From Sraffa: backwards, to a better understanding of Marx's Values/Prices 'transformation'; