The Gravitation of Market Prices as a Stochastic Process

Saverio M. Fratini and Alessia Naccarato

Università di Roma Tre

Abstract

The theory of value has been based ever since Adam Smith on the idea that the market prices of commodities, those at which actual trade takes place, gravitate around a central position known as natural prices. This paper seeks to develop a statistical idea of the process in question and suggests in particular that market prices can be said to gravitate around natural prices if the probability of their means being very close to natural prices after t observations tends to 1 as t tends to infinity. A set of possible conditions leading to that result is also presented.

JEL codes: B51, C32, C62, D46

Keywords: market price, natural price, gravitation, random variables, consistency.

1. Introduction

In explaining the concept of price he was adopting in *Production of Commodities by Means of Commodities* (1960), Sraffa wrote that '[s]uch classical terms as "necessary price", "natural price" or "price of production" would meet the case, but value and price have been preferred as being shorter and in the present context (which contains no reference to market prices) no more ambiguous' (Sraffa, 1960, p. 9).

In this passage at least, Sraffa thus refers explicitly to the distinction between two different conceptions of price, namely natural price and market price, which has certainly been adopted in the theory of value since Adam Smith but probably for even longer.

The study of market prices, the prices at which commodities are actually sold at a given moment, cannot be addressed theoretically because they can be affected by so many circumstances that every specific combination of them actually occurring is a sort of unique and unrepeatable phenomenon. The market price of a commodity, understood as the price at which trade actually takes place, can only be observed and studied *ex-post*, from a historical rather than theoretical viewpoint.

At the same time, however, according to Smith's analysis, the market price of each commodity tends to 'gravitate' around a central position, which constitutes the 'natural' or 'normal' price of that commodity. The theory of value is therefore concerned with investigating the determinants of natural prices, and this sort of investigation is exactly what Sraffa intended to carry out in his book, which accordingly 'contains no reference to market prices'.

Attention will instead be focused here on the idea that market prices gravitate around natural prices and, in particular, on different possible conceptions of the same. Our starting point (sec. 2) will clearly be the analyses of Smith and the Classical economists, but since these have already been studied extensively, it will suffice to recall some key features here.

Section 3 will then consider the way in which the gravitation of market prices has mostly been addressed since the publication of Sraffa's book, namely through the construction of dynamical models where the market prices obtaining at any date are understood not as actual prices but as states of a dynamical system generated by some differential equations (or difference equations) starting from a given initial state.[1] Once the market price trajectory is determined, the gravitation consists in its tendency to move towards the natural position.

Though interesting for various reasons, this kind of analysis cannot be considered fully convincing or satisfactory because the results obtained are inevitably dependent on the validity of the special assumptions upon which the model is built. As shown in sec. 4, a different way of understanding gravitation, based on a 'statistical concept of equilibrium', was therefore introduced by Parrinello (1990).

This paper constitutes an attempt to develop Parrinello's idea. In particular, since the market prices obtaining at a certain date can be treated as random variables, their means after t observations are random variables as well. The assertion is thus put forward in section 5 that market prices gravitate around natural prices if the probability of their means being very close to natural prices after t observations tends to 1 as t tends to infinity. Section 6 then presents a set of possible conditions leading to that result.

[1] For basic concepts about dynamical systems and their use in economics, see Frisch (1936).

2. The classical economists' view of gravitation

The starting point for the study of the gravitation of market prices around their natural or normal level is unquestionably the analysis that Smith developed primarily in his *Lectures on Jurisprudence* and *Wealth of Nations* (1976, vols. 2, 3 and 6). Since a great deal of attention has already been focused on these passages in Smith's works,[2] we shall confine ourselves here to recalling only a few key points.

As Smith wrote, '[o]f every commodity there are two different prices, which tho' apparently independent will be found to have a necessary connection, viz. the natural price and the market price' (Smith, 1979, vol. 6, p. 494 – LJ(B) 224).

The natural price of a commodity is 'what is sufficient to pay the rent of the land, the wages of the labour, and the profits of the stock employed in raising, preparing, and bringing it to market, according to their natural rates' (Smith, 1979, vol. 2, p. 72 – WN I.vii.4). In other words, if the prices of commodities are at their natural levels, then labourers with the same skills, land of the same quality and capital can respectively receive the same rates of wages, rent and profit regardless of the sphere of activity in which they are employed.[3]

As some have argued (see in particular Vianello, 1989, p. 99), natural prices can also be viewed as necessary prices in the sense that there is no incentive for workers, landowners or capitalists to change their sector of employment *only if* commodity prices are at their natural level. Natural prices are therefore necessary, under the hypothesis of free competition, for the physical composition of output in the economic system to be in a position of rest.

The market price is instead '[t]he actual price at which any commodity is commonly sold' and 'is regulated by the proportion between the quantity which is actually brought to market, and the demand of those who are willing to pay the natural price of the commodity', i.e. 'the effectual demand' (Smith, 1979, vol. 2, p. 73 – WN I.vii.7, 8).

It is therefore evident that the natural price and the market price of a commodity are different in nature and depend on different forces: (a) the former is a theoretical variable and the latter an actual (observed) magnitude; (b) the former depends – for a given technique – on the 'ordinary or average' rates of wages, rents and profits, which in turn depend on 'the general circumstance of the society' (Smith, 1979, vol. 2, p. 72 – WN I.vii.1, 2), whereas the

[2] See in particular, for a textual analysis of Smith's observations on market prices and the process of gravitation, Aspromourgos (2007) and (2009).

[3] According to Aspromourgos (2007, p. 29), '[i]n latter-day terms, natural price is a notion of opportunity cost: it is the price which just enables payment to the owners of the employed inputs, the remuneration normally available in alternative uses.'

determination of the latter is a market phenomenon based, according to Smith, on the quantity actually brought to market and effectual demand. Regardless of how 'seemingly independent they appear to be', however, the natural price and the market price 'are necessarily connected' (Smith, 1979, vol. 6, p. 496 – LJ(B) 229).

