega$ if $f(\mathbf{P} \setminus P'_i, X) P_i f(\mathbf{P}, X)$. f is manipulable if it is manipulable by

[2]As pointed out by Blin and Satterthwaite (1978), this definition of a SCF, due to Karni and Schmeidler (1976), is different from the definition proposed by Gibbard (1973) and Satterthwaite (1975), according to which the only argument of a SCF is represented by preference profiles.

[3]The notion of a INOA social choice function was introduced by Karni and Schmeidler (1976).

some individual $i \in I$ at some $\mathbf{P} \in \Omega^n$ and $X \in \mathcal{A}$ via some $P'_i \in \Omega$. f is Strategy-Proof (SP) if it is not manipulable.

f is dictatorial if there exists an individual d such that, for all $\mathbf{P} \in \Omega^n$ and $X \in \mathcal{A}$, $f(\mathbf{P}, X)P_d y$, for all $y \in X$. f is Non-Dictatorial (ND) if it is not dictatorial.

A SWF on Ω, w, is said to underlie a SCF on Ω, f, if, for all $\mathbf{P} \in \Omega^n$ and for all $X \in \mathcal{A}$, $f(\mathbf{P}, X)w(\mathbf{P})y$, for all $y \in X$.

A SCF on Ω, f, is Rational (R) if there exists a SWF on Ω, w, which underlies it.

Finally, consider the following set of definitions, crucial for Kalai and Muller's characterization results (see p. 462).

Let $T = \{(x, y) \in A \times A : x \neq y\}$, $TR = \{(x, y) \in T :$ there exist no $P, P' \in \Omega$ such that xPy and $yP'x\}$, and $NTR = T \setminus TR$.

A set $S \subseteq T$ is closed under decisiveness implication if, for every two pairs $(x, y), (x, z) \in NTR$, the following two conditions are satisfied.

DI1. If there exist $P, P' \in \Omega$ with $xPyPz$ and $yP'zP'x$, then

DI1a. $(x, y) \in S$ implies that $(x, z) \in S$,

DI1b. $(z, x) \in S$ implies that $(y, x) \in S$.

DI2. If there exists $P \in \Omega$ with $xPyPz$, then

DI2a. $(x, y) \in S$ and $(y, z) \in S$ imply that $(x, z) \in S$,

DI2b. $(z, x) \in S$ implies that $(y, x) \in S$ or $(z, y) \in S$.

Ω is decomposable if there exists a set S such that $TR \subset S \subset T$[4], which is closed under decisiveness implication.

3 Characterization theorems with infinite sets of alternatives

Kalai and Muller, in their Theorems 1 and 2, dealt with the characterization of the domains admitting SWF which are PO, IIA, and ND, while in their Theorem 3 they provided a characterization of the domains admitting SCF which are PO, INOA, SP, and ND.

In both cases, these authors did not specify for which sets of alternatives their results are true. As regards the theorems on SWF, it can be easily verified that the arguments on which their proofs rely hold both for finite

[4]The symbol $\subset$ denotes strict set inclusion.

and infinite sets of alternatives. We do not repropose here those arguments, but simply restate Kalai and Muller's theorems making this point explicit.

Theorem 1. *Let $|A| \leq \infty$. There exists a SWF on Ω, w', which is PO, IIA, and ND for $n = 2$, if and only if there exists a SWF on Ω, w'', which is PO, IIA, and ND for $n > 2$.*

Theorem 2. *Let $|A| \leq \infty$. Ω is decomposable if and only if there exists a SWF on Ω, w, which is PO, IIA, and ND for $n \geq 2$.*

Let us consider now Kalai and Muller's Theorem 3 which concerns the characterization of the domains admitting SCF which are PO, INOA, SP, and ND. It establishes that these domains are the same allowing for the existence of SWF which are PO, IIA, and ND. Nonetheless, the question related to the cardinality of the set of alternatives is, in this case, more subtle. The following example shows that, when the set of alternative is infinite, even if Ω is decomposable and, consequently, there exists a SWF which is PO, IIA, and ND, there may not exist any SCF which is PO, INOA, SP, and ND.

Example 1. *Let $|A| = \infty$ and $\Omega = \{P, P'\}$, where $P \notin \overline{\Omega}$ and $P' \neq P$. Then, Ω is decomposable and there exists no SCF on Ω, f, which is PO, INOA, SP, and ND for $n \geq 2$.*

Proof. We first show that there exists a SWF on Ω, w, which is PO, IIA, and ND for $n = 2$. Let $w : \Omega^2 \to \mathcal{P}$ be a function such that $w(P, P) = P$, $w(P, P') = P$, $w(P', P) = P$, and $w(P', P') = P'$. It is immediate to verify that w is a SWF on Ω which is PO, IIA, and ND for $n = 2$. Then, Theorem 2 implies that Ω is decomposable. Suppose that there exists a SCF on Ω, f, which is PO, INOA, SP and ND for $n \geq 2$. Let $\mathbf{P}$ be a preference profile such that $P_i = P$, for all $i \in I$ and let $f(\mathbf{P}, A) = x$. Consider an alternative $y \in A$ such that yP_ix, for all $i \in I$. Then, $f(\mathbf{P}, A) \neq x$, as f is PO, a contradiction. This implies that there exists no SCF on Ω, f, which is PO, INOA, SP, and ND for $n \geq 2$. ■

The argument used in the proof of Example 1 can be used to prove the following proposition.

Proposition 1. *Let $|A| = \infty$. If there exists a SCF on Ω, f, which is PO, INOA, SP, and ND for $n \geq 2$, then $\Omega \subset \overline{\Omega}$.*

Example 1 and Proposition 1 follow from the requirement that f is PO.

We consider now the implication of the requirement that f is INOA. The following proposition shows that, for $|A| \leq \infty$, INOA and R are equivalent.

Proposition 2. *Let $|A| \leq \infty$. A SCF on Ω, f, is INOA for $n \geq 2$ if and only if it is R for $n \geq 2$.*

Proof. Suppose that f is a SCF on Ω which is INOA for $n \geq 2$. Let $w : \Omega^n \rightarrow \mathcal{P}$ be a function such that, for all $\mathbf{P} \in \Omega^n$ and for all $x, y \in A$, $xw(\mathbf{P})y$ if and only if $f(\mathbf{P}, \{x, y\}) = x$. We already know from Karni and Schmeidler (1976) (see Proposition 2, p. 490), that w is a SWF on Ω. Suppose now that w does not underly f. Then, there exist $\mathbf{P} \in \Omega^n$, $X \in \mathcal{A}$, and $y \in X$ such that $yw(\mathbf{P})x$, where $x = f(\mathbf{P}, X)$. But, since f is INOA, we have $f(\mathbf{P}, \{x, y\}) = x$, a contradiction. This implies that f is R for $n \geq 2$. Suppose that f is a SCF on Ω which is R for $n \geq 2$. Then, there exists a SWF on Ω, w, which underlies it. Suppose that f is not INOA. Then, there exist a preference profile $\mathbf{P} \in \Omega$ and two feasible sets $X, Y \in \mathcal{A}$ such that $Y \subseteq X$, $f(\mathbf{P}, X) \in Y$ and $f(\mathbf{P}, X) \neq f(\mathbf{P}, Y)$. Then, $f(\mathbf{P}, X), f(\mathbf{P}, Y) \in X$, and $f(\mathbf{P}, Y)w(\mathbf{P})f(\mathbf{P}, X)$, a contradiction. This implies that f is INOA for $n \geq 2$. ■

This proposition implies the following corollary.

Corollary. *Let $|A| = \infty$. If there exists a SCF on Ω, f, which is INOA for $n \geq 2$, then there exists a SWF on Ω, w, such that $w(\mathbf{P}) \in \overline{\Omega}$, for all $\mathbf{P} \in \Omega^n$.*

Proof. Suppose that there exists a SCF on Ω, f, which is INOA for $n \geq 2$. Then, there exists a SWF on Ω, w, which underlies f as, by Proposition 2, f is R. Suppose that there exists a preference profile $\mathbf{P} \in \Omega^n$ such that $w(\mathbf{P}) \notin \overline{\Omega}$. Then, there exists an alternative $y \in A$ such that $yw(\mathbf{P})f(\mathbf{P}, A)$, a contradiction. ■

Proposition 1 and the Corollary to Proposition 2 show that, when $|A| = \infty$, $\Omega \in \overline{\Omega}$ and the existence of a SWF, w, such that $w(\mathbf{P}) \in \overline{\Omega}$, for all $\mathbf{P} \in \Omega^n$, are necessary conditions for the existence of a SCF on Ω, f, which is PO, INOA, SP, and ND for $n \geq 2$. Hence, they show that Kalai and Muller's Theorems 2 and 3 are, in a sense, asymmetric as the former holds when $|A| \leq \infty$ whereas the last holds only when $|A| < \infty$. It can therefore be stated as follows.

Theorem 3. *Let $|A| < \infty$. Ω is decomposable if and only if there exists a SCF on Ω, f, which is PO, INOA, SP, and ND for $n \geq 2$.*

In what follows, we consider two ways of extending Theorem 3 to the case where $|A| = \infty$. The first identifies a set of domains admitting SCF which are PO, INOA, SP, and ND when $|A| = \infty$. The second proposes a new definition of a SCF which allows us to state Kalai and Muller's Theorem 3 when $|A| \leq \infty$, restoring the symmetry between the theorems on SWF and SCF.

Let us first show that all the domains belonging to a subset of the set of single-peaked domains as defined by Black (1948) admit SCF which are PO, INOA, SP, and ND when $|A| = \infty$. To this end, we first provide the following definitions.

Given a preference ordering $Q \in \mathcal{P}$, a preference ordering $P \in \mathcal{P}$ is said to be single-peaked à la Black (see Black (1948)) relative to Q if there is an alternative $x \in A$ such that xPy, for all $y \in A$, and, for all alternatives $y, z \in A$, $xQyQz$ implies yPz, and, $zQyQx$ implies yPz.

Given a preference ordering $Q \in \mathcal{P}$, a preference ordering $P \in \mathcal{P}$ is said to be single-peaked à la Arrow (see Arrow (1963)) relative to Q if, for all alternatives $x, y, z \in A$, $xQyQz$ and xPy implies yPz, and, $zQyQx$ and xPy implies yPz.

The following proposition says that the set of the single-peaked preferences à la Black relative to a preference ordering is a subset of the set of the single-peaked preferences à la Arrow relative to the same preference ordering.

Proposition 3. *Given a preference ordering $Q \in \mathcal{P}$, if a preference ordering P is single-peaked à la Black relative to Q, then it is single-peaked à la Arrow relative to Q.*

Proof. Given a preference ordering $Q \in \mathcal{P}$, let P be a preference ordering which is single-peaked à la Black with respect to Q and let v be the alternative such that vPy, for all $y \in A$. Moreover, let $x, y, z \in A$ be three alternatives such that $xQyQz$. Consider the following cases. (i) if $x = v$, then xPy implies yPz as $vQyQz$ implies yPz; (ii) if $y = v$, then xPy cannot occur as vPx; (iii) if $v = z$, then xPy cannot occur as $xQyQv$ implies yPx; (iv) if $vQxQyQz$, then xPy implies yPz as $vQyQz$ implies yPz, (v) if $xQyQzQv$, then xPy cannot occur as $xQyQv$ implies yPx. Let now $x, y, z \in A$ be three alternatives such that $zQyQx$. Then, xPy implies yPz by considering, *mutatis mutandis*, the same cases above. ■

The converse of this proposition does not hold as, when $Q \notin \overline{\Omega}$, Q is a preference profile which is single-peaked à la Arrow but not à la Black with respect to itself. The same holds for Q^{-1} when $Q \notin \underline{\Omega}$. The following example shows that, when $|A| = \infty$, the converse of Proposition 3 does not hold even in the case where $Q \in \overline{\Omega} \cap \underline{\Omega}$.

Example 2. *Let A be the closed interval on the real line $[0, 1]$. Let $Q \in \mathcal{P}$ be a preference ordering such that if $x, y \in [0, 1]$ and $x > y$, then xQy. Then, there exists a preference ordering P which is single-peaked à la Arrow with respect to Q and is not single-peaked à la Black with respect to Q.*

Proof. Let P be a preference ordering such that if $x, y \in [0, \frac{1}{2})$ and $x > y$, then xPy; if $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then xPy; if $x, y \in [\frac{1}{2}, 1]$ and $x > y$, then yPx. Then P is single-peaked à la Arrow with respect to Q but it is not single-peaked à la Black with respect to Q as there is no alternative $x \in A$ such that xPy, for all $y \in A$. ■

Kalai and Muller, in their Example 2, proved that the set of preference profiles which are single-peaked à la Arrow relative to a preference profile Q is decomposable and therefore admits SWF which are PO, IIA, and ND, by their Theorem 2. Then, they also claimed (see p. 468) that, as a consequence of their Theorem 3, the same set of preferences admits SCF which are PO, INOA, SP, and ND. Example 2 shows that, when $|A| = \infty$, this claim does not hold even in the case where $Q \in \overline{\Omega} \cap \underline{\Omega}$.

The following proposition restores the validity of Kalai and Muller's claim when $|A| = \infty$, restricting their Example 2 to the set of domains which consists of preference orderings which are single-peaked à la Black with respect to a preference ordering $Q \in \overline{\Omega} \cap \underline{\Omega}$.

Proposition 4. *Let $|A| = \infty$. Let Ω_Q be the set of all preference orderings à la Black relative to a preference ordering $Q \in \overline{\Omega} \cap \underline{\Omega}$. Then, there exists a SCF, f, on Ω_Q which is PO, INOA, SP, and ND for $n \geq 2$.*

Proof. Let Ω_Q be the set of all preference orderings à la Black relative to a preference ordering $Q \in \overline{\Omega} \cap \underline{\Omega}$. We first show that Ω_Q is decomposable, following an argument used by Kalai and Muller in the proof of their Example 2. Let $S_1 = \{(x, y) \in T : xQy\}$. Then $TR = \emptyset$ as $Q, Q^{-1} \in \Omega_Q$. Moreover, it follows immediately from the definition of S_1 that $TR \subset S_1 \subset T$. Consider two pairs $(x, y), (x, z) \in NTR$. Suppose that $(x, y) \in S_1$ and that there exists $P, P' \in \Omega_Q$ with $xPyPz$ and $yP'zP'x$. Then, $zQxQy$ cannot occur as $zP'x$, $yP'x$, and P' is single-peaked à la Arrow, by Proposition 2, and, $xQzQy$ cannot occur as xPz, yPz, and P is single-peaked à la Arrow, by

Proposition 2. But then, $(x,z) \in S_1$ as $xQyQz$. Hence, S_1 satisfies DI1a. Suppose that $(z,x) \in S_1$ and that there exists $P, P' \in \Omega_Q$ with $xPyPz$ and $yP'zP'x$. Using, *mutatis mutandis*, the same argument above, $(y,x) \in S_1$ as $zQyQx$. Hence, S_1 satisfies DI1b. Suppose that $(x,y) \in S_1$, $(y,z) \in S_1$, and that there exists $P \in \Omega_Q$ with $xPyPz$. Then, $(x,z) \in S_1$ as $xQyQz$. Hence, S_1 satisfies DI2a. Suppose that $(z,x) \in S_1$ and that there exists $P \in \Omega_Q$ with $xPyPz$. Then, $yQzQx$ cannot occur as yPz, xPz, and P is single-peaked à la Arrow, by Proposition 2. Then, $(z,y) \in S_1$ as either $zQyQx$ or $zQxQy$. Hence, S_1 satisfies DI2b. This completes the proof that Ω_Q is decomposable as we have shown that S_1 is closed under decisiveness implication. We show now that there exists a SWF on Ω_Q, w, which is PO, IIA, and ND for $n = 2$, and which is such that $w(\mathbf{P}) \in \overline{\Omega}$, for all $\mathbf{P} \in \Omega_Q^2$. Let $S_2 = \{(x, y \in T : (y,x) \notin S_1\}$. Then, $S_1 = S_2$. Let $w : \Omega_Q^2 \to \mathcal{P}$ be a function such that, for all $\mathbf{P} \in \Omega_Q^2$ and for all $x, y \in A$, $xw(\mathbf{P})y$ if xP_iy, for $i \in I$, or, xP_1y and $(x,y) \in S_1$, or, xP_2y and $(x,y) \in S_2$. Then, w is a SWF on Ω_Q which is PO, IIA, and ND for $n = 2$, by the argument used by Kalai and Muller in the proof of their Theorem 2. It remains to show that $w(\mathbf{P}) \in \overline{\Omega}$, for all $\mathbf{P} \in \Omega_Q^2$. Given a preference profile $\mathbf{P} \in \Omega_Q^2$, let x and y be the alternatives such that xP_1v, for all $v \in A$, and yP_2v, for all $v \in A$. Suppose there exists an alternative z such that $zw(\mathbf{P})x$ and $zw(\mathbf{P})y$. Consider the following cases. (i) If $P_1 = P_2$, then $x = y$, zP_1x, and zP_2x, a contradiction; (ii) if $P_1 \neq P_2$, $Q = P_1$, and $xw(\mathbf{P})y$, then zP_2x and zQx, a contradiction; (iii) if $P_1 \neq P_2$, $Q = P_2$, and $xw(\mathbf{P})y$, then zQy, a contradiction; (iv) if $P_1 \neq P_2$, $Q \neq P_1$, $Q \neq P_2$, and $xw(\mathbf{P})y$, then yP_2zP_2x and $zQxQy$, a contradiction. Therefore, there exists no alternative $z \in A$ such that $zw(\mathbf{P})x$ and $zw(\mathbf{P})y$. Hence $w(\mathbf{P}) \in \overline{\Omega}$, for all $\mathbf{P} \in \Omega_Q^2$. We complete now the proof following the argument proposed by Kalai and Muller in their proof of Theorem 3. Given a preference profile $\mathbf{P}$ and a feasible set $X \in \mathcal{A}$, let f' be a SCF such that $f'(\mathbf{P}, X)w(\mathbf{P})y$, for all $y \in X$. Then, by the argument used by Kalai and Muller, f' is a SCF on Ω_Q which is PO, INOA, SP, and ND for $n = 2$. Finally, let $f'' : \Omega_Q^n \times \mathcal{A} \to A$ be a function such that, for each $\mathbf{P} \in \Omega_Q^n$ and for each $X \in \mathcal{A}$, $f''(P_1, \ldots, P_n, X) = f'(P_1, P_2, X)$. It is straightforward to verify that f'' is a SCF on Ω_Q which is PO, INOA, SP, and ND for $n > 2$. Then, there exists a SCF on Ω_Q which is PO, INOA, SP, and ND for $n \geq 2$. ■

We consider now a second way of restoring the validity of Kalai and

Muller's Theorem 3 when $|A| = \infty$ which maintains the symmetry between the theorems on SWF and SCF. To this end, we propose the following new definition of a SCF.

Let $\mathcal{F}$ denote the set of all the nonempty and finite subsets of A. A SCF on Ω is a function $f^* : \Omega^n \times \mathcal{F} \rightarrow A$ such that, for all $\mathbf{P} \in \Omega$ and for all $X \in \mathcal{F}$, $f^*(\mathbf{P}, X) \in X$.

f^* will inherit, *mutatis mutandis* - i.e., when $\mathcal{A}$ is replaced by $\mathcal{F}$ - all the properties previously introduced on f. Therefore, it is straightforward to verify that all the arguments used in the proof of Kalai and Muller's Theorem 3 for $|A| < \infty$ hold for f^* also when $|A| = \infty$.

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