Cold War, Dynamic Programming, and the Science of Economizing: Bellman Strikes Gold in Policy Space[1]

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> General facts, or, if you please, the general laws which facts follow, are styled *principles,* whenever it relates to their application; that is to say, the moment we avail ourselves of them in order to ascertain the rule of action of any combination of circumstances presented to us.
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> Jean-Baptiste Say, [1803] 1880, 13,

> Programming, of course, means *allocation*.
>
> Stuart Dreyfus 1956, 3

> The theory of mathematical programming is largely the theory of duality…. Duality has two faces. One the one hand, it has contributed significantly to computational methods…. On the other hand, duality has contributed almost all of the significant economic interpretations.
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> H. W. Kuhn 1968, 50

How Applied Mathematics became a Science of Economizing

The title of this introductory section plays on the title of E. Roy Weintraub's book on *How Economics became a Mathematical Science*. Weintraub (2002), Philip Mirowski (2002), Robert Leonard (forthcoming), Esther Mirjam Sent (1997), and other authors in John Davis's (1997) and Mary Morgan and Malcolm Rutherford's (1998) edited volumes, have emphasized the economist's appropriation and adaptation of

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mathematical tools of expression and problem solving in the mid-twentieth century. [2] My investigation is the dual to their quest to answer why and how economics shed its prewar pluralism and became so dependent on mathematical analysis after World War II.[3] This chapter from my forthcoming book, *Protocols of War: The Mathematical Nexus of Economics, Statistics, and Control Engineering,* scrutinizes the history of dynamic programming from 1950 to 1960 to understand how wartime exigencies forced not only economists but also mathematicians and engineers to incorporate a science of economizing.

In his essay on "Economic Reasoning and Military Science", Nobel-laureate Thomas Schelling (1960, 4) asserted that during World War II and the cold war, the US military employed researchers, such as himself, to practice the "science of economizing." Although World War II was portrayed as a conflict between Allied capitalism and the state corporatism and the cold war was portrayed as a war between the free-market USA versus the centrally-planned Soviet Union, both wars forced the US government to embrace planning within a competitive structure of war. This wartime need to plan, innovate, and command an efficient allocation of resources in a rivalry with a competitor nation engendered remarkably similar developments in applied mathematics between, for example, the USA and the Soviet Union during the

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[2] Roger Backhouse (1998, 93) notes that the use of algebra in theoretical articles in the *American Economic Review* rose from only 25%in 1940 to 80% in 1960. Backhouse (1998, 105) also points out that the dominant orthodoxy of "the use of formal optimizing methods that are tested using econometric techniques" was not achieved until after 1960. From the late 1940s to the 1960s, optimization tools such as dynamic programming were being forged in military decision science.

[3] During the Napoleonic wars, Jean Victor Poncelet , a French military engineer languishing in a Russian prison imagined tangential lines constructing a dual approach to solving geometric problems. After returning to Paris, Poncelet (1822) published his prisoner-of-war notes as *Traité des propriétés projectives des figures*. In Poncelet's projective geometry, dual theorems are theorems that remain valid if the words "point" and "line" are interchanged in reference to a plane: the line on the point is the dual to the point on the line.

1950s and 1960s. Indeed an American National Research Council report (1956, 2) credited a revolutionary change in applied mathematics in the USA to "the growing power, endurance, and precision of the systems which are being engineered under the pressure of industrial competition and military rivalry." A glance at the differences between the effects of industrial competition and military rivalry (highlighted in Table 1) will help us understand the wartime context that transformed applied mathematics into a science of economizing.

Table 1 Comparison of Competition Types

	Competition in **Markets**	Competition in **Warfare**
Between	Firms Similar products in same industry often same country	Nations Different products in different industries in different countries
Metaphors	Grab for market share Invisible Hand	Teeter-totter of offense and defense Military--Industrial Complex
Change in Products	Rapid "upgrading"	Rapid obsolescence
Maximization of	Profit	Military worth
Top efficiency in	Average costs	Time
Government Stance on Firms Cooperating	Discourages	Encourages
Effects	Lower average costs	Rapid deskilling Rapid technological Innovation
Economist	Explains how market achieved optimality	Prescribes how to optimize
Returns to scale	Constant or increasing	Increasing

In market competition one company's automobile competes against another company's automobile, during war one country's tanks compete against another country's tanks, but there is also competition between one nation's fighter plane and another nation's bomber. This competition between offensive weapons and defensive weapons in the pursuit of superiority led to minimizing time – time to design, time to produce, time to compute lead firing angles, time to get projectile to enemy target - as the top priority. Indeed minimization of time took priority over minimization of monetary cost, and it fed the wartime fire of rapid technological innovation:

> Whatever is used, the point must be made that it is not sufficient to develop an answer to the immediate problem as it must be assumed that the enemy is constantly improving, and we must therefore continuously develop to maintain superiority. Superiority in time is the important objective. We therefore regard ourselves as primarily a manufacturing laboratory whose responsibility it is to find the problem, develop the solution, prove it, design it for production and accomplish its initial production, all in a minimum of time and all in the expectation that it will be changed while in the course of production. Time is the essence of our problem; me must maintain superiority; the enemy is striving to beat us and will unless our momentum of development is greater than his. (Gillmor memo 1943, 3-4)

So where did World War II and cold war competition with its intranational firm cooperation, offence-versus-defense technological spur, and the emphasis on time efficiency take applied mathematics from 1940 to 1960s? The more the military could rely on mathematics to design mechanisms and operations, the more they could save on time and scarce resources. Control engineering is a discipline of designing. Even before their mobilization day (M-day) in World War II, the American military forces were encouraging the use of models for design and testing and by the 1960's the cutting edge theories in control engineering were theories about equation system models rather than theories about physical phenomena.[4]

Faced with a situation of scarcity but no markets for setting prices, the military relied on economists and mathematicians to model and solve for the efficient allocations of resources. Mathematicians and statisticians made major leaps in the mathematics of decision making when asked to improve ordnance quality (see chapter 3) and control of inventories (see chapter 4). US military sponsorship was responsible for a considerable quantity of the published research in the 1950s and 1960s on new techniques of management science and normative microeconomics.

In his *American Economic Review* essay, Paul Homan (1946, 870) asserted that the analytical traits developed in economists' training were particularly suited to the state mobilization for World War II: "In this respect, the way opened for economists in wartime service was, in degree, quite beyond that for persons brought up in any other academic discipline except, perhaps, for physical scientists in various fields of research, especially that of atomic research." Homan argued that the wartime

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[4] For example in *Topics in Mathematical Systems Theory*, "To put it more bluntly, control theory does not deal with the real world, but only with mathematical models of certain aspects of the real world; therefore the tools as well as results of control theory are mathematical. There is a close analogy between this situation and the evolution of probability theory into a strictly mathematical discipline." (Kalman et. al. 1969, 27)

experience insured "a continuing large place for economists in the public service" (Homan 1946, 870). The US military encouraged an economic way of thinking not just in economists but also in mathematicians and engineers. In the 1940s and 1950s "economic theory for a non-market economy", as RAND economist Charles Hitch (1958, 206) described it, was commonly practiced in the interdisciplinary research groups for the US armed forces.

A key objective of this paper, which is a draft of chapter 5 of *Protocols of War*, is to show how the US military needs steered the mathematician Richard Bellman to model allocation and control processes in a form consistent with an economic way of thinking.[5] Wartime planning took money and markets out of the equations and transferred the priority of modeling allocation systems from a positive-economic description of "what is" to a normative, welfare-economic statement of "what ought to be." As his functional equation theory of dynamic programming matured, Bellman and his co-authors at the RAND Corporation increasingly appropriated the vocabulary of economics and an economic way of thinking as they suggested to the US military how they should allocate atomic bombs in nuclear warfare, manage their vast inventories of spare parts, and control guided missiles. Bellman's modeling process in turn lent itself to agile appropriation by micro and macro economists.

[5] Richard Bellman was born in Brooklyn in 1920 and died in 1984. During World War II, Bellman worked in the Theoretical Physics Division at Los Alamos. He had training in electronics and earned his Ph.D. in mathematics at Princeton in 1946. Bellman worked at the RAND Corporation in the summers of 1948-1951 and full-time from 1952-1965. In 1965 he took up an appointment as Professor of Mathematics, Electrical Engineering and Medicine at the University of Southern California. In the year of his death, he published an autobiography, *Eye of the Hurricane* (1984). Richard Roth (1986) published a collection of Bellman's key works, having to select from among the 620 research papers that Bellman produced in addition to his 40 books and seven monographs.

During the early 1950s, Bellman generalized the functional equation approach of recursive optimization that Pierre Massé (1944) and Kenneth Arrow, Ted Harris, and Jacob Marschak (1951) had independently presented to their wartime clients for multistage economical control of stocks and flows (see chapter 4). Both studies illustrated Alfred Marshall's observation that dynamic, rather than static, economic analysis was often needed to manage stocks because "the exhaustion of the stock will disrupt the statical rest." (Marshall 1898, 46). During the brief war with Germany from September 1939 to June 1940 and under the Vichy regime, Massé, a French hydroelectric engineer stationed in the Pyrenees, faced the problem of determining how much water should be taken from the reservoirs each month in order to minimize the current and future use of coal - the extremely scarce alternative source of electrical power. The challenges to determining the optimal flow of water included an uncertain future inflow of water from precipitation, uncertain future final demand for electrical power, and the fact that one's optimal decision this month was really conditional upon the decisions of the manager in future months. Massé's approached what he called the "jeu des réservoirs"(the operation or game of the reservoirs) through a recursive search for and application of a rule of action: increase the flow of water from the reservoirs to the turbines if the marginal utility of the flow from the reservoir is greater than the marginal expectation of the water kept in the reservoir; decrease the flow of water from the reservoirs to the turbines if the marginal utility of the flow from the reservoir is less than the marginal expectation of the water kept in the reservoir. It was in meeting the challenge of determining the marginal expectation of the stock of the reservoir that Massé took from his *polytechnicien'*s tool bag the

Markov chains of Maurice Fréchet, the log-normal distributions identified by Paul Levy and Robert Gibrat, the stochastic gaming of Emile Borel, and the recursive approximation techniques of Blaise Pascal.

The US Office of Naval Research funded Arrow, Harris, and Marschak's study into how to determine the optimal flow into the stock of the inventory of spare parts for mechanisms of the cold war. Their starting point was the rule of action of a two-bin inventory policy (*S, s*) that the military was already using. Arrow, Harris and Marschak modeled the optimal maximum stock (*S*) and best reordering point (*s*) as functions of the distribution of the random variable of demand, the cost of reordering, and the penalty for a shortage.

The features common to both the first French and American articulations of recursive optimization was a wartime need for normative rules of action and a functional equation spelling out an economic criterion in the face of a multistage decision process that included the dynamic element of unused stock being used in the next stage, the stochastic element of uncertain demand, and costs for deviation from an optimal policy. Bellman embraced these features and generalized the functional equation approach for many types of multistage decision processes.

Bellman first articulated his mathematical approach to planning for the allocation of scarce resources in the early 1950s when the RAND Corporation, a US Air Force (USAF) think factory, employed him. As we saw in chapter three on sequential analysis during WWII, when the wartime government was the client, statisticians, economists and engineers tailored their mathematical protocols to yield numerical solutions that made efficient use of scare computing resources. This institutional context gave rise to similar features in the Statistical Research Group's (SRG) sequential analysis and the RAND Corporation's dynamic programming:

- Economic criterion - minimization of costs (losses, time) or maximization of utility (military worth, damage to the enemy)
- Assumption of uncertainty
- Mathematically-guided decision-making process in discrete steps
- Decision-making protocol coded as an algorithm
- Solution to the mathematical problem was a policy that dictated the optimal action to take

This chapter will highlight these qualities, but also two features that were new to the cold war applied mathematics: the characteristics of the early generations of digital computers dictated computational feasibility which in turn molded theoretical formulation, and mathematicians had more license to work on basic research of abstract problems. The SRG's protocol of sequential analysis had been algorithmic enough to be encoded into a sophisticated adding machine in which the quality control inspector added new information with each additional item sampled. Mathematicians, however, determined optimal parameters before constructing the operational charts, tables or setting up the gears on the machine. The machine for sequential analysis was essentially no different than a table designed for a specific inspection. In the latter half of the 1950s and 1960s, mathematicians at the RAND Corporation and the Research Institute of Advanced Study (RIAS) made full use of digital computers. The protocols they designed were recursive algorithms that achieved solutions (policies, strategies) through numerical approximation. The computer was often the optimizer and an integral part of design processes.

As we saw in the previous chapter, the members of the World War II Applied Mathematics Panel worked on specific projects - for example, determining slewing routines for a particular gun sight, testing the strength of new alloys, or minimizing sample size in destructive testing of ordnance quality. The scope of the military research projects funded during the cold war was often much broader – for example, the science of production planning and multistage decision-making or the design of new weapon systems. This larger scope was related to the fact that a cold war allowed for a longer time horizon for projects to come to fruition than a hot war. George Dantzig (2007), for example, noted that it was eight years before the Air Force could make full use of his suggested linear programming/simplex method for mechanizing the planning process.

Dynamic programming is a modeling mindset. It disciplines the users to properly frame the criterion equation so as to achieve computationally-feasible solutions. This protocol for modeling problems that occur in a sequence of time periods or stages includes an equation that forces an imperative mood –a normative, ought-to-be, approach: your total return as a function of the state of your resources should be the maximum gained after N stages, and that maximum is equal to the return that results from your decision on allocating resources in the first stage plus the maximum gains you earned in the rest of the *N-1* stages given that in the first stage you allocated the resources so as to maximize your return in the remaining stages. In one of his earliest descriptions of his yet-to-be named protocol Bellman explained:

> We wish to discuss a general class of multi-stage problems involving a sequence of operations between each of which information is acquired which can be used to guide the subsequent operations. This class of problems is characterized by the fact that at each time the problem may be described by a set of parameters which change from operation to operation, which is to say that each operation performs a mapping of the parameter space upon itself, and secondly, that the purpose of the operations is to optimize according to a criterion which has the important property that after any initial number of operations, starting from the state one finds oneself in, one optimizes according to the same criterion.
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> The importance of this last point is two-fold. It allows a description of the optimum sequences of operations to be given very simply in terms of best first moves, and it allows a mathematical formulation by means of recurrence relations which are very useful both theoretically and computationally. (Bellman 1951a, 1)

In dynamic programming there is a duality between the solving the criterion functional equation (determining the value of the maximum returns or minimum total costs) and solving for the optimal policy that would result in satisfying the criterion equation. The user can choose which one to solve for and derive the other from that solution. This license to broaden the scope of scientific and mathematical research while working for a client enabled Bellman to identify and fully explore *policy space*. The notion of a well-defined workspace – such as Euclidean space, non-Euclidean space, Hilbert space, or state space - permitted mathematicians to distinguish different geometries, different dimensions, or different functional specifications and solution processes. The USAF wanted rules of action; they preferred solutions in the form of a policy –a sequence of decisions for each stage. Bellman realized that solving for the optimal policy that would maximize military worth or minimize military costs was often more feasible than solving the equation for the criterion function itself. He also demonstrated that whatever the first move is, the subsequent moves would have to be optimal. This meant that often one could concentrate on just solving for the rule of action on the first move (Bellman 1951b, 3). His method of solving problems by approximations in what he called "policy space" not only made good use of digital computers it also opened the door to solving problems in the calculus of variations and control engineering that mathematicians had been unable to solve with classical analytical techniques.

As we will see, in the early 1950s the context for Bellman's first formulation of a dynamic programming protocol was the uniqueness of nuclear warfare, including decision-making with regard to strategic bombing and allocation of scarce fissionable material for atomic bombs. In the latter half of the 1950s, the major application of Bellman's marriage of dynamic programming and control engineering was determining and maintaining optimal trajectories – particularly those formulated for a race of the two superpowers –the USA and the USSR –in outer space. Although these applications were the fruit of Air Force seed money, Bellman and his colleagues also had the resources and the inclination to take their algorithms beyond specific applications and explore abstract and philosophical qualities of their new mathematics. The solution to one planning problem was a policy, but Bellman also asked how sensitive was that policy to small changes in initial conditions? What was the structure of policy solutions obtained from different parameters? So, for example, even if computational feasibility dictated solving problems by discrete approximation, Bellman used traditional analysis of continuous functions to explore the structure of solutions. At some conjunctures, the model became more important than the physical process modeled. By the mid-1960s Bellman, Rudolf Kalman, and others working under contract from the US Air Force were advertising their control theory or systems theory as a *theory of models* of certain types of processes - not a theory of the physical processes that were modeled. To be more specific it was a theory of models for designing systems. So what we observe in this story is the development of a new mathematical discipline that carried on some of the threads of a seemingly pure older mathematics (e.g. the calculus of variations) but which developed through continual referencing to the modeling and computational needs of physical processes and the economizing needs of the US military. It is only within this cold-war funded feedback process between abstract and real that economists, engineers and mathematicians crafted recursive optimization for modeling multistage decision processes.

We will begin this chapter by looking at the programming imperative of the US military that led Bellman to solve mathematical problems of allocating scarce resources by approximating policy solutions stage by stage. Table 2 outlines the chronological, thematic structure we will use to understand the development of Bellman’s functional equation approach. We follow the course of one model in particular –the “gold-mining” equation to see what Bellman learned from adapting

Table 2. Developments in Bellman's Application of the Functional Equation Theory of Dynamic Programming

Military Problem	**Multi-strike nuclear warfare**		**Logistics**	**Aerospace race**	
Original Study Dates	1951	1953	1953, 1954	1954	1958
Model Problem	Discrete Gold-mining equation for allocation of scarce nuclear bombs	Continuous gold-mining equation	Production smoothing, inventory control, bottlenecks	Bang-bang control for minimizing time for stabilization	Self-correcting optimal trajectory for missile or rocket
Mathematical Reference	Sequential analysis	Calculus of variations	Variational problems in mathematical economics & stochastic control	Variational problems in deterministic control theory	Variational problems in adaptive control theory
Novel Approach	Generalized recursive functional equation for economizing criterion solved by approximations in multistage policy space	Treated problem as an initial value problem with *N*-stages instead of 2-point boundary problem with *N*-dimensions	Replaced continuous variational problems with discrete multistage decision processes	Modeled control as a function of state in multistage decision process	Modeled control as a function of hidden state revealed through multistage learning process

that model. That takes us, as it did Bellman, to his formal comparison of his dynamic programming approach with the calculus of variations, which mathematicians had been using for over two centuries to examine solutions for a maximum or minimum function in a family of functions. That comparison led Bellman to take up variational problems that economists had posed in their work on US Navy and US Air Force research projects in logistics. In the course of that work on stochastic control problems, such as inventory control in the face of uncertainty, Bellman realized that he could apply his same state-transition formulation of multistage decision processes to problems in deterministic and adaptive optimal control.

Bellman recognized in his early allocation problems for nuclear warfare that solving for the optimal policy was the dual of solving for the value of the maximum gain that results from an optimal policy. Once he turned his attention to variational problems in mathematical economics, such as optimal inventory control, Bellman perceived another duality: the rule of action that enables one to sequentially formulate the tangent lines of an envelope of an optimal curve is the mathematical dual to the classical mathematician's instantaneous scheduling of the variational curve as a locus of points:

> ….when proving geometric theorems we have a choice of using either the point-line form or the line-point form, whichever is more convenient or more intuitive to us.
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> Thus, we can consider a curve to be a locus of points or an envelope of tangents. The calculus of variations corresponds to a curve being taken as a locus of points; dynamic programming views a curve as an envelope of tangents (Bellman 1968, 36-37).

Most of the researchers I have highlighted in *Protocols of War* acknowledged they were developing mathematics for action. For example, Massé explained that the solution to the wartime dynamic allocation problem he posed for generating electricity in war-torn France was a "rule of action" – an "operational rule" or "strategy" (Massé 1944 and [1959] 1962). Similarly, Arrow, Harris, and Marschak

(1951, 251) spoke of inventory policies as “rules of action.” Bellman called policy space a “domain for activists.” Action involves setting a goal, figuring out a strategy for reaching that goal, and making sure along the way you are staying on target to achieving that goal. An activist is frequently eyeballing the situation and making a quick comparison between the present state and the desired goal. The ocular comparison leads to a policy to transform that actual state to the desired state by taking the path of least resistance from where you are to where you want to be. At each stage the past history of the system is irrelevant; the decision maker follows a rule of action based only on the current state and the remaining resources and time at hand. At the end, if you were to reflect back on all the states achieved in the pursuit of your goal, you could gaze at a curve constructed by these tangent lines of action at each stage. It is as if your many quick comparisons of where you are to where you wanted to be were tangents to the curve of the states you were at various times. The classical, “primal” calculus of variations approach had been to find the optimal curve among a family of nearby curves at the beginning of the process and stick with that schedule point-by-point. It is only with the “dual” approach formulated in cold war programming that we see the active mathematical process of recursive optimization, mathematical models of feedback, and the mathematical analogies of the rational decision-maker thinking at the margin for every new initial value of resources. In the last sections of this chapter we will briefly examine Bellman’s application of dynamnic programming to optimal trajectories and the economist’s appropriation of dynamic programming (these two are covered in more detail in chapters six and seven respectively).

Planning for the USAF

For the creators of the phrases “dynamic programming,” “linear programming,” and “mathematical programming”, the term “programming” was synonymous with planning. In many cases, as Bellman’s coder and co-author Stuart Dreyfus pointed out, programming was synonymous with allocation: “In the linear programming model limited resources are *allocated* to various activities. In dynamic programming resources are *allocated* at each of several time periods.” (Dreyfus 1956, 3-4).

Long before digital computers existed and even longer before the expression became a description for writing code for computers, the US military used the term programming to refer to constructing plans of action.[6] Indeed, in the early years of digital computers, one “coded” rather than “programmed”, and computer operators more than likely adopted the latter term because so many of the early coding operations were done for military planning purposes. In June of 1946, the Pentagon hired George Dantzig to “mechanize the planning process”. According to Dantzig, his immediate goal was to “speed up what was being done by hand.” (Dantzig [2000] 2007). By the summer of 1947 Dantzig and his colleagues in the Air Force-funded Project SCOOP (Scientific Computation of Optimum Programs) had generalized Wassily Leontief's descriptive inter-industry model, which had a singular set of outputs, to describe a variety of output scenarios and to give the criterion for choosing the optimal one.[7] All computations were done on hand-held calculators by a team of

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[6] In a 2000 interview with Irving Faustig, George Dantzig was explicit about his use of the term “programming” in the 1940s and 1950s: “It is a word used by the military for a plan, a schedule, or in general for a program of actions…. The list of instructions to be executed by a computer was first called a "code". The military refer to their plans as "programs"…”

[7] According to Martin Kohli’s (2001) history, Wassily Leontief constructed his first input-output table in 1936 to explore the interdependence among the various parts of the economic system and to see

about 10 arithmeticians (often female “computers”). Dantzig and his colleagues in Project SCOOP and at the RAND Corporation certainly foresaw that machines would eventually be able to compute their programs and they designed their algorithmic protocols with that in mind, but it was not until 1955 that they could comfortably rely on digital computers to solve complex linear programs.

Most of the developmental work on mathematical programming in the 1950’s was done at the RAND Corporation in Santa Monica California. As David Jardini (1998) explains in his thorough history of the institution, the US Army Air Forces initiated a contract with the Douglas Aircraft Company in March 1946, with the objective of obtaining recommendations for “preferred techniques and instrumentalities” for intercontinental warfare.[8] By mid 1947, RAND employed 100 full-time researchers divided into six sections (Evaluation of Military Worth, Administration and Services, Rocket Vehicles, Airborne Vehicles, Electronic Communications and Nuclear Physics). The work of these sections was channeled into “systems analysis” which was the core interdisciplinary work of RAND.[9]

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what would be the effect of a change in industrial productivity or savings on relative prices among the 44 sectors of the economy. It was an exercise that paid homage to François Quesnay’s representation of a system in a *tableau économique* and to the neoclassical pursuit of the relative price analysis. During World War II, the US Bureau of Labor Statistics worked with Leontief to change the units of analysis from relative to absolute prices and quantities and to greatly expand the model, including adding a separate government sector, for the purpose of simulating the effects of changes in government spending and war demobilization, on the economy. With these changes the input-output table became a quantitative model used for planning. After the war, Dantzig’s task was to optimize this planning approach (see Dantzig 1963, 18 and Klein 2001).

[8] Philip Mirowski (2002) also writes on the history of RAND in his account of how economics became a cyborg science. Mirowski emphasizes John von Neumann’s influence on RAND, the institution’s cultivation of game theory, and the extent to which RAND influenced the Cowles Commission, and through Cowles, mathematical neoclassical economics, in the 1950s. Although RAND has a far less explicit role in his book on *How Economics Became and Mathematical Science,* Roy Weintraub (2002) explores the axiomization of mathematical economics during this same time period.

[9] In 1948, formal ties with the Douglas Aircraft Company were severed and project RAND turned into the autonomous non-profit RAND Corporation, with major funding still coming from contracts with the Air Force. With that transformation the “sections” became departments along the lines of academic

According to Jardini, this practice was inherited from the wartime rudimentary quantitative comparisons of alternative aircraft systems at Douglas Aircraft, but RAND researchers greatly expanded the scope to include, for example, recommendations for new weapon systems. In his briefing to the Army Air Forces in April 1947, Arthur Raymond, the vice president of engineering for the Douglas Aircraft Company explained the goal and benefits to the military of the RAND contract between the aircraft company and the military:

> RAND is not a production contract, an experimental contract, nor a research contract…It is a planning contract. Its purpose is to mobilize certain special engineering and scientific skills to assist in arriving at sound conclusions fundamental to the development of your programs. Its particular field is that of long range air warfare. (Raymond quoted in Jardini 1988, 28)

The practice of government planning was strongly associated with the enemy ideologies of fascism and communism.[10] As we will see in sections that follow, consultants at the RAND Corporation sometimes had to use euphemisms to describe the work they did under their "planning" contract, but some defended their planning as a new scientific approach that had transcended the arbitrary authoritarianism of old. At the second International Conference on Operational Research in Aix-en-

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departments in universities and the interdisciplinary core of general systems analysis gradually lost ground to the more specific projects of the departments. According to Jardini (1998), the role of central systems analysis was further diminished by the Air Forces critical response to the type of analysis that came from RAND in the late 1940s.

[10] Friedrich Hayek, for example, warned in his popular 1944 treatise on *The Road to Serfdom*, that through planning British and American democracies were unknowingly traveling the totalitarian road.

Provence in 1960, George Dantzig reflected on the rapid growth of the type of planning initiated at the Rand Corporation:

> It is a real mystery that a field that is so full of applications to everyday problems had created so little scientific interest in the prewar period. Historically planning in this world of uncertainty has traditionally endowed those in authority with the 'wisdom' to decide among undecidables….However, in the last twenty to thirty years many of the obstacles that made planning an authoritarian task rather than a scientific task have been overcome. These are *first* the formulation of objectives and familiar relationships in mathematical terms; *second*, the development of reporting systems and the systematizing of statistical information so that basic data was at hand; *third*, techniques for solutions of the mathematical models were developed (i.e. theory, numerical analysis, electronic computers). Dantzig 1960, 283.

The planning done at RAND was not central government planning of the entire US economy. The closest analogy is planning done at the level of the firm – as if the Air Force or the Navy was a large corporation. One of the problems facing firms is how to allocate scarce resources within the firm given the absence of markets for intra firm transactions. So, for example, in the absence of a market in scarce fissionable material to guide its allocation among various nuclear weapons projects, economists at RAND tried Bellman's dynamic programming allocation equations or Dantzig's approach to linear programming to determine the shadow prices of Uranium 235 (see

Enke 1954 and Klein 2001). Indeed, it was the military's need in the late 1940s to plan a rational allocation of scarce atomic bombs in the event of a nuclear war with the Soviet Union that first quickened Bellman's functional equation approach to multistage decision processes.

Mining for Gold

The essential ingredients of the dynamic programming protocol include a framing the problem in a functional equation stating an economic criterion that results from an optimal policy, and a recursive algorithm for specifying the equation and guiding the approximation of a policy solution. Bellman's articulation of this protocol, like his mathematical algorithm, proceeded in discrete stages, so a chronological examination of his conquest of policy space will take us far in understanding his mathematics, its context, and its legacy. In his 1984 reflection on his career in mathematics, Bellman asserted that *effective numerical solutions* were a central theme of his research. His first exposure to such a solution came during his first stay at RAND in the summer of 1948. There he became acquainted with Dantzig and his approach to linear programming. By that time, Dantzig was perfecting his algorithm for maximizing a linear equation subject to linear constraints. Dantzig's "simplex method" for solving linear programming problems was by intention very compatible with numerical approximations methods of digital computing.[11]

The pay and intellectual stimulation at RAND were such, that after his summer of 1948 residence, Bellman decided to work at RAND during his Christmas, Easter,

[11] As early as 1953 RAND researchers could program their algorithms in machine language on a Johnniac, a digital computer based on a design by John von Neumann, but Dantzig said he and others could not fully rely on digital computation of major programming problems until 1955.

and summer vacations from his full time teaching job at Stanford University. Ed Paxson recommended that Bellman's focus at RAND should be on *multistage decision processes*, and in the summer of 1949, Bellman worked with Dave Blackwell on optimal strategies in games (Bellman's favorite was the mathematical analysis for the "red dog" game). Bellman stayed at Stanford in the summer of 1950, but returned to RAND for the Fall quarter. It was during that sojourn that he struggled with a name for his emerging approach to multistage decision processes:

> The 1950's were not good years for mathematical research. We had a very interesting gentlemen in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word, research. I'm not using the term lightly; I'm using it precisely. His face would suffuse, he would turn red, and he would get violent if people used the term, research, in his presence. You can imagine how he felt, then, about the term, mathematical. The RAND Corporation was employed by the Air Force, and the Air Force had Wilson as its boss, essentially. Hence, I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. What title, what name, could I choose? In the first place, I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons, I decided therefore to use the word, "programming". I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying – I thought let's kill two birds with

> one stone. Let's take a word which has an absolutely precise meaning, namely dynamic, in the classical physical sense. It also has a very interesting property as an adjective, and that is it's impossible to use the word, dynamic, in the pejorative sense. Try thinking of some combination which will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities. (Bellman 1984, 159).

When he returned to RAND in summer of 1951, Bellman worked with Hal Shapiro, and Ted Harris on the applications of *functional equations* to decision and learning processes (see Bellman, Harris and Shapiro 1953, and Bellman 1951a). The equations they used were similar to those presented by Arrow, Harris and Marschak at the RAND logistics conference the previous year and the recursive optimization equations that Arrow, J. D. Blackwell and M. A. Girshick had formulated at RAND in the summer of 1948 to examine Wald's sequential analysis from a more decision-theoretic perspective. In functional equations the unknown one is solving for is a function rather than a number - so functions appear on both the left and the right side of the equations. What the functional equations in the RAND research projects had in common was their recursive optimization: the criterion function for N stages called itself up for $N-1$ stages.

Bellman's work in the summer of 1951 on one functional equation in particular highlighted a duality in the modeling of multi-stage descion-making that made computation more feasible. The duality was between solving for the maximum value of returns explained in the functional criterion equation and solving for the optimal policy that would yield maximum returns. The approximation techniques the RAND mathematicians used were usually more effective if one first solved for the optimal policy and then derived the value of the function. In subsequent published accounts, Bellman called the equation, which inspired him in the summer of 1951 to seek solutions in *policy space* rather than *functional space*, “the gold mining equation”.[12] Here’s one of several published descriptions of the problem that passed security clearance:

> *Efficient Gold Mining.* We are fortunate enough to possess two gold mines, Amerconda and Bonanza, the first of which contains an amount x of gold, while the second possesses an amount y. In addition we have a rather delicate gold-mining machine which has the property that if used to mine gold in Amerconda, there is a probability p_1 that it will mine a fraction of r_1 of the gold there and remain in working order, and a probability $(1- p_1)$ that it will mine no gold and be damaged beyond repair. Similarly Bonanza has associated the probabilities p_2 and $(1- p_2)$ and the fraction r_2.
>
> We begin by using the machine in either the Amerconda or Bonanza mine. If the machine is undamaged, we again make a choice of using the machine in either of the two mines, and continue in this

[12] In his earliest RAND research memoranda Bellman also used the name “strategy space” for what he later labeled exclusively “policy space.” Bellman acknowledged that policy space was a functional space, but his narrow use of the term “functional space” meant what I would call criterion space.

> way, making a choice before each mining operation, until the machine is damaged.
>
> What sequence of choices maximizes the amount of gold mined before the machine is damaged? (Bellman 1954a 275).

On the surface, it seems a strange problem for the US Air Force to pay top dollar to a brilliant mathematician to solve. Why and how does one discretely bundle "each mining operation?" Why should we have to assume that the only mining machine available is so delicate one must factor in the probability that it will be damaged beyond repair and mine no gold? In the summer of 1951, Edward Quade, Ed Paxton's assistant in the systems analysis program, was directing a project comparing bomber types and multi-strike attack plans. For Bellman the "gold mining equation" was a unclassified euphemism for the problem he worked on in multi-strike analysis (see for example Bellman 1951a). We just have to substitute an expensive aircraft carrying an atomic bomb and vulnerable to antiaircraft fire for what Bellman, in a double-entendre, called the gold-mining machine "of a sensitive nature" and substitute the amount of damage done to two different targets (in North Korea or the Soviet Union) for the amount of gold obtained from the two mines.

In his reminiscence on the history of operations research, Quade ([1989] 1999) described the novel problems of atomic weapons that he, Bellman and other mathematicians at RAND studied in the early 1950s:

> During World War II many bombers were sent to the target armed with HE [high explosives]. However, with the advent of the atomic bomb, only one bomber was needed with an atomic bomb to attack a

> target. But, to send out just one bomber would be suicide, since all the defenders could concentrate on that one target and it would never get through. Conceptually the study therefore looked at sending escorts to attack enemy fire; they looked just like bombers, but were empty. In those days atomic weapons were considered to be free. However, the Atomic Energy Commission directed that a limit be placed on the number of bombs because they were too scarce. Of course in the 6-8 years that it would take for the bombers [Boeing's B-52s] to become operational all the bombs needed could be built.
>
> Another problem with the study was the fact that if a bomber were shot down, say 30 miles short of the target, there was no credit given for the destruction it would cause or the damage from fall out. The bomb had to be dropped right on the target. (Quade [1989] 1999, 5)

Thus, the probability that the delicate mining machine would fail before any gold was mined had to be factored in. Likewise, Bellman and others had to apply their mathematics to the allocation of scarce fissionable material in an atomic weapons program in which there was no market for the primary raw material.[13] Bellman's first two applications of dynamic programming were to the problem of allocation processes within multi strike analysis.

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[13] In Project RAND's Fourth Annual Report (1950, 7), Paxson defined the problem of planning for nuclear warfare as: "Given a fixed amount of fissile material and a fixed sum of money with which to procure, operate, and maintain a strategic striking force at strength for a 4-year period, specify the atomic bombs and aircraft which will maximize the damage of an initial bombing strike." In 1951 in response to Air Force criticism, RAND changed the definition of the problem from one deadly initial strike to multi-strike analysis, but the allocation of scarce fissile material remained part of the planning task.

Where were the values of the probabilities for successful hits and losses of bombers to come from? Given that military aircraft had only ever dropped two nuclear bombs, one might suspect they were close to subjective probabilities. As Warren Weaver, however, explained to a conference of Social Scientists in September 1947, economists and mathematicians at the RAND Corporation were quantifying military worth "to see to what extent it is possible to have useful quantitative indices for a gadget, a tactic, or a strategy, so that one can compare it with available alternatives and guide decision by analysis" (Weaver quoted in *Conference of Social Scientists* 1948, 7). Jardini (1998) describes the great extent that the Evaluation of Military Worth Section went to compute expected damage to target and damage to US aircraft given type of aircraft, type of bomb, weather conditions, enemy geography, and a variety of other factors. In 1947, the Evaluation of Military Worth Section devised a physical "coverage problem machine" to simulate bombing routines and a random digit generator to produce errors. As early as 1946, the RAND corporation gave a contract to Bell Telephone Laboratories, to compute for various bombers the probability they might avoid anti-aircraft missiles. John von Neumann's estimation that the bombing coverage calculations would require at least 50 million multiplications, inspired RAND to be one of five organizations in 1950 to begin building a copy of the Princeton Institute for Advanced Study's Computer based on a design by Arthur W. Burks, Herman H. Goldstine and von Neumann (1947) of an electronic computing instrument (see Gruenberger 1968, Ware 2002, Jardini 1998, and Edwards 1996). The "JOHNNIAC" computer would not be operational until 1953, but by 1950, Paxson and his colleagues working on the strategic bombing assessment had, with other machines, calculated expected outcomes for over 400,000 bomber configurations. This major undertaking to evaluate military worth and quantify the *p*'s and *r*'s of Bellman's and others equations was the first major stimulus to RAND's path breaking work in hardware construction and software design.[14] . The link between the attempt to measure military worth and economic notions of utility is evident in Jacob Marschak's and M. R. Mickey's (1951,1) RAND report on "Notes on the Optimal Choice of Weapons:"

> We want to choose a weapon system that, subject to a given cost constraint, will maximize the mathematical expectation of the military utility. Military utility, as a function of damage done to the enemy, can be interpreted as the probability of victory. This concept is embedded into the more comprehensive concept, that of social utility. A monotone non-decreasing military utility function is assumed….[15]

When sections were transformed into more powerful departments in 1948, the RAND economics department emerged from the Evaluation of Military Worth Section. The US military service's interest in quantifying expected damage,

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[14] In his study of computers in cold-war America, Paul Edwards (1996) describes how RAND's needs shaped computer science and inspired IBM to commit to digital computer development.

[15] Marschak and Mickey (1951, 9,10) wanted to model their social utility function on the

> "Jeffersonian triad of life, liberty (which would be lost through defeat) and the pursuit of happiness (achieved through consumption)…For example, one might devise assumptions that would enable us to use the expected loss of lives as a controlled variable, and to construct the efficient set in the space whose coordinates are: the budget K, the expected probability of victory U, and the expected loss of lives. Such an assumption would probably imply that social utility w is linear in lives, and this is hardly plausible. We shall not pursue this matter further in the present paper, and shall confine our considerations to the expected probability of victory (U) that can be bought for a fixed amount of money (K)."

Marschak and Mickey were not the only RAND researchers to exclude military lives lost from their analysis and as Jardini (1998) points out this neglect, or even worse the attempts to put a price tag servicemen's lives lost, was one of the major criticisms that the Air Force client had of RAND's quantitative evaluations of military worth.

complemented by von Neumann's and Oscar Morgenstern's (1944) linking of their notion of expected utility as a payoff from strategies in game theory, was a major stimulus to post World War II work in economic utility theory (see for example, Bracken 1962).

So RAND economists, statisticians and computers ensured that the *p*'s and *r*'s (probabilities of success and returns likely if there was success) were readily at hand to calculate the solutions to Bellman's "gold-mining" equation for a myriad of bombing scenarios.

With regard to the multi-strike problem, Bellman's approach was to begin with a functional criterion equation defining the total returns one could expect from an optimal stagewise mining (bombing) plan:

$$f(x, y) = \left\{ \begin{array}{l} \text{expected amount of gold mined before the machine is damaged} \\ \text{when } A \text{ has } x, B \text{ has } y, \text{ and an optimal policy is employed} \end{array} \right\} \qquad (1)$$

In other words:

$$f(x, y) = \max \left[f_a(x, y), f_b(x, y) \right] \qquad (2)$$

Bellman then reasoned that if one arbitrarily started in stage 1 at *A*, the Amerconda mine, then the expected value of the amount mined in that initial stage would be $p_1 r_1 x$. He assumed that the optimal policy would continued to be pursued from that stage onward so the expected amount mined from the second stage onward would be $p_1 [f(1-r_1) x,y]$. To get the problem going, Bellman assumes that whatever the decision taken in the initial stages, the remaining decisions must be an optimal policy built on that first decision. That assumption forces a recursion. The recursive, stage-by-stage reasoning yields two equations, depending on which mine one started with, representing the total expected amount mined if one pursued an optimal policy.

$$f_a(x,y) = p_1 \left[r_1 x + f\left[(1-r_1)x, y\right]\right] \tag{3}$$

$$f_b(x,y) = p_2 \left[r_2 y + f\left[x, (1-r_2)y\right]\right] \tag{4}$$

Bellman substituted equations 3 and 4 into equation 2, and ended up with the functional equation:

$$f(x,y) = \max \begin{Bmatrix} A: p_1 \left[r_1 x + f\left[(1-r_1)x, y\right]\right] \\ B: p_2 \left[r_2 y + f\left[x, (1-r_2)y\right]\right] \end{Bmatrix} \tag{5}$$

Bellman’s solution was at each stage:

$$\begin{Bmatrix} \text{if } p_1 r_1 x/(1-p_1) > p_2 r_2 y/(1-p_2), \text{ take the } A \text{ choice,} \\ \text{if } p_1 r_1 x/(1-p_1) < p_2 r_2 y/(1-p_2), \text{ take the } B \text{ choice} \\ \text{if } p_1 r_1 x/(1-p_1) = p_2 r_2 y/(1-p_2), \text{ either choice is optimal} \end{Bmatrix} \tag{6}$$

The rule of action for an optimal policy that Bellman derived from work on the gold-mining equations was to choose at each stage the target (the mine) that has the highest ratio of immediate expected gain to expected loss. When he first worked on this problem in 1951, Bellman was struck by the fact that the path of least resistance to an effective numerical solution was a declaration of what action to take rather than the solving first for the value of maximum total returns: “The solution was given most easily not in terms of the unknown function, but in terms of an action or decision. This intrigued me, because I had never seen this phenomenon before.” (Bellman 1984, 160). The optimal policy determined the value of the function, and the function determined the optimal policy, so the optimizer had a choice as to which way to solve the problem. Computation by approximation made the policy route more viable as an approach to a solution partly because with approximation one has to start somewhere and an appropriate starting point for an optimal policy was usually more apparent from experience than an arbitrary guess on the functional value.[16] Bellman hoped that work on simplified versions of models like this would reveal a pattern in the solution and thus “some metaphysical concept such as ‘the principle of least action’ that we can apply to problems of a more complicated type” (Bellman 1954a, 284-285).

As with the military use of sequential analysis we can see the priority of heuristics over analytical elegance. This policy rule-of-thumb to follow on each bombing mission was not only the easiest solution to obtain; it was also more likely to be what the client wanted. The Air Force was ultimately more interested in knowing *how* to maximize damage to enemy targets (solutions in policy space) than the actual value of the damage done by optimal operations (solutions in functional criterion space since the function started with was the total gold mined). At the very least, the USAF was not going to be content with classical mathematical analysis that often only led to declarations of the existence and uniqueness of a solution rather than a solution itself. Also the military often had a preference for approximate solutions because exact solutions took too much time and scare computational resources.

The needs of the military client and the capacity for fast computation created a new technique to solve mathematical problems – *numerical approximations in policy space*. Bellman first approached this as a paid-for summer-time occupation, but it

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[16] Bellman (1954b, 148) also demonstrated that successive approximations, a technique that could be used on approximating the optimal policy, yielded better results than solving for the maximum criterion value.

soon became a career-time commitment. Like most mathematicians of his day, Bellman was at first scornful of such applied mathematics:

> Numerical solution was considered the last resort of an incompetent mathematician. The opposite, of course is true. Once working in the area, it is very quickly realized that far more ability and sophistication is required to obtain a numerical solution that to establish the usual existence and uniqueness theorems. It is far more difficult to obtain an effective algorithm than one that stops with a demonstration of validity. A final goal of any scientific theory must be the derivation of numbers. (Bellman 1984, 184-185).

In the summer of 1952, Bellman faced a major decision: whether to be “a traditional intellectual” working on the pure mathematics of number theory at Stanford or to be a full-time applied mathematician at RAND, “using the results of my research for the problems of contemporary society” (Bellman 1984, 173). Bellman resigned his faculty position at Stanford and took a permanent position at RAND to work in the areas of dynamic programming, control theory, and time-lag processes.

In June of 1952, John von Neumann communicated Bellman’s first public paper on the theory of dynamic programming to the National Academy of Sciences. Bellman acknowledged that his functional equations approach for optimization in decision processes was intimately related to Abraham Wald’ sequential analysis (see chapter 3) and the work of his RAND colleagues, including Kenneth Arrow, David Blackwell, and Meyer Girshick’s (1949) Bayesian approach to sequential decision problems and Arrow, Ted Harris, and Jacob Marschak's (1951) work on optimal inventory control (see chapter 4). Bellman explained that his theory was applicable to cases where each operation in a sequence of operations gives rise to a stochastic event, which then determines subsequent operations: "In many cases, the problem of determining an optimal sequence of operations may be reduced to that of determining the first operation" (Bellman 1952, 716.)

Within a few months of that first public paper on dynamic programming, Bellman named the technique that he had been using over and over to derive recursive functional equations the *principle of optimality*: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." (Bellman 1954a, 285 but repeated word for word in hundreds of Bellman's publications). The mathematical transliteration of the verbal principle of optimality yielded a new class of functional equations and Bellman's new functional equation theory of dynamic programming. It also turned many of the multistage decision problems into "Markovian" decision processes where the past history added no additional useful information –at each stage one only needed to know the new initial condition of the state, which had resulted from the previous stage.[17]

By the end of 1952, Bellman had articulated the key features of dynamic programming: the pursuit of effective numerical policy solutions to recursive functional equations that maximized yield or minimized costs over several stages of a process guided by a principle that ensured the policy decision at each stage was the

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[17] For more on the history of Andrei Markov's probability theory see chapter 10 of Klein 1997.

right one in the context of the entire process. A key to understanding Bellman's theory is to realize how general it is. Bellman's dynamic programming *exhorts* the user to maximize or minimize recursively one stage at a time. The principle of optimality is a guideline for solving a problem by helping the practicing optimizer write an equation to solve, but it does not hand the user an equation, let alone a solution, on a plate. Abiding by the principle of optimality forces the user to express the total returns (as a function of the state of resources) as a recurrence relation such that the total returns from an optimal strategy are equal to the returns from the first stage plus the cumulative return of all later stages arising from an optimal strategy with the state of resources that exists as a result of the decision taken in the first stage. A variety of specifications of the functional equation are compatible with this philosophy as are a variety of approximation techniques to solve the functional equation once it is specified. In championing the vagueness of his approach, Bellman argued that it was "the spirit of the problem rather than the letter that is significant. A certain amount of ingenuity is always required in attacking new questions and no amount of axiomatics and rigid prescriptions can ever banish it" (Bellman, 1954a, 285). The generality was intimately connected to the criterion of computational feasibility and it opened the door for solving a variety of mathematical problems:

> Our aim is to present a simple, readily applicable technique, requiring no mathematical background beyond elementary calculus, which can be used to compute the solution of a variety of problems in a routine fashion, with no regard to linear or nonlinear, stochastic or

deterministic features of the underlying processes. (Bellman 1957b, 277).

From Policy Space to Optimal Control via a Calculus of Variations

Bellman gave the gold-mining equation a life of its own, autonomous from its humble beginning in multi-strike analysis for bombers with atomic payload. In the process he learned the advantages and limitations of approximating in policy space.[18] Bellman was a mathematician at heart, as opposed to a scientist applying mathematics, so he took put his gold-mining equation to work to explore the structure of classes of policy solutions. He tried, for example, a non-linear utility function (Bellman 1953a). His colleagues at RAND, Samuel Karlin and Hal Shapiro (1952), came up with a counterexample involving 3 mines (targets) for which the Bellman's rule of action did not fit. That inspired Bellman to look at further variations on the gold-mining equation.

In December 1953, von Neumann communicated Bellman and Sherman Lehman's work on a *continuous* gold-mining equation to the National Academy of Sciences. Faced with the challenge of articulating a continuous mixed strategy (as opposed to putting together a sequence of policy decisions from consecutive stages), Bellman and Lehman suggested, "for mathematical purposes, mixing at a point is to an arbitrary degree of approximation equivalent to mixing pure strategies in an

[18] In their edited collection of essays on *Models as Mediators*, Mary Morgan, Margaret Morrison and the other contributors make a strong case for acknowledging the autonomy of models and investigating what the creators learn from their models. In their initial essay, Mary Morgan and Margaret Morrison (1999, 36) assert that in mediating between data and theory, models gain an autonomy worthy of the consideration of historians of science: "models should no longer be treated as subordinate to theory and data in the production of knowledge. Models join with measuring instruments, experiments, theories and data as one of the essential ingredients in the practice of science."

interval about the point." (Bellman and Lehman 1954, 116) They thus ended up with a problem in which in essence fractions of the delicate mining machine (bomber with atomic bomb) could be used in several mines (targets) simultaneously and the values of the respective fractions could change instantaneously. These conditions were obviously impossible for planning a bombing strategy but they did inspire Bellman and Lehman to make new connections between dynamic programming and the calculus of variations, which mathematicians, and to a lesser extent economists, had been using since the nineteenth century to find optimal functions:

> As essential feature of our research lies in viewing a policy in its extensive rather than normal form, to borrow the terminology of game theory. Another way of stating this is that instead of determining the complete solution for one set of initial parameters, which would correspond to determining the extremal cruve in the classical theory of the calculus of variations, we attack our problem by imbedding it into the family of problems of this type with arbitrary initial parameters. This is the approach used throughout the theory of dynamic programming. Having done this we determine an optimal continuation from each position, which upon being carried through yields an optimal policy. (Bellman and Lehman 1954, 115)

Since the work of Gotfreid Liebenitz and Issac Newton in the seventeenth century, mathematicians had access to calculus to determine minimum and maximum points of a function. With the work of Leonard Euler and Joseph Louis Lagrange in the eighteenth century, further developed by Carl Gustav Jacobi and William Rowan

Hamilton in the nineteenth century, mathematicians could use a *calculus of variations* to determine the minimum and maximum functions (or curves) in a set of functions (or curves). The calculus of variations however, was quite limited for practical purposes if there were too many independent variables (dimensions) or too many constraints. A common way to solve calculus of variation problems was to first specify and then solve appropriate Euler–Lagrange equations. Solving these second-order differential equations often required a trial-and-error approximation of the two-point boundary conditions, which in itself was often a formidable task. If there were constraints, as is typical in optimization problems, there could be intervals over which no solution could be obtained. If the function depended on many variables, dimensionality became an insurmountable problem because there would have to be trial-and-error approximations on two boundary conditions for each variable. The calculus of variations depended on deterministic differential equations so incorporating uncertainty into models was no easy task. Also analogue computers could solve some very specific differential equations, but digital computers required that differential equations be transformed into difference equations in order to transform transcendental operations into arithmetic operations. For decades, most mathematicians had been content with using the calculus of variations to reveal analytical structure and to indicate if solutions existed and if so were they were unique. Rarely did the ivory tower mathematician need the numerical solution so often out of reach with the calculus of variations.

Within a cold war, however, the military client was demanding maximization subject to constraints and searching for quantitative rules of action to follow in uncertain conditions. What Bellman did, starting with his work on the continuous gold-mining equation, was to transform the two-point boundary variational problem into an initial-value problem that was much easier to solve by approximation- indeed the presence of constraints made initial value problems easier rather than more difficult to solve because constraints helped to ensure convergence in the approximation process.

Ironically, dynamic programming, designed with computational feasibility in mind, became, became a tool for obtaining solutions to more general problems then a variational approach could achieve. Bellman accomplished that by imbedding the individual problem into a family of problems of the same type:

> In the majority of cases, when we are interested in a particular solution, we are not so much interested in a particular solution as we are in the dependence of the solution upon the parameters describing the phenomena. For example, we are interested in the dependence of the solution upon the initial resources and the duration of the process.
>
> In other terms, in the main, we are more interested in the structure of policies than we are in numerical solutions. If we can discern a uniform structure common to a number of approximations, of increasing degree of complexity, to a specific process, then we can feel that we have derived something of value from our mathematical abstraction. (Bellman 1957b, 287)

Bellman's transformation of a two-point boundary problem in the calculus of variations into an initial value dynamic programming problem led Bellman to realize, as he speculated an economist intuitively would, that "the way one utilized resources depended critically upon the level of these resources, and the time remaining in the process" (Bellman 1984, 180). Bellman and his colleagues increasing drew on the vocabulary and analogies of economic theory and around the same time that he initiated work on the continuous gold mining equation, Bellman took up variational problems in mathematical economics. In successfully tackling variational problems with dynamic programming rather than the classical calculus of variations for which the type of problems were named, Bellman once again acknowledged that the desired solution "was not merely a set of functions of time, or a set of numbers, but a rule telling the decision maker what to do: a policy" (Bellman 1984, 181). In 1953, John von Neumann communicated to the National Academy of Sciences the work of Bellman, Irving Glicksberg, and Oliver Gross (1953) on variational problems minimizing the linear or quadratic costs for maintaining a system at a specified state. A few months later von Neumann communicated Bellman's (1953a) use of dynamic programming to analyze bottlenecks in production and allocation processes. Bellman, Glicksberg, and Gross expanded their work on variational problems in production smoothing (see for example, 1954) and also provided a new proof (1955) that the two-bin inventory policy was optimal if costs were directly proportional to the amounts ordered (see chapter 4). In his 1957 book on dynamic programming Bellman included several chapters on dynamic programming problems in mathematical economics, including allocation processes, inventory control, and bottlenecks, and Bellman circulated within RAND a 46-page paper on "Dynamic Programming and Its Application to Variational Problems in Mathematic Economics" (1956).

Bang-Bang, Optimal Trajectories, and Control Engineering

Revelations arising from his work on several variational problems in mathematical economics directed Bellman's approach to solving a key problem in control engineering - the "bang-bang" problem (discussed in more detail in the chapter 6). The problem was to determine how to restore a system to its equilibrium as quickly as possible subject to constraints on the restoring force. The first mathematical treatments (see, for example, McDonald 1950, Rose 1953 and LaSalle 1954) addressed the problem in the context of quickly achieving stability in a servomechanism (see section on optimization and Harold Hotelling's contribution in chapter 1). The bang-bang problem, and the related one of quickly obtaining and then maintaining optimal trajectories, acquired renewed impetus in the space race that picked up considerable speed in 1957 with the Soviet Union's launching of the Sputnik satellite into outer space. Bellman turned the bang-bang and the optimal trajectory problems, which other mathematicians had been approaching with a calculus of variations, into a multistage decision-making process (see for example Bellman, Glicksberg, and Gross 1956b).

Bellman also took to heart what he had learned from transforming variational problems into dynamic programming problems. Bellman's key contribution to optimal control theory was to construct a model that incorporated the notion: "the way one utilized resources depended critically upon the level of these resources, and the time remaining in the process." Bellman's discrete state-variable model (or state-transition model) made control (the policy decision) a function of state. The transition from one state to the next, achieved by changing an input under your control (for example, turning on the propulsion engine of a spacecraft to correct velocity), depended upon the state you were in. Bellman was encouraged to pursue this state-transition model (in which the next state depends on the current state and on a controlled input, which in turn depends on the current state) by the fact that it meshed perfectly with the notion of feedback in control processes and Bellman's state-transition model became the first mathematical representation of that process.

One visual image that Bellman used to compare his approach to optimal control with that of the calculus of variations is reproduced in Figure 1. The rabbit is following a straight course along the horizontal axis. The calculus of variations mathematician with his textbook of differential equations determines the best curve of pursuit. The dynamic programming advocate, however, uses staged visual feedback in chasing the rabbit to change her course accordingly and in essence is at each glance following a line tangent to the pursuit curve. Bellman's trajectory as an envelope of tangent lines drawn sequentially at different stages is the dual of the calculus of variations trajectory as a locus of points drawn as a schedule ahead of time (see section on duality in chapter 1). If the rabbit maintains a known constant velocity, both approaches are mathematically equivalent. If, however, the rabbit's course is variable and/or uncertain and learning is necessary in the chase, the tangential approach based on staged feedback is more likely to succeed. With each tangent, there is a new stage and a new initial condition and the pursuer is determining the optimal strategy based only on where she is now – one is still approximating in policy space, but policy is now an endogenous variable - *control is a function of state*.

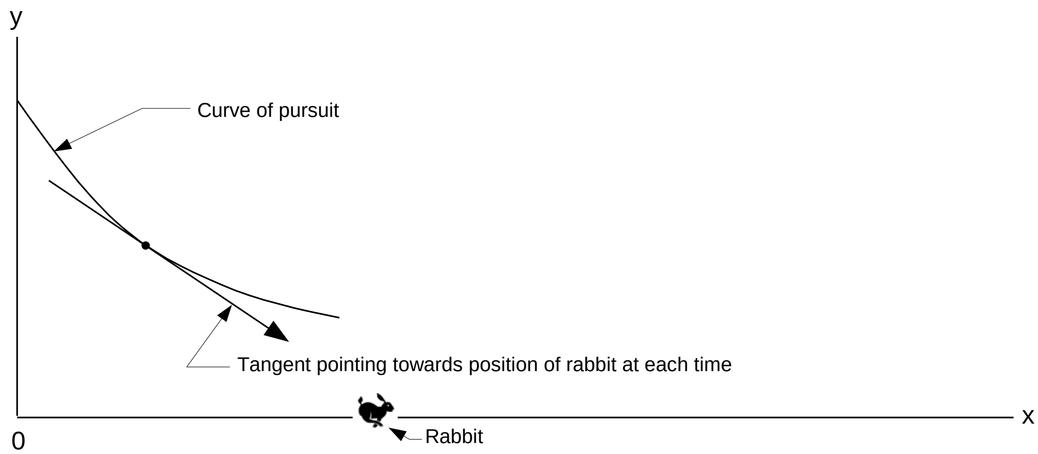

Figure 1. Bellman 1961, 21

As we will see in the next chapter, Bellman's heuristic state-transition approach to control theory and Lev Semenovich Pontryagin's elegant maximum principle for solving control problems with a calculus of variations ushered in a mathematically-guided space age, a theoretical discipline of optimal control, and a systems theory based on the design of systems (as opposed to just assessing and incrementally improving the stability of an existing one). Rudolf Kalman's assessment of Bellman's key contribution to control theory indicates how far Bellman got with his approximations in policy space and making policy a function of state. It also indicated the key obstacle to further extensions (how does one define the state of the system one is trying to model and deal the problem of unobservable or hidden states?) that Kalman himself attempted to tackle (see chapter 6 and for example, Kalman 1960 and Kalman and Bucy 1961).

> The principle that the inputs should be computed from the state was enunciated and emphasized by Richard Bellman in the mid-1950s. *This is the fundamental idea of control theory…*

> Our definition of a control law implicitly assumed that we know what the state of the plant is at each instant, in other words, that all the internal variables of the plant can be read out as outputs. In most practical cases we should not expect this to be true. Indeed, we should always think of the state as an abstract quantity which represents inaccessible variables inside the plant. This is the first difficulty we must come to grips with in trying to apply Bellman's recipe: *control is a function of state.* Kalman et.al.1969, 46, 50

From Normative Microeconomics to Positive Macroeconomics

The Econometric Society Chicago meeting in December 1952 was one of the first public forums for Bellman's introduction of his dynamic programming approach. He subsequently presented on dynamic programming at the 1953, 1955, and 1956 meetings of the Econometric Society annual meetings and published articles in the society's journal *Econometrica* (see Bellman 1954c and 1955). In the contexts of management science and operations research, microeconomists, usually with funding from the Office of Naval Research or the US Air Force's contract with the RAND Corporation, were the first economists to use Bellman's generalized recursive optimization (see for example, Simon 1956 and1959, Whitin 1960, Dorfman 1960). As the space race heated up in the late 1950s, optimal control theory presented two approaches that economists could appropriate for their own theories on economic growth: Pontryagin's elegant application of his maximum principle to the calculus of variations and Bellman's heuristic application of his principle of optimality to dynamic programming. At first, most macroeconomists chose the former (see for example, Samuelson 1970 and Dorfman 1969), partly because of economists' prewar uses of the calculus of variations (e.g. Hotelling 1925 & 1931, Evans 1925, Roos 1927, and Ramsay 1928). It was not until after Robert Lucas and Edward Prescott's 1971 *Econometrica* article on "Investment under Uncertainty" that macroeconomists began to systematically apply Bellman's state-transition methods of recursive optimization to their theories of rational expectations, business cycle theory, and economic growth. [19]

The history of Bellman's military-inspired application of an economic way of thinking sketched in the previous sections of this chapter is the dual to the late twentieth-century history of the macroeconomists' appropriation of mathematical programming. Mathematical models of allocative recursive dynamics were ripe for Robert E. Lucas Jr.'s, Edward C. Prescott's, and Thomas J. Sargent's picking in the 1970s and 1980's because the military needs during the cold war had steered mathematics toward the Markovian-like principle of ignoring sunk costs, a decision rule for thinking at the margin in each stage of a multistage process, a functional equation formulating an economic criterion of maximizing gains or minimizing losses, a notion of state that ensured no wastage of information, and the duality of solving for maximum function and solving for the optimal policy rule .

The heuristic, dual protocol of dynamic programming sent out a broad invitation to economic interpretation. The breadth of this invitation eventually encompassed the normative microeconomic theory for the would-be rational producer, the normative macroeconomic theory for the rational social planner, and the positive

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[19] We will explore this appropriation more thoroughly in chapter seven (see also Ljungqvist and Sargent 2000, *xxxii-xxiv* and Bouman 2006).

macroeconomic theory that like an invisible hand, the system as a whole maximizes welfare through the micro-level individual actors all rationally following consistent optimal policy rules of action. The latter application was not perceived until two decades after Bellman's first generalization of the functional equation approach of dynamic programming. For example, while acknowledging the usefulness of dynamic programming in a normative context of planning, Dale Jorgenson could not imagine the interpretation of the system as a model of optimal capital accumulation having any place in positive macroeconomic theory:

> This interpretation is of interest for applications of the dynamic input-output system to the planning of economic development. There is not evidence at present that a model of capital accumulation involving explicit maximization would be useful in explaining the operation of any actual system. (Jorgenson 1960, 417)

Economists such as Lucas, Prescott, and Sargent, took dynamic programming from the normative theory of the firm to the positive theory of the entire system with the vehicles of rational expectations and competitive industry equilibria. In his 1961 article on "Rational Expectations and the Theory of Price Movements," John Muth (1961, 316) had suggested the hypothesis that "expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory." No information was wasted in the simultaneous determination of actual and expected future prices. Muth was modeling price expectations of a firm for the Office of Naval Research's project on "Planning and the Control of Industrial Operations," (see chapter 4), but he added the adjective "rational" to the noun “expectations” in order to make clear that his theory was positive economics: “At the risk of confusing this purely descriptive hypothesis with a pronouncement as to what firms ought to do, we call such expectations "rational." (Muth 1961, 316).

In the 1960’s, Lucas was a colleague of Muth at the Graduate School of Industrial Administration at Carnegie Tech (now Carnegie Mellon) –the home of the Navy’s major production planning project. In a 1998 interview with Bennett McCallum, Lucas spoke of the influence of that institutional mission on rotating his perspective of normative economics as a starting point for positive economics:

> Jack [Muth] was the junior author in the Holt, Modigliani, Muth, and Simon monograph *Planning Production, Inventories, and Workforce.* This was a normative study –operations research—that dealt with the way managers should make decisions in light of their expectations of future variables, sales, for example…
>
> The power of thinking of allocative problems normatively, even when one’s aim is explaining behavior and not improving it, was one of the main lessons I learned from Carnegie, from Muth and perhaps more from Dave Cass. The atmosphere at Chicago when I was a student was so hostile to any kind of planning that we were not taught to think: How *should* resources be allocated in this situation? How *should* people use the information available to them to form expectations? But these *should* be an economist’s first questions.
>
> Lucas quoted in Samuelson and Barnett 2007, 61-62

A decade after Muth's first articulation of the concept of rational expectations, Lucas and Prescott (1971) used a similar assumption of a probability distribution common to both expected and actual output prices to model anticipated future demand of firms. Lucas and Prescott assumed these firms were operating in a competitive industry facing random shifts in the industry demand curve. In the course of determining competitive equilibrium time paths of investments and prices for the industry, Lucas and Prescott paired Bellman's dynamic programming protocol to Harold Hotelling's (1931) assumption that the competitive industry in essence maximized consumer surplus.

Recursive optimization, including Bellman's functional equation theory of dynamic programming, enabled Lucas, Sargent and others to construct a "new classical macroeconomics" that embraced the micro foundations of the rational expectations of consumers and firms and dynamic competitive equilibria in the macroeconomic demonstration that money is neutral and anticipated effects of countercyclical government policy often render such intervention impotent and harmful. With their appropriated mathematical models of intertemporal decisions processes, they demonstrated that economic actors could rationally anticipate the future effect of many government policy changes and that anticipated monetary expansions would only lead to higher inflation and not to increased output or higher employment (see for example Lucas 1972 and Sargent and Wallace 1975). These mathematically elegant demonstrations in positive macroeconomics thus carried the normative prescription that governments should avoid fine tuning the economy even with the good intention of lowering unemployment, and should instead stick to steady, rule-based guidelines for changing the money supply.

Rules of action also became important in the new classical notion of equilibrium. The classical macroeconomists assumed that the actors in the economy were engaged in multistage decision processes and that the principle of rational expectations “forces the modeler toward a market equilibrium point of view.” (Lucas 1996, 671). Instead of having to rely on what Lucas (1996 669,671) called “church-supper” models that “patched in” to a static general equilibrium theory a dynamic assumption that a system always in disequilibrium was tending toward equilibrium, they used the stationarity assumptions of their time series data, the recursive features of their approximations in policy space, and modern general equilibrium theory (in the spirit of Arrow and Gerard Debreu) to craft a new way of seeing equilibrium. In the words of Nancy L. Stokey, Lucas, and Prescott (1989, 441):

> there is a wide class of situations in which the “invisible hand” ensures that the sets of Pareto-optimal allocations and competitive equilibrium allocations coincide exactly. In these situations we can interpret certain normative models of optimal decision-making (from the point of view of a hypothetical “benevolent social planner”) as positive models of equilibrium outcomes.

The universally applied policy rules of action for consumers and firms ensure that the economic system as a whole invisibly solves the functional equation criterion equation maximizing welfare.[20] As one review put it:

> Then an equilibrium can be sees as a system of well-defined, optimal, and mutually consistent policy (behaviour) rules that determine what the individuals and institutions in the economy do as a function of the variables that describe the state of the world. The beauty of this equilibrium concept is that it can be applied to dynamic economics and still allow economic theories with strong positive implications. (Faig 1990, 966)

Once again the dynamic programmers struck gold in policy space, and Lucas, Herbert Simon, and Finn Kydland's use of recursive optimization helped them achieve Nobel Prizes in Economic Sciences.

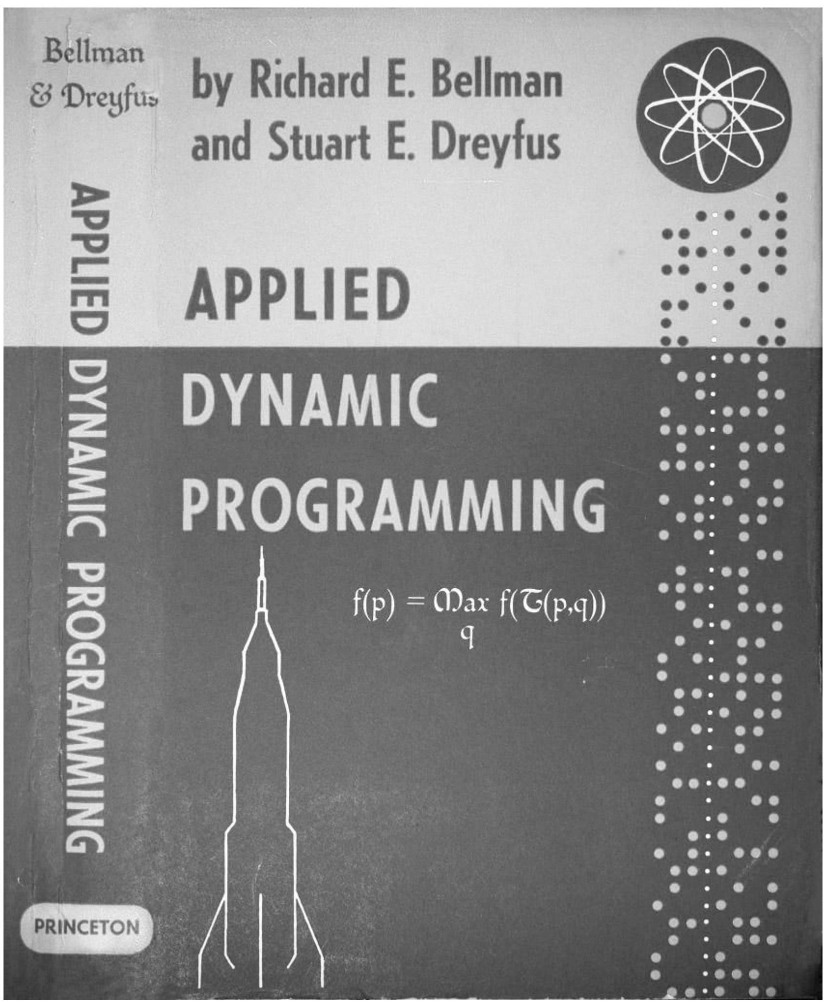
Conclusion

The dust cover for Bellman and Dreyfus's *Applied Dynamic Programming* (Figure 2) illustrates some of the stimulants to key developments in dynamic programming: the military need for effective computation, efficient allocation of nuclear bombs, and control in optimal trajectories in space travel. Bellman initiated his technique by looking at military allocation processes. That and most of his subsequent studies drew

[20] The compelling conformity of rules of action in the new classical macroeconomic concept of equilibrium is related to what Sargent called the "communism of models" in the rational expectations framework: "All Agents inside the model, the econometrician, and God share the same model. The powerful and useful empirical implications of rational expectations—the cross-equation restrictions and the legitimacy of the appeal to a law of large numbers in GMM estimation—derive from that communism of models." Interview with Sargent in Samuelson and and Barnett 2007, 312 (see also Sent 1998)

on what the science of economizing. For example, in his study on the "Optimal Use of Guided Missiles against a Fixed Target System- Maximum Expected Damage," Bellman claimed that with his functional recursive criterion equation theory of dynamic programming "a decision can be reached concerning the number of missiles that should be allocated and the number of targets that should be attacked as well as the optimal assignment of the missiles to the targets." (Bellman 1957c, 4)

Figure 2 Bellman 1961, 21



Such was the language in Bellman's research memoranda circulated within RAND and the US Air Force. In more public forums such as the National Academy of Sciences, academic journals in mathematics, or management journals on operations research, Bellman used allocation euphemisms such as "gold-mining." The origins of the "gold-mining" equation were in Bellman's work in summer of 1951 for a RAND study of multi-strike bombing. Even though the story of the mining machine "of a sensitive nature," which accompanied published versions of the word problem, seemed a trivial and incongruous representation of gold mining, Bellman's protocol for this problem took on a life of its own. Bellman himself published variations of the gold-mining problem, including one with a non-linear utility equation and continuous version in which what used to be a bomber is divided into fractional parts which are simultaneously used with the fractions changing instantaneously. Over forty-five years later there are still mathematical military papers expanding on the gold-mining equation.[21]

Working for war-oriented government clients, Abraham Wald, Pierre Massé, Kenneth Arrow, Theodore Harris, Jacob Marschak, and Richard Bellman developed similar path-breaking protocols in recursive optimization in the 1940s and 1950s. What struck Bellman early on in his first work on the functional equation theory of dynamic programming was the duality of the process of solving the criterion equation and the determination of the optimal policy that fulfilled the criterion. Client needs, approximation techniques and intuition on initial guesses were such that solutions in policy space usually came easier than solutions in functional (criterion) space. This

[21] A recent study in the *Naval Research Logistics* journal took Bellman's word problem literally "The 'gold-mining' decision problem is concerned with the efficient utilization of delicate mining equipment working in a number of different mines….Our first model is concerned with developing an operational policy where the equipment may be switched from one mine to the other at most once during a finite horizon…The models developed here may have potential applications in other areas including production of items requiring the same machine or choosing a sequence of activities requiring the same resource." (Karkul and Parlar quoted in MathSciNet 2002j;90097).

realm of policy space did not exist in the prewar mathematical approaches to finding optimal points or optimal functions. War, with its competition between national powers and its government funding of client-oriented mathematics, gave mathematicians a new space to conquer - policy space.

Bellman made it clear that his decision between a teaching career at Stanford and an applied research career at RAND was a decision on what type of mathematics he wanted to pursue. Prolonged work in policy space was only possible with the US military client dictating planning problems and footing the bill. The original problems were related to how to efficiently bomb the enemy, maintain large inventories of spare parts for military vehicles, or put a spy satellite into orbit in outer space. The physical processes to be represented mathematically were apparent, measurements of the variables were available -thanks to the input of considerable resources-, and the major constraint on the process of developing mathematics to study these problems was computational feasibility.

Exploring the history of Bellman's mathematical dynamic programming is necessary in developing a full understanding of the history of recursive macroeconomic theory. The recursive methods now at the forefront of macroeconomic dynamics are grounded in the cold war competition between two superpowers. In the 1950s and 1960s, economists working at the RAND Corporation and on grants from the Office of Naval Research applied dynamic programming to operations research and managerial economics. Macroeconomists, however, only took up dynamic programming with Robert Lucas and Edward Prescott's (1971) application of dynamic programming to the modeling of rational expectations.

Bellman (1956, 1) himself commented on "the array of mathematical techniques which have been borrowed, begged, stolen or improvised" to meet the challenges within mathematical economics. One cannot help but note, however, the irony of a government's decades-long planning contract underwriting the Nobel-prize winning demonstration that the rationality of consumers often renders government intervention impotent and harmful.

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