This connection is based on competition, which plays a dual role. In the first place, competition among buyers (or sellers) causes the market price to rise above (or fall below) the natural price when the produced quantity of a commodity is below (or above) the effectual demand. Because of the discrepancy between natural and market prices, the rates at which labour, land and capital are paid cannot be uniform across sectors. As a result, labour, land and capital flow from one industry to another in search of higher rates of remuneration. There is thus a second mechanism of competition which ensures that the rates of wages, rent and profits in different sectors tend to balance.

Due to competition, the quantities produced thus tend to adjust to the effectual demand for commodities and the market prices of commodities 'gravitate' around the natural prices at the same time (cf. Smith, 1976, vol. 2, p. 75 – WN I.vii.15).

In describing this process, however, Smith initially seems to place the three distribution variables on the same level. In particular, he says that if the market price of a commodity is above its natural price, then at least one of the three component parts of its price must be above the amount calculated at its natural or ordinary level:

> [i]f it is rent, the interest of all other landlords will naturally prompt them to prepare more land for the raising of this commodity; if it is wages or profit, the interest of all other labourers and dealers will soon prompt them to employ more labour and stock in preparing and bringing it to market. [Smith, 1976, vol. 2, p. 75 – WN I.vii.13]

He then goes on, however, to add that:

> [t]he occasional and temporary fluctuations in the market price of any commodity fall chiefly upon those parts of its price which resolve themselves into wages and profit. That part which resolves itself into rent is less affected by them. [Smith, 1976, vol. 2, p. 76 – WN I.vii.18]

Without entering into the details of Smith's argument about rent, we can say that what matters here is that the rates of remuneration of land, labour and capital can respond to the fluctuations of market prices differently. This opens the door to a further step taken by

Ricardo and Marx.[4] Since capitalists are usually entrepreneurs, either directly or indirectly, the deviation of market prices from natural ones is reflected primarily in the rates of profit obtained in different activities,[5] which diverge from the natural or ordinary rate of profits.[6]

Under this specification, the second mechanism of competition mentioned above turns into the competition of capital for the most profitable employment. In Ricardo's words: [7]

> [i]t is then the desire, which every capitalist has, of diverting his funds from a less to a more profitable employment, that prevents the market price of commodities from continuing for any length of time either much above, or much below their natural price. It is this competition which so adjusts the exchangeable value of commodities, that after paying the wages for the labour necessary to their production, and all other expenses required to put the capital employed in its original state of efficiency, the remaining value or overplus will in each trade be in proportion to the value of the capital employed [Ricardo, 1951, vol. 1, p. 91].

This is indeed the view that has prevailed since Smith. In particular, it is also the way, as shown in the following section, in which the allocation of productive agents amongst sectors and the resulting adjustment of produced quantities are considered within the current analysis of gravitation.

3. Gravitation as a dynamical model: a brief overview

As stated above, market prices are the prices at which commodities are actually sold. Unlike the Walrasian mechanism of *tâtonnement*,[8] for example, classical gravitation is therefore

---

[4] For the comparison of Smith's ideas about gravitation with those of Ricardo and Marx, see in particular Vianello (1989), Salvadori & Signorino (2013) and Signorino (2012).

[5] The capitalist, as the organizer of production, hires workers and leases land. The wage rate and the rent rate are, consequently, fixed by contract and therefore -- unless they are 'indexed' in some specific way -- can be varied only by signing new contracts. Moreover, following this reconstruction, the bargaining of wage and rent rates takes place before the market prices of the commodities produced by the employment of those workers and that land are known.

[6] Moreover, the assumption that the divergence of market prices from natural prices entails obtained rates of profit differing from the ordinary rate (while wages and rent remain at their natural levels) is a useful simplification making it possible to resolve the indeterminacy of the effect of a certain system of market prices on distribution variables.

[7] Similar claims can be found in Marx: 'competition levels the rates of profit of the different spheres of production into an average rate of profit […]. This is accomplished by continually transferring capital from one sphere to another, in which the profit happens to stand above the average for the moment. […] These incessant emigrations and immigrations of capital, which take place between the different spheres of production, […] create a tendency to reduce the rate of profit everywhere to the same common and universal level.

'This movement of capitals is caused primarily by the stand of the market-prices, which lift profits above the level of the universal average in one place and depress them below it in another' [Marx, 1909, vol. 3, p. 243].

assumed to take place in the real world; it is an actual phenomenon and not merely a theoretical construction.

Since the market phenomena occurring day by day in the real world are so complex as to be unpredictable,[9] it is impossible to pinpoint the forces governing the determination of market prices with the same degree of generality as in the case of natural prices. The classical economists appear in particular to have been well aware that market prices cannot be addressed theoretically, 'except for the *sign* of their deviation from their natural or normal levels' (Garegnani, 2002, p. 393).[10]

As in every science, however, a great deal of the real world's complexity can be eliminated by constructing a model in which the central phenomena alone are represented. This is in fact the approach adopted by most scholars in studying gravitation, and the literature thus contains a variety of models. A number of their key features are presented here, but a complete survey lies decidedly beyond the scope of this paper.

Let us consider a world with N commodities labelled n = 1, 2, ..., N, no joint production, no alternative techniques and no natural resources.

The technical coefficients of production are arranged in a matrix A and a (column) vector $\ell \in \mathbb{R}^N_{++}$, in which a_{nj} and ℓ_n respectively represent the amount of commodity j and the amount of labour for the production of each unit of commodity n.[11] The 'final' effectual demand, expressed by the (row) vector $y \in \mathbb{R}^N_{++}$, is taken as given, so that the 'gross' effectual demand is $q = y \cdot (\mathrm{I} - A)^{-1}$. As in Sraffa (1960, p. 10), it is assumed that $q \cdot \ell = 1$.

With the composite commodity y adopted as the numéraire (cf. Sraffa, 1960, p. 11) and given a wage rate (or share) w, the (column) vector[12] of natural prices π is determined together with the natural or ordinary rate of profits r by means of the usual Sraffian system:

[8] For some comparisons of classical and neo-classical theories of value and distribution with specific reference to gravitation and price adjustments, see also Ciccone (1999), Garegnani (2002) and Fratini & Levrero (2011).

[9] If this were not so, i.e. if the prices determined for example in a trading session at the Chicago Mercantile Exchange were predictable, no speculative transaction would ever be possible.

[10] A similar view is expressed by Salvadori & Signorino (2013): 'Smith devotes much care to determining natural values and to the gravitation process of market magnitudes to their natural counterparts. The same cannot be maintained as regards the question of market price determination' (p. 161); and yet 'while the classical authors extensively investigated long-period, natural values and gravitation, they were more sketchy on market price determination in situations of market disequilibrium.' (p.170).

[11] As all the means of production are circulating capital goods and there is no fixed capital in this model, no question connected with the degree of capacity utilisation emerges in the case considered here. See Ciccone (2011) for a study of the possibility of variations in the degree of capacity utilisation during the process of gravitation.

[12] In this paper, as a general rule, vectors of quantities are row vectors and vectors of prices are column vectors. Row and column vectors can in any case be distinguished quite easily on the basis of context.

$$\pi = (1+r)A \cdot \pi + w\ell \qquad [1]$$

$$y \cdot \pi = 1 \qquad [2]$$

If an arbitrary vector of market prices $\pi_t = [\pi_{1,t}, \pi_{2,t}, \ldots, \pi_{N,t}]$ is instead taken, with $y \cdot \pi_t = 1$, the rate of profits will not be uniform across sectors. A 'market rate of profits'[13] can then be defined for each sector:

$$r_{n,t} = \frac{\pi_{n,t} - \sum a_{n,j}\pi_{j,t} - \ell_n w}{\sum a_{n,j}\pi_{j,t}} =: r_n(\pi_t)\ ;\ \forall\ n = 1, 2, \ldots, N. \qquad [3]$$

There is thus, in every period, a vector of market rates of profits $r_t(\pi_t) = [r_{1,t}(\pi_t), r_{2,t}(\pi_t), \ldots, r_{N,t}(\pi_t)]$ associated with the market price vector π_t.

Now, as Adam Smith argued, given the effectual demand for commodities, the deviation of market prices from natural prices depends on the quantities actually produced. Let $q_t = [q_{1,t}, q_{2,t}, \ldots, q_{N,t}]$, with $q_t \cdot \ell = 1$, be the vector of produced quantities in period t. It is thus possible to write:

$$\pi_t = \pi + \phi(q_t) \qquad [4]$$

where $\phi(.)$ is assumed to be a continuous function such that: i) $\phi(q) = [0, 0, \ldots, 0]$ and ii) $y \cdot \phi(q_t) = 0\ \forall\ q_t : q_t \cdot \ell = 1$. This means that: i) market prices correspond to normal prices when the quantities brought to market equal the effective demand; ii) market relative prices are expressed in terms of the same numéraire as normal relative prices, i.e. $y \cdot \pi_t = y \cdot \pi = 1$.

As regards the quantities produced and brought to market, according to the classical idea of gravitation, they depend on the sectoral rates of profits $r_t(\pi_t)$. In particular, it is possible to write:

$$q_{t+1} = q_t + \psi[r_t(\pi_t)] \qquad [5]$$

[13] The expression 'market rate of profits' is used by Garegnani (1990, p. 334) to denote the rate of profits obtainable in a sector in a given 'market position', i.e. if the market prices emerging in period t were assumed to persist in the following periods too. This market rate of profits is what is usually regarded in the literature on gravitation as the sectoral rate of profits when the prices are not natural prices. It cannot, however, be viewed as the rate of profits actually obtained in period t. The rate of profits of sector n obtained in period t depends in actual fact both on the market prices π_t, at which the outputs are sold, and on the market prices π_{t-1}, at which the means of production were paid. Therefore: $r_{n,t}(\pi_{t-1}, \pi_t) := (\pi_{n,t} - \Sigma\, a_{n,j} \cdot \pi_{j,t-1} - \ell_n \cdot w) / \Sigma\, a_{n,j} \cdot \pi_{j,t-1}$. See also: Lager (1998) and Bellino (2011, pp. 63-4).

where $\psi(.)$ is assumed to be a continuous function such that: i) $\psi(r) = [0, 0, ..., 0]$ and ii) $\psi(r_t)\cdot\ell = 0\ \forall\ r_t$. In other words, i) produced quantities are stationary if the rate of profits is uniform across sectors, i.e. $r_{n,t} = r\ \forall\ n = 1, 2, ..., N$, and ii) the total employment of labour does not change during the process[14] but $q_{t+1}\cdot\ell = q_t\cdot\ell = 1$.

Specification of the functional forms $\phi(.)$ and $\psi(.)$ gives a model describing the dynamics of the variables π_t and q_t. Therefore, given an arbitrary initial vector of produced quantities q_0, the system formed by difference equations [4] and [5] makes it possible to determine prices and quantities at every moment of time. In particular, a solution of the system (if it exists) can be denoted as $\pi(q_0,t)$ and $q(q_0,t)$.

Because of the assumption made, the natural prices π and the quantities corresponding to the effectual demand for commodities q constitute an equilibrium (or a stationary state) in the sense of rational mechanics for the dynamical system considered here. In fact, as can be easily proved, when the initial vector of produced quantities is set equal to the effectual demand for commodities, i.e. $q_0 = q$, neither the quantities nor the prices change over time and, in particular, $\pi(q,t) = \pi$ and $q(q,t) = q\ \forall\ t \geq 0$. It becomes possible at this point to study the gravitation of market prices around their natural position in terms of equilibrium (asymptotic) stability. In other words, the natural prices π are regarded as a 'centre of gravitation' for market prices if $\pi(q_0,t) \to \pi$ as $t \to \infty$, for every q_0.

There is no need here to go any further along this path, as it has already been widely explored. We shall therefore just recall that while it is possible to construct models in which the natural position is an unstable equilibrium, there are reasonable assumptions bringing about equilibrium stability. Readers are referred to Boggio (1990) and Bellino (2011) for an overview of these results.[15]

4. Random disturbances and expectations in Parrinello's contribution

When the complexity of the real world is addressed by means of a simplified model, this unquestionably leads to an error of approximation. In order to evaluate the significance of the

[14] As stressed by Garegnani (1990, p. 332), 'the *aggregate* economic activity (on which the effectual demand of the individual commodities evidently depend) can be taken as given in analysing market prices'. Garegnani therefore assumed the level of aggregate labour employment as constant in his paper, just as we are doing here.
[15] For a critical analysis of gravitation models, see also Dupertuis and Sinha (2009b).

error made, it is necessary to know what aspects are being neglected and how important they may prove in explaining the phenomenon in question. In this respect, the main problem with dynamical models of gravitation is the fact that it is impossible to specify with a sufficient degree of generality all the forces and mechanisms that may be involved in the actual determination of market prices and therefore impossible to have a complete picture of what is being overlooked.

It is therefore essentially impossible to evaluate the error precisely. It could be extremely serious or insignificant. In both cases, according to Parrinello (1990), there are logical problems:

> [i]f we admit that the disequilibrium model also contains some error of specification, then this model is also unable to explain market prices exactly. A serious cumulation of errors over a sequence of rounds or iterations might make the test not reliable. […] By contrast, if we believe that the model describes with precision the dynamics of the system and it "proves" that gravitation of market prices toward production prices exists and has the properties required, the method of long period states would become legitimate, but, at the same time, useless. In fact, should this stage be achieved, we would resort directly to the "perfect" disequilibrium model and the method of approximation based on attractors should be dismissed as a non necessary approximation. (p. 114)

Parrinello therefore suggests that the gravitation of market prices around the natural position should be addressed from another angle. Instead of tackling gravitation with the tools of the rational mechanics, those used in the neoclassical theory of equilibrium, and studying gravitation as the rest position of a dynamical process, he suggests that the revival of the classical approach can benefit from the adoption of a 'statistical concept of equilibrium' (p. 115) whereby market prices and quantities are regarded as random variables and gravitation as a stochastic process.

Since Parrinello's contribution constitutes the starting point of our analysis, as presented in the next two sections, it should be recalled here before proceeding any further.

In his *Wealth of Nations*, Adam Smith identifies two different kinds of random shocks that may affect the gravitation process. First, there are 'accidental variations in the demand' that can alter the market prices of commodities, as in the case of public mourning leading to a rise in the price of black cloth (cf. Smith, 1976, vol. 2, pp. 76 and 132 – WN I.vii.19 and I.x.b.46). Second, there are variations in the quantity obtained by the same employment of productive agents because of seasonality: '[t]he same quantity of industry, for example, will, in different years, produce very different quantities of corn, wine, hops, sugar, tobacco, etc' (Smith, 1976, vol. 2, pp. 132, 4 – WN I.x.b.46).

Parrinello therefore includes two random disturbances in his analysis – namely $\mu_{n,t}$ and $\varepsilon_{n,t}$, both assumed to have zero mean and to be normally distributed and serially uncorrelated – in order to take into account the effects of these accidental shocks on the market price and the produced quantity of a certain commodity. Moreover, instead of considering the quantity actually produced in t as related to market prices in t – 1, Parrinello suggests that reference should be made to the market price *expected* by entrepreneurs for period t.

In particular, referring to a certain commodity n and using the same symbols as in the previous section, Parrinello's gravitation model is made up of the following equations:

$$\text{sign}\ (\pi_{n,t} - \pi_n - \mu_{n,t}) = -\ \text{sign}\ (q_{n,t} - q_n) \qquad [6]$$

$$\text{sign}\ (q_{n,t} - q_n - \varepsilon_{n,t}) = \text{sign}\ (\pi_{n,t}^e - \pi_n). \qquad [7]$$

The meaning of the equations is quite simple: i) the direction of deviations of market prices from the natural level depends both on the sign of the deviation of the quantity brought to market from the effectual demand and on the effect of the random shock on demand; ii) the sign of the deviation of the quantity brought to market from the effectual demand depends in turn on the sign of the gap ($\pi_{n,t}^e - \pi_n$) between expected and natural prices. There is, however, also the effect of seasonality.

Since $E[\mu_t] = E[\varepsilon_t] = 0$, equations [6] and [7] entail, as Parrinello points out, that:

$$\text{sign}\ (E[\pi_{n,t}] - \pi_n) = -\ \text{sign}\ (\pi_{n,t}^e - \pi_n). \qquad [8]$$

Within this model, it is therefore possible to obtain $\pi_{n,t}^e = E[\pi_{n,t}]$ – i.e. rational expectations – if and only if $\pi_{n,t}^e = E[\pi_{n,t}] = \pi_n$. This is the primary conclusion: the theoretical level of price corresponds to what the agents must expect in order to be rational.

5. Market prices and quantities as random variables

Parrinello's contribution has the unquestionable merit of introducing two new elements into the debate on gravitation. First, market prices and the quantities actually produced are to be treated as random variables, which is in fact precisely how magnitudes that cannot be exactly predicted are addressed. Second, decisions about the quantities to bring to market are based on expectations about future prices and hence on sectoral rates of profits.

Let us take this second element as our starting point. In the usual dynamical models of gravitation, as seen in sec. 3, the quantities actually brought to market in period t depend on the market prices obtaining in period $t - 1$. Parrinello instead regards these decisions as based on expected prices rather than past prices. In particular, he defines the expected price of a commodity $\pi_{n,t}^e$ (with $n = 1, 2, \ldots, N$) as 'the price which, on average, the producers expect to rule at time t on the basis of their information at time $t - 1$' (Parrinello, 1990, p. 117).[16]

Under the 'law of unique price' or some other principle,[17] there is just one price for each commodity on the market at a certain date. If different agents have different arbitrary beliefs, however, then there are as many different expected prices for the same commodity, in the same moment, as producers. In this case, it is therefore impossible to write anything resembling either the difference equation [5] or Parrinello's sign condition [7]. We are instead forced to admit that since individual decisions are essentially unpredictable and since the quantities of commodities actually produced are the result of many individual decisions, nothing much can be said about them.[18]

It can just be said that in our analysis, the quantities q_t are random vectors – not only because of seasonality but also for deeper reasons – that take values in the set $Q = \{q_t \in \mathbb{R}^N_+ : q_t \cdot \ell = 1\}$. Moreover, if it is assumed that producers, in the aggregate, do not make systematic errors, then $E[q_t] = q$.

As regards market prices, the previously introduced equation [4] can be taken as a starting point with the inclusion on its RHS of a vector of random disturbance $\mu_t = [\mu_{1,t}, \mu_{2,t}, \ldots, \mu_{N,t}]$ such that i) $\mu_{n,t}$ is white noise, $\forall\ n = 1, 2, \ldots, N$ and ii) $y \cdot \mu_t = 0$. This gives us:

$$\pi_t = \pi + \phi(q_t) + \mu_t. \qquad [9]$$

---

[16] Parrinello assumes in particular that producers have the same information in $t - 1$, so that $\pi_{n,t}^e = E[\pi_{n,t}|I_{t-1}]$, where I_{t-1} denotes the information available at $t - 1$.

[17] As in the case of Adam Smith in particular. As noted by Aspromourgos (2007, pp. 33, 34), when Smith identifies the market price with the actual price at with transactions occur, he refers to 'the most common actual price'.

[18] In the dynamical models discussed in sec. 3, given the determinants of natural prices and the quantities initially brought to market q_0, there is only one possible path of quantities: $q(q_0,t)$. There is, in other words, just one possible configuration of the disequilibrium allocation of productive agents amongst industries for every period of time. It is instead assumed here that the gravitation can follow many different paths for the same initial conditions and data. In the absence of specific information about the way in which the quantities actually produced are decided period by period, various different patterns must be regarded as possible.

It is worth noting that as long as the function $\phi(.)$ is not specified – it being simply assumed that $\phi(.)$ can be any function such that i) $\phi(q) = [0, 0, \ldots, 0]$ and ii) $y \cdot \phi(q_t) = 0 \; \forall \; q_t \in Q$ – equation [9] requires no assumption stronger than Parrinello's 'sign relation' [6]. It is indeed even more general than equation [6] because: a) it only asserts that deviations of market prices from natural prices depend on the quantities actually produced and on random disturbance, without any particular restriction as regards the sign of those relationships;[19] b) according to equation [9], as well as equation [4], the market price of each commodity may depend – although it does not necessarily do so – on the vector of produced quantities q_t and not on its quantity $q_{n,t}$ alone (with $n = 1, 2, \ldots, N$).

As a result of equation [9] and the assumptions made about function $\phi(.)$ and the random vectors q_t and μ_t, π_t is a random vector that takes values in $\Pi = \{\pi_t \in \mathbb{R}^N_+ : y \cdot \pi_t = 1\}$, with $E[\pi_t] = \pi$.

Again, little is known about the random vector π_t. In particular, we do not know its distribution function and therefore have no idea of how likely π_t is to remain in a certain neighbourhood of π, or whether this likelihood arises for large enough t. Far more definite results could be obtained, however, by focusing attention on the vector of average market prices after t observations instead of the vector of market prices observed in t.

As Ciccone (1999, p. 65) rightly observes, the reference to average market prices is particularly important in Ricardo's and Marx's analyses of gravitation. For example, Marx wrote as follow:[20]

> the fluctuations of supply and demand do not explain anything but the deviations of market-prices from the prices of production. These deviations balance mutually, so that in the course of long periods the *average market-prices* correspond to the prices of production [Marx, 1909, vol. 3, p. 419, emphasis added].

This observation will be interpreted here in a specific, statistical, way.

---

[19] In particular, Parrinello assumed that, without disturbance, the deviation of prices and the deviation of quantities are opposite in sign. As shown by Steedman (1984), however, this is so in the case of only two commodities.

[20] A similar idea can be found in Adam Smith's analysis, in which, as noted by Aspromourgos, 'the temporary and occasional price' is opposed to 'the average or ordinary price'; and 'equality of natural price and average/ordinary price is an expression of the conviction that competition will indeed prevail upon actual prices, tending to bring them into line with opportunity cost, so that actual prices at least on average approximate natural price' (Aspromourgos, 2007, p. 30, footnote).

Let $\overline{\pi}_{n,t}$ be the average of the market prices of commodity *n* (with *n* = 1, 2, …, *N*) after *t* observations, that is:

$$\overline{\pi}_{n,t} = \frac{1}{t} \cdot \sum_{\tau=1}^{t} \pi_{n,\tau} \qquad [10]$$

The (column) vector of average market prices after t observations is therefore $\overline{\pi}_t = [\overline{\pi}_{1,t}, \overline{\pi}_{2,t}, \ldots, \overline{\pi}_{N,t}]$.

It is now possible to put forward some observations on the characteristics of the random vector $\overline{\pi}_t$. In particular, if $|\overline{\pi}_t - \pi|$ is the Euclidean distance between $\overline{\pi}_t$ and π, and $pr(|\overline{\pi}_t - \pi| < \delta)$ the probability that this distance is smaller than a given real number δ, it can be said that natural prices are a centre of gravitation for market prices if the following condition is satisfied:

$$\lim_{t \to \infty} pr(|\overline{\pi}_t - \pi| < \delta) = 1 \qquad [11]$$

This means that the average market prices converge in probability to the natural prices[21] or, in more technical terms, that $\overline{\pi}_t$ is a consistent estimator of π.[22] The conditions for this kind of gravitation are discussed in the next section.

6. Gravitation and the consistency of average market prices

In accordance with the notation already introduced in the previous sections, π_n is the natural price of commodity *n*, with *n* = 1, 2, …, *N*, and $\pi = [\pi_1, \pi_2, \ldots, \pi_N]$ is the vector of natural prices determined as solution of the system of equations [1] and [2].

The market price of commodity *n* at the moment of time τ, with $\tau = 1, 2, \ldots, t$, is denoted by $\pi_{n,\tau}$, and assumed to be a random variable such that $E[\pi_{n,\tau}] = \pi_n$ and $\sigma^2_{\pi_{n,\tau}} = \sigma^2_n \; \forall \tau$. Therefore,

[21] The idea of convergence in probability used here as 'gravitation' around natural prices seems to have little to do with the 'equilibrium entropy prices' proposed by Foley (2003). In both cases, however, the central prices do tend to emerge statistically through the repetition of transactions.

[22] We can incidentally remark that this idea of 'centre of gravitation' seems to be exempt from some critiques raised against the standard notion, as those in Dupertuis and Sinha (2009a) and (2009b).

if $\pi_\tau \in \Pi$ is, as above, the vector of market prices of the N commodities obtaining at time τ, with τ = 1, 2, ..., t, then $E[\pi_\tau] = \pi$ and $E\left[(\pi_\tau - \pi)(\pi_\tau - \pi)^{\mathrm{T}}\right] = S$, with:

$$S := \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix}. \qquad [12]$$

It is worth stressing that there is no need to assume matrix S to be a diagonal matrix. On the contrary, we expect the market prices ruling in period t to be correlated among themselves. First, relative prices are considered here in terms of a numéraire commodity and the N prices are therefore not independent: once a numérairaie commodity is adopted, given N – 1 prices, the N-th can be deduced from them. Second, it is quite possible to imagine that the N commodities include some pairs of complementary or substitutes commodities, whose market prices are therefore non-independent.

However, in order to simplify the analysis,[23] we made the following assumption:

Assumption. Matrix S is constant over time, meaning that it does not change with τ.

The constancy of matrix S is not a very strong assumption. In particular, with this assumption, we are not ruling out autoregresive forms, neither univariate nor multivariate. The prices ruling in a certain period may very well depend on the prices emerged in previous

[23] Actually, S is just one of the possible structures that the variance-covariance matrix of π_τ can assume. In order to clarify this point, let us start by defining the variable $\xi_{n,\tau} = (\pi_{n,\tau} - \pi)$, with n = 1, 2, ..., N and τ = 1, 2, ..., t. The deviations can then be arranged in a (N×t) matrix such that:

$$\xi = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1t} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{N1} & \xi_{N2} & \cdots & \xi_{Nt} \end{bmatrix}.$$

Application of the vec-operator (Magnus & Neudecker, 1999, p. 34-36) to ξ yields $vec[\xi] = \tilde{\xi}$, whose matrix of variances and co-variances is $E\left[\tilde{\xi}\tilde{\xi}^{\mathrm{T}}\right]$ and assumes different forms under different hypotheses.

In particular, $E\left[\tilde{\xi}\tilde{\xi}^{\mathrm{T}}\right]$ is a block diagonal matrix with t matrices S as diagonal elements in the case considered here, where the variability of market prices is assumed to remain constant through time,. If the variability of market prices is instead not constant through time, then $Var[\pi_\tau] = S_\tau$, with τ = 1, 2, ..., t, where S_τ is the τ-th diagonal block of the matrix $E\left[\tilde{\xi}\tilde{\xi}^{\mathrm{T}}\right]$, while the off-diagonal blocks are (or not) matrices of zeros if market prices are not (or are) serially correlated.

periods, although, according to our assumption, this influence, if exists, must be recursive. In fact, what we are assuming is that the structure of the mechanisms potentially acting on market prices is persistent over time.

In the previous sections we have also defined, by equation [10], an average market price $\overline{\pi}_{n,t}$ for each commodity n = 1, 2, …, N, which is the mean of the market prices after t observations. As a result of the well-known properties of the sample mean (see Casella & Berger, 2002, p. 330–32), and of the assumptions made above about the random variable $\pi_{n,\tau}$:

$E[\overline{\pi}_{n,t}] = \pi_n$ and $\sigma^2_{\overline{\pi}_{n,t}} = \dfrac{\sigma_n^2}{t}$.

When the average market prices of the N commodities are organised in a vector $\overline{\pi}_t = [\overline{\pi}_{1,t}, \overline{\pi}_{2,t}, \ldots, \overline{\pi}_{N,t}]$, then $E[\overline{\pi}_t] = \pi$ and $E\left[(\overline{\pi}_\tau - \pi)(\overline{\pi}_\tau - \pi)^{\mathrm{T}}\right] = S_{\overline{\pi}_t} = \frac{1}{t} S$. It therefore follows from the assumption posited on matrix S that $S_{\overline{\pi}_t}$ is a non-diagonal matrix whose entries tend towards zero as t tends to infinity.

