Non-convex Aggregate Technology and Optimal Economic Growth

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Abstract

This paper examines a model of optimal growth where the agregation of two separate well behaved and concave production technologies exhibits a basic non-convexity. Multiple equilibria prevail in an intermediate range of interest rate. However, we show that the optimal paths monotonically converge to the one single appropriate equilibrium steady state.

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1 Introduction

Problems in the one-sector optimal economic growth model where the production technology exhibits increasing return at first and decreasing return to scale afterward have received earlier attention. Skiba (1978), examined this question in continuous time and provided some results, which were further extended rigorously in Majumdar and Mitra (1982) for a discrete time setting. With a convex-concave production function, it has been shown that the time discount rate plays an important role: when the future utility is heavily discounted, the optimal program converges monotonically to the "low"

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steady state (possibly the degenerated state characterized by vanishing long run capital stock) while in the opposite case, it tends in the long run to the optimal steady state, usually referred to as the Modified Golden Rule (MGR) state in the literature. Dechert and Nishimura (1983) further showed that the corresponding dynamic convergence is also monotonic. They also pointed out that, unlike the case of a high or low discount factor, this convergence now depends upon the initial stock of capital if the rate of interest falls into an intermediate range of future discounting.

In the present paper, we put emphasis on the existence of many technology-blueprint books, where each technology is well behaved and strictly concave, but the aggregation of theses technologies gives rise to some local non-convex range. Consider two Cobb-Douglas technologies depicted in Figure 1 where output per capita is a function of the capital-labor ratio. The intersection of the production graphs is located at point C where $k = 1$.Therefore the α-technology is relatively more efficient than the β-technology when $k \leq 1$, but less efficient when $k \geq 1$. The two production graphs have a common tangent passing through A and B. Thus, the aggregate production which combines both the α-technology and the β-technology exhibits a non-convex range depicted by the contour ACB. Beside the degenerate state (0,0) in Figure1, there may exist two MGR long run equilibria $\widehat{k}_\alpha$ and $\widehat{k}_\beta$. In this case, we must ask which of these two states will effectively be the equilibrium, and how the latter will be attained over time.

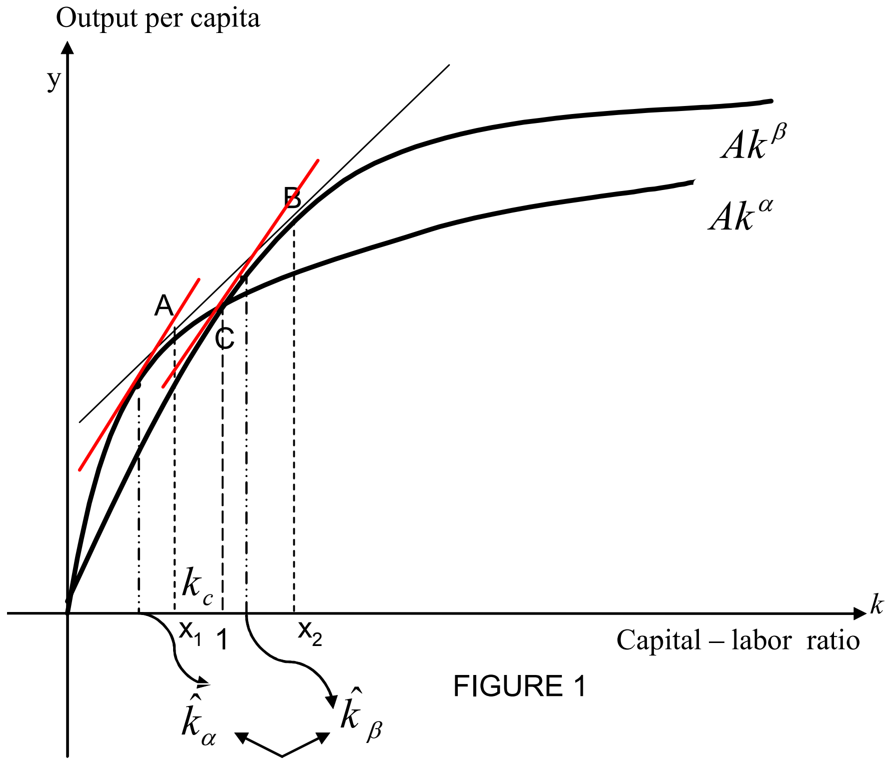

FIGURE 1

We shall show that when future discounting is high enough, the equilibrium is the optimal steady state $\widehat{k}_{\alpha}$ corresponding to the technology that is relatively more efficient at thelow capital per head. Conversely, when future discounting is low, the equilibrium is the optimal steady state $\widehat{k}_{\beta}$ corresponding to the technology that is relatively more efficient at high capital per head. For any initial value of the initial capital stock in these cases, the convergence to the optimal steady state equilibrium is monotonic. In contrast, when future discounting is in some intermediate range, there might exist two optimal steady states and the dynamic convergence now depends on the initial stock of capital k_0. We show that there exists a critical value k_c such that every optimal path from $k_0 < k_c$ will converge to $\widehat{k}_{\alpha}$, and every optimal path from $k_0 > k_c$ will converge to $\widehat{k}_{\beta}$.

The paper is organized as follows. In Section 2, we specify our model. In Section 3, we provide a complete analysis of the optimal growth paths, and in section 4, we summarize our findings and provide some concluding comments.

2 The Model

The economy produces a homogeneous good according two possible Cobb-Douglas technologies, the $\alpha-$technology $f_\alpha(k) = Ak^\alpha$, and the β-technology $f_\beta(k) = Ak^\beta$ where k denotes the capital per head and $0 < \alpha < \beta < 1$. The efficient technology will be $y = \max\left\{Ak^\alpha, Ak^\beta\right\} = f(k)$.

The convexified economy is defined by $cof(k)$ where co stands for convex-hull. It is the smallest concave function minorized by f. Its epigraph, i.e. the set $\{(k,\lambda) \in R_+ \times R_+ : cof(k) \geq \lambda\}$ is the convex hull of the epigraph of f, $\{(k,\lambda) \in R_+ \times R_+ : f(k) \geq \lambda\}$ (see figure 1). One can check that $cof = f_\alpha$ for $k \in [0, x_1]$, $cof = f_\beta$ for $k \in [x_2, +\infty[$, and affine between x_1 and x_2. More explicitly, we have

$$\alpha A x_1^{\alpha-1} = \beta A x_2^{\beta-1} = \frac{Ax_1^\alpha - Ax_2^\beta}{x_1 - x_2}$$

which implies

$$x_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{1-\beta}{\beta-\alpha}}$$

and

$$x_2 = \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}} \left(\frac{1-\alpha}{1-\beta}\right)^{\frac{1-\alpha}{\beta-\alpha}}.$$

In our economy , the social utility is represented by $\sum_{t=0}^{t=+\infty} \gamma^t u(c_t)$ where γ is the discount factor and c_t the consumption. In period t, this consumption is constrained by the net output $f(k_t) - k_{t+1}$, where k_t denotes the per head capital stock available at date t.

The optimal growth model can be written as

$$\max \sum_{t=0}^{+\infty} \gamma^t u(c_t)$$

under the constraints

$$\forall t \geq 0, c_t \geq 0, k_t \geq 0, c_t \leq f(k_t) - k_{t+1}, \text{ and } k_0 > 0 \text{ is given.}$$

We assume that the utility function u is strictly concave, increasing, continuously differentiable, $u(0) = 0$ and (Inada Condition) $u'(0) = +\infty$. The discount factor γ satisfies $0 < \gamma < 1$.

Let V denote the value-function, i.e.

$$V(k_0) = \max \sum_{t=0}^{+\infty} \gamma^t u(c_t)$$

under the constraints

$$\forall t \geq 0, c_t \geq 0, k_t \geq 0, c_t \leq f(k_t) - k_{t+1}, \text{ and } k_0 \geq 0 \text{ is given.}$$

Remark 1: Before proceeding the analysis, we wish to say that our technology specification used for aggregation purpose in this paper is not restrictive. Indeed, consider the following production function $f(k) = \max\{Ak^\alpha, Bk^\beta\}$, with $A \neq B$. Define $\widetilde{k} = \frac{k}{\lambda}$, $\widetilde{c} = \frac{c}{\lambda}$, $v(c) = u(\frac{c}{\lambda})$, where λ satisfies $A\lambda^\alpha = B\lambda^\beta$. Let $A' = A\lambda^\alpha = B\lambda^\beta$. It is easy to check that the original optimal growth model behind becomes

$$\max \sum_{t=0}^{+\infty} \gamma^t v(\widetilde{c}_t)$$

under the constraints

$$\forall t \geq 0, \widetilde{c}_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \leq \widetilde{f}(\widetilde{k}_t) - \widetilde{k}_{t+1}, \text{ and } \widetilde{k}_0 > 0 \text{ is given.};$$

where $\widetilde{f}(x) = \max\{A'x^\alpha, A'x^\beta\}$.

3 Analysis of the optimal growth paths

The preliminary results are summarized in the following proposition.

Proposition 1 *(i) For any $k_0 \geq 0$, there exists an optimal growth path $(c_t^*, k_t^*)_{t=0,\ldots,+\infty}$ which satisfies:*

$$\forall t, 0 \leq k_t^* \leq M = \max\left[k_0, \widetilde{k}\right], 0 \leq c_t^* \leq f(M),$$

where $\widetilde{k} = f(\widetilde{k})$.

(ii) If $k_0 > 0$, then $\forall t, c_t^* > 0, k_t^* > 0, k_t^* \neq 1$, and we have Euler equation

$$u'(c_t^*) = \gamma u'(c_{t+1}^*) f'(k_{t+1}^*).$$

(iii) Let $k_0' > k_0$ and $(k_t'^*)$ be an optimal path associated with k_0'. Then we have: $\forall t, k_t'^* > k_t^*$.

(iv) The optimal capital stocks path is monotonic and converges to an optimal steady state. Here, this steady state will be either $\widehat{k}_\alpha = (\gamma A \alpha)^{\frac{1}{1-\alpha}}$ or $\widehat{k}_\beta = (\gamma A \beta)^{\frac{1}{1-\beta}}$.

Proof. (i) The proof of this statement is standard and may be found in Le Van and Dana (2003), chapter 2. (ii) From Askri and Le Van (1998), the value-function V is differentiable at any $k_t^*, t \geq 1$. Moreover, $V'(k_t^*) = u'(f(k_t^*) - k_{t+1}^*) f'(k_t^*)$ and this excludes that $k_t^* = 1$ since 1 is the only point where f is not differentiable. From Inada Condition, we have $c_t^* > 0, k_t^* > 0, \forall t$. Hence, Euler Equation holds for every t.

(iii) It follows from Amir (1996) that $k_0' > k_0$ implies $\forall t, k_t'^* > k_t^*$. From Euler Equation we have

$$u'(f(k_0) - k_1^*) = \gamma V'(k_1^*)$$

and

$$u'(f(k_0') - k_1^*) = \gamma V'(k_1'^*).$$

If $k_1^* = k_1'^*$ then $k_0 = k_0'$: a contradiction. Hence, $k_1^* < k_1'^*$. By induction, $\forall t > 1, k_t'^* > k_t^*$.

(iv) First assume $k_1^* > k_0$. Then the sequence $(k_t^*)_{t \geq 2}$ is optimal from k_1^*. From (iii), we have $k_2^* > k_1^*$. By induction, $k_{t+1}^* > k_t^*, \forall t$. If $k_1^* < k_0$, using the same argument yields $k_{t+1}^* < k_t^*, \forall t$. Now if $k_1^* = k_0$, then the stationary sequence $(k_0, k_0, ..., k_0, ...)$ is optimal.

We have proved that any optimal path (k_t^*) is monotonic. Since, from (1), it is bounded, it must converge to an optimal steady state k^s. If this one is different from zero, then the associated optimal steady state consumption c^s must be strictly positive from Inada Condition. Hence, from Euler Equation, either $k^s = \widehat{k}_a$ or $k^s = \widehat{k}_b$ since it could not equal 1.

It remains to prove that (k_t^*) cannot converge to zero. On the contrary, for t large enough, say greater than some T, we have $u'(c_t^*) > u'(c_{t+1}^*)$ since $f'(0) = +\infty$. Hence, $c_{t+1}^* > c_t^*$ for every $t \geq T$. In particular, $c_{t+1}^* > c_T^* > 0$, $\forall t > T$. But $k_t^* \to 0$ implies c_t^* : a contradiction. ■

We obtain the following corollary:

Corollary 2 *If $\gamma A\alpha > 1$, then any optimal path from $k_0 > 0$ converges to $\widehat{k}_\beta$. If $\gamma A\beta < 1$, then any optimal path from $k_0 > 0$ converges to $\widehat{k}_\alpha$.*

Proof. In Proposition 1, we have shown that any optimal path (k_t^*) converges either to $\widehat{k}_\alpha$ or to $\widehat{k}_\beta$. But when $\gamma A\alpha > 1$, we have $\widehat{k}_\alpha > 1$, $f(\widehat{k}_\alpha) = A(\widehat{k}_\alpha)^\beta$ and $f'(\widehat{k}_\alpha) = \beta A(\widehat{k}_\alpha)^{\beta-1} \neq \frac{1}{\gamma}$. Consequently, $\widehat{k}_\alpha$ could not be an optimal steady state. Therefore, (k_t^*) cannot converge to $\widehat{k}_\alpha$. From the statement (iv) in Proposition 1, it converges to $\widehat{k}_\beta$.

Similarly, when $\gamma A\beta < 1$, any optimal path from $k_0 > 0$ converges to $\widehat{k}_\alpha$.
■

In Figure 1, when $\widehat{k}_\alpha \geq 1, \alpha-$ technology is clearly less efficient than $\beta-$technology, thus $\widehat{k}_\alpha$ is not the optimal steady state. Similarly for $\widehat{k}_\beta \leq 1$. In these cases, there will be an unique optimal steady state. But when the discount factor is in an intermediate range defined by $\gamma A\alpha \leq 1 \leq \gamma A\beta$, there exists more than one such state. We now give an example where $\widehat{k}_\alpha$ and $\widehat{k}_\beta$ are both optimal. Since x_1 and x_2 are independent of A and γ, we can choose A and γ such that

$$\alpha A x_1^{\alpha-1} = \beta A x_2^{\beta-1} = \frac{1}{\gamma}, \text{ with } 0 < \gamma < 1.$$

It is easy to check that x_1 and x_2 are optimal steady states for the convexified technology and hence for our technology. Since $x_1 = \widehat{k}_\alpha$, $x_2 = \widehat{k}_\beta$, we have found two positive optimal steady states.

Let now $\widehat{k}_\alpha$ and $\widehat{k}_\beta$, depicted in Figure 1, be two optimal steady states and ask the question which of them will be the long run equilibrium in the optimal growth model. We first get an immediate result in:

Proposition 3 *Assume $\gamma A\alpha \leq 1 \leq \gamma A\beta$. If $\gamma A\alpha$ is close to 1, then any optimal path (k_t^*) from $k_0 > 0$ converges to $\widehat{k}_\beta$. If $\gamma A\beta$ is close to 1, then (k_t^*) converges to $\widehat{k}_\alpha$.*

Proof. First, observe that when $\gamma A\alpha \leq 1$ then $f(\widehat{k}_\alpha) = A\left(\widehat{k}_\alpha\right)^\alpha$ and when $1 \leq \gamma A\alpha$, $f(\widehat{k}_\beta) = A\left(\widehat{k}_\beta\right)^\beta$. Now consider the case $\gamma A\alpha = 1 < \gamma A\beta$. We have $\widehat{k}_\alpha = 1$ and $A > 1$.

It is well-known that given $k_0 > 0$, there exists a unique optimal path from k_0 for the β-technology. Moreover, this optimal path converges to $\widehat{k}_\beta$.

Observe that the stationary sequence $\left(\widehat{k}_\alpha, \widehat{k}_\alpha, ..., \widehat{k}_\alpha, ...\right)$ is feasible from $\widehat{k}_\alpha$, for the β-technology, since it satisfies $0 \leq \widehat{k}_\alpha = 1 < A\left(\widehat{k}_\alpha\right)^\beta = A$. Hence, if $\left(\widetilde{k}_t\right)$ is an optimal path for β-technology starting from $\widehat{k}_\alpha$ and if (k_t) is an optimal path of our model starting also from $\widehat{k}_\alpha$, we will have

$$\sum_{t=0}^{\infty} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) < \sum_{t=0}^{\infty} \gamma^t u(f(\widetilde{k}_t) - \widetilde{k}_{t+1}) \leq \sum_{t=0}^{\infty} \gamma^t u(f(k_t) - k_{t+1}) = V(\widehat{k}_\alpha).$$

That shows that $\widehat{k}_\alpha$ can not be an optimal steady state. Hence, any optimal path from $k_0 > 0$ must converge to $\widehat{k}_\beta$.

Since $\widehat{k}_\alpha$ is continuous in γ, V continuous and since $\sum_{t=0}^{\infty} \gamma^t u(f(\widehat{k}_\alpha) - \widehat{k}_\alpha) < V(\widehat{k}_\alpha)$ when $\gamma A\alpha = 1$, this inequality still holds when $\gamma A\alpha$ is close to 1 and less than 1. In other words, $\widehat{k}_\alpha$ is not an optimal steady state when $\gamma A\alpha$ is close to 1 and less than 1. Consequently, any optimal path with positive initial value will converge to $\widehat{k}_\beta$.

Similar argument applies when $\gamma A\beta$ is near one but greater than one. ∎

What then happens when the discount factor is within an intermediate range? We now would like to show :

Proposition 4 *Assume $\gamma A\alpha < 1 < \gamma A\beta$. If both $\widehat{k}_\alpha$ and $\widehat{k}_\beta$ are optimal steady states then there exists a critical value k_c such that every optimal path from $k_0 < k_c$ will converge to $\widehat{k}_\alpha$, and every optimal path from $k_0 > k_c$ will converge to $\widehat{k}_\beta$.*

Proof. Consider at first $k_0 < \widehat{k}_\alpha$. Since $\widehat{k}_\alpha$ is optimal steady state, we have $k_t^* < \widehat{k}_\alpha$, $\forall t > 0$. Since the sequence (k_t^*) is increasing, bounded from above by $\widehat{k}_\alpha$, it will converge to $\widehat{k}_\alpha$. Similarly, when $k_0 > \widehat{k}_\beta$, any optimal path converges to $\widehat{k}_\beta$.

Let $\overline{k} = \sup \left\{k_0 : k_0 \geq \widehat{k}_\alpha\right\}$ such that any optimal path from k_0 converges to $\widehat{k}_\alpha$. Obviously, $\overline{k} \leq \widehat{k}_\beta$, since $\widehat{k}_\beta$ is optimal steady state.

Let $\underline{k} = \inf \left\{k_0 : k_0 \leq \widehat{k}_\beta\right\}$ such that any optimal path from k_0 converges to $\widehat{k}_\beta$. Obviously, $\underline{k} \geq \widehat{k}_\alpha$, since $\widehat{k}_\alpha$ is optimal steady state.

We claim that $\overline{k} = \underline{k}$.

It is obvious that $\overline{k} \leq \underline{k}$. Now, if $\overline{k} < \underline{k}$, then take k_0, k_0' which satisfy $\overline{k} < k_0 < k_0' < \underline{k}$. From the definitions of $\overline{k}$ and $\underline{k}$, there exist an optimal path from $k_0, (k_t^*)$, which converges to $\widehat{k}_\beta$ and an optimal path from $k_0', (k_t'^*)$, which converges to $\widehat{k}_\alpha$. For t large enough, $k_t'^* < k_t^*$, which is impossible since $k_0 < k_0'$ (see Proposition 1, statement (iii)).

Posit $k_c = \overline{k} = \underline{k}$ and conclude. ■

Remark 2: The existence of critical value is standard since the paper by Dechert and Nishimura (1983). See also, for the continuous time setting. Askenazy and Le Van (1999). But in these models, the technology is convex-concave. The low steady state is unstable while the high is stable. An optimal path converges either to zero or to the high steady state. In our model, with a technology, say concave-concave, any optimal path converges either to the high steady state or the low steady state.

4 Concluding comments

It is shown in this paper that when future discounting is high enough, precisely when $\gamma A \beta < 1$, the resulting long run equilibrium is the optimal steady state $\widehat{k}_\alpha$. For any value of the initial capital stock, the convergence to this equilibrium is monotonic. On the other hand, when future discounting is relatively low, precisely when $\gamma A \alpha > 1$, the same result will be obtained but with the equilibrium optimal steady state $\widehat{k}_\beta$. When future discounting is in some middle rang, i.e. when $\gamma A \alpha < 1 < \gamma A \beta$, there might exist two optimal steady states and the dynamic convergence will depend on the initial stock of capital. We show that there is a critical capital stock k_c such that every optimal paths from $k_0 < k_c$ will converge to $\widehat{k}_\alpha$, and every optimal paths from $k_0 > k_c$ will converge to $\widehat{k}_\beta$.

Several useful remarks can be made. First, it is conceivable that the results obtained in this paper are unaffected when either one or both production technologies entails some fixed costs, i.e. positive output is made possible only if the capital per capita exceeds a threshold level, but their aggregation exhibits the kind of non-convexity depicted in Figure 1. Second, for the economist-statisticians, this paper hopefully highlights the importance of informations other than those contained in the technology-blueprint book. Under either high or low future discounting, only one technology is relevant in the sense that it is the chosen technology in long run equilibrium.

This certainly helps identifying the production function for data aggregation task. If the future discount rate falls in a range defined by $\gamma A\alpha < 1 < \gamma A\beta$, then the computation of the critical capital stock k_c is essential in view of the determination of the relevant production technology at stake. Third, when there are several production technologies, it is possible to proceed with pair-wise aggregation in order to determine the relevant technology for long run equilibrium. Assume that we have a third technology, say the ε- technology, to take into account. Pair-wise aggregation of α and β-technology allows us to eliminate the α- technology, say. Therefore, we now have to perform the same analysis with β - technology and ϵ-technology, and so on so forth when several technologies are at stake. Pair-wise consideration in this way would help determining the relevant technology corresponding to the optimal steady state. Fourth, under the regular conditions of concavity in Ramsey model, the long run equilibrium could be achieved with a decentralized market mechanism. Recall that non-convexity is thought to be the main cause for market failure. But in the case we consider in this paper, the economy attains one (and only one) long run equilibrium corresponding to a well-behaved concave production technology. Therefore, despite the non-convexity arising from technology aggregation, there is no market failure and decentralized allocation will indeed implement the Modified Golden Rule State.

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