Proposition. If $\overline{\pi}_t$ is the vector of average market prices after t observations and it is assumed that $E[\overline{\pi}_t] = \pi$ and $E\left[(\overline{\pi}_\tau - \pi)(\overline{\pi}_\tau - \pi)^{\mathrm{T}}\right] = S_{\overline{\pi}_t} = \frac{1}{t} S$, then $\lim_{t \to \infty} pr\left(\left|\overline{\pi}_t - \pi\right| < \delta\right) = 1$.

Proof. Let us start by defining the random vector $X := \left(\overline{\pi}_t - \pi\right)$ and the function $h(X) = X^{\mathrm{T}} X$. Given a strictly positive real number δ, we define the set $D = \{X \in \mathbb{R}^N : h(X) \geq \delta\}$ and its complement $\overline{D} = \{X \in \mathbb{R}^N : h(X) < \delta\}$. Then, if the unknown density function of X is denoted as $g(X)$, the following expectation:

$$\begin{aligned} E[h(X)] &= \int_{R^N} h(X) \cdot g(X)\, dX_1 dX_2 \ldots dX_N = \\ &= \int_D h(X) \cdot g(X)\, dX_1 dX_2 \ldots dX_N + \int_{\overline{D}} h(X) \cdot g(X)\, dX_1 dX_2 \ldots dX_N \end{aligned} \qquad [13]$$

can be written as the sum of two positive quantities so that it immediately implies:

$$E[h(X)] \geq \int_D h(X) \cdot g(X)\, dX_1 dX_2 \ldots dX_N \,. \qquad [14]$$

Because of the definition of the set D given above, the inequality can be enforced by replacing $h(X)$ with its minimum δ in D:

$$E[h(X)] \geq \delta \int_D g(X)\, dX_1 dX_2 \ldots dX_N = \delta\ pr(X \in D) \quad [15]$$

and rearranging the terms:

$$pr(X \in D) \leq \frac{1}{\delta} E[h(X)] \quad [15']$$

which implies:

$$pr(X \in \overline{D}) \geq 1 - \frac{1}{\delta} E[h(X)]. \quad [16]$$

As a result of equation [16]:

$$\lim_{t \to \infty} pr(X \in \overline{D}) \geq \lim_{t \to \infty} \left(1 - \frac{1}{\delta} E[h(X)]\right). \quad [17]$$

Since $h(X)$ = $X^{\mathrm{T}} X = \sum_{n=1}^{N} (\overline{\pi}_{n,t} - \pi_n)^2$, then $E[h(X)] = \sum_{n=1}^{N} \sigma^2_{\overline{\pi}_{n,t}} = \frac{1}{t} \sum_{n=1}^{N} \sigma^2_n$, which implies: $\lim_{t \to \infty} \frac{1}{\delta} E[h(X)] = 0$.

It follows that:

$$\lim_{t \to \infty} pr(X \in \overline{D}) = 1 \quad [18]$$

or in other terms:

$$\lim_{t \to \infty} pr(h(X) < \delta) = 1. \quad [18']$$

Therefore, given that $h(X) < \delta$ entails $|\overline{\pi}_t - \pi| < \delta$, then:

$$\lim_{t \to \infty} pr(|\overline{\pi}_t - \pi| < \delta) = 1. \ \square \quad [19]$$

6. Conclusions

As defined by Adam Smith and other classical economists, market prices are those at which trade actually takes place at a certain moment. The complexity of the real world and the exceptional character of the events that may occur at any given moment, therefore, preclude the possibility of a theory of the determination of market prices.

There are, however, two possible approaches. The first involves representing reality by means of a simplified model, thereby radically reducing the complexity of the real world, and studying the determination of market prices within the model. The second instead means accepting our ignorance[24] about the forces governing the determination of market prices and regarding them as random variables whose actual values can only be known *ex-post*.

The first has so far been adopted by most scholars in analysing the gravitation of market prices around a theoretical central position, i.e. natural prices, perhaps with the only relevant exception of Parrinello's contribution (1990) discussed in sec. 4. Gravitation has therefore been addressed as the stability of the rest position of a dynamical system (sec. 3).

The second is instead adopted in this paper, which suggests that the gravitation of market prices should be seen as a stochastic process. This, clearly, does not means neglecting the existence of forces acting with regularity on market prices, but we recognize that there are also other influences which we are not able to predict with sufficient certainty.[25] Therefore, sec. 5 introduces a new conception of the gravitation of market prices whereby natural prices are a centre of gravitation for market prices if the probability of their means being very close to natural prices after t observations tends to 1 as t tends to infinity. This is consistent, as stated above, with the view of Smith, Ricardo and Marx that due to competition, actual prices come to approximate natural prices (or production prices) on average over a long enough period of time.

Finally, assuming that entrepreneurs, in the aggregate, do not make systematic errors about produced quantities, and that the structure of market prices determination is persistent over time, then, as shown in sec. 6, the gravitation of market prices, in the sense specified here, occurs.

---

[24] According to the well-known sentence by Poincarré, '[c]hance is only the measure of our ignorance' because '[f]ortuitous phenomena are, by definition, those whose laws we do not know' (1913, p. 395).

[25] Something similar can be said for the throwing of a dice. It is clear that it is under the influence of some regularities, such as the laws of gravity, the properties of the materials, ... and so on and so forth; but these are not enough to predict the result of the throw.

Acknowledgements

The authors thank T. Aspromourgos, E. Bellino, R. Ciccone and L. Pieraccini for their comments and suggestions but remain wholly responsible for any errors or imperfections in the paper.

References

Aspromourgos, T. (2007) 'Adam Smith's Treatment of Market Price and their Relation to «Supply» and «Demand»', *History of Economic Ideas*, 15(3): 27-58.

Aspromourgos, T. (2009) *The Science of Wealth: Adam Smith and the framing of political economy*, London and New York: Routledge.

Bellino, E. (2011) 'Gravitation of Market Prices towards Natural Prices', in R. Ciccone, C. Gehrke & G. Mongiovi (eds) *Sraffa and Modern Economics*, London and New York: Routledge.

Bellino, E. & Serrano, F. (2011) 'Gravitation analysis: beyond cross-dual models and back to Adam Smith', mimeo.

Boggio, L. (1990) 'The dynamic stability of production prices: a synthetic discussion of models and results', *Political Economy—Studies in the Surplus Approach*, 6(1-2), pp. 47-58.

Casella, G. & Berger, R.L. (2002) *Statistical Inference, 2nd Edition*, Pacific Grove: Duxbury-Thomson Learning.

Ciccone, R. (1999) 'Classical and neoclassical short-run prices: a comparative analysis of their intended empirical content', in G. Mongiovi & F. Petri (eds) *Value, Distribution and Capital*, London: Routledge.

Ciccone, R. (2011) 'Capacity utilization, mobility of capital and the classical process of gravitation', in R. Ciccone, C. Gehrke & G. Mongiovi (eds) *Sraffa and Modern Economics*, London and New York: Routledge.

Dupertuis, M-S. & Sinha, A. (2009a) 'Sraffa's System: Equal Rate of Profits and the Notion of Centre of Gravitation', *Journal of Economic Behaviour and Organization*, 71, pp. 495-501.

Dupertuis, M-S. & Sinha, A. (2009b) 'A Sraffian Critique of the Classical Notion of the Centre of Gravitation', *Cambridge Journal of Economics*, 33(6), pp. 1065-1087.

Foley, D. (1994) 'A Statistical Equilibrium Theory of Markets', *Journal of Economic Theory*, 62 (2), pp. 321-345.

Foley, D. (2003) ‘Statistical Equilibrium in Economics: Method, Interpretation, and an Example’, in F.H. Hahn & F. Petri (eds) *General Equilibrium. Problems and Prospects*, London: Routledge.

Fratini, S.M. & Levrero, E.S. (2011) ‘Sraffian indeterminacy: a critical discussion’, *Cambridge Journal of Economics*, 35(6), pp. 1127–1149.

Frisch, R. (1936) ‘On the notion of equilibrium and disequilibrium’, *The Review of Economic Studies*, 3(2), pp. 100-105.

Garegnani, P. (1984) ‘Value and Distribution in the Classical Economists and Marx’, *Oxford Economic Papers*, 36(2), pp. 291-325.

Garegnani, P. (1990) ‘On some supposed obstacles to the tendency of market prices toward natural prices’, *Political Economy—Studies in the Surplus Approach*, 6(1-2), pp. 329-359.

Garegnani, P. (2002) ‘Sraffa’s Price Equations: a Stationary Economy or “Normal Positions”?’, in S. Boehm, C. Gehrke, H.D. Kurz & R. Sturn (eds) *Is There Progress in Economics?*, Cheltenham and Northampton: Edward Elgar.

Lager, C. (1998) ‘On the notion of the rate of profit’, *Review of Political Economy*, 10, pp. 447-458.

Magnus, J.R. & Neudecker, H. (1999) *Matrix Differential Calculus with Applications in Statistics and Econometrics, 2nd Edition*, Chichester: John Wiley & Sons.

Marx, K. (1909 [1867-1894]) *Capital*, 3 vols, Chicago: Charles H. Kerr & Company.

Parrinello, S. (1990) ‘Some Reflections on Classical Equilibrium, Expectations and Random Disturbance’, *Political Economy—Studies in the Surplus Approach*, 6(1-2), pp. 113-124.

Petri, F. (2011) ‘On some aspects of the debate on the gravitation of market prices to long-period prices’, in R. Ciccone, C. Gehrke & G. Mongiovi (eds) *Sraffa and Modern Economics*, London and New York: Routledge.

Poincarré, H. (1913) *Foundations of Science: Science and Hypothesis, The Value of Science, Science and Method*,

Ravagnani, F. (2002), ‘Produced Quantities and Returns in Sraffa’s Theory of Normal Prices: Textual Evidence and Analytical Issues’, in S. Boehm, C. Gehrke, H.D. Kurz & R. Sturn (eds) *Is There Progress in Economics?*, Cheltenham and Northampton: Edward Elgar.

Ricardo, D. (1951-1973) *Works and Correspondence of David Ricardo*, P. Sraffa (ed.), 11 vols, Cambridge: Cambridge University Press.

Roncaglia, A. (1990) ‘Is the Notion of Long-Period Positions Compatible with Classical Political Economy?’, *Political Economy—Studies in the Surplus Approach*, 6(1-2), pp. 103-112.

Salvadori, N. & Signorino, R. (2013) 'The Classical Notion of Competition Revisited', *History of Political Economy*, 45(1), pp. 149-175.

Schefold, B. (1989) *Mr. Sraffa on Joint Production and Other Essays*, London: Unwin.

Signorino, R. (2012) 'David Ricardo on natural an market prices', *MPRA Paper No. 39212*.

Smith, A. (1976) *The Glasgow Edition of the Works and Correspondence of Adam Smith*, 7 vols, Oxford: Clarendon Press.

Sraffa, P. (1960) *Production of Commodities by Means of Commodities*, Cambridge: Cambridge University press.

Steedman, I. (1984) 'Natural prices, differential profit rates and the Classical competitive process', *The Manchester School*, 52, pp. 123 – 140.

Vianello, F. (1989) 'Natural (or normal) prices: some pointers', *Political Economy—Studies in the Surplus Approach*, 5(2), pp. 89-105.

Authors contact informations

Saverio M. Fratini
Università degli Studi Roma Tre
Dipartimento di Economia
via S. D'Amico, 77
00145 - Roma
e-mail: saveriomaria.fratini@uniroma3.it

Alessia Naccarato
Università degli Studi Roma Tre
Dipartimento di Economia
via S. D'Amico, 77
00145 - Roma
e-mail: alessia.naccarato@uniroma3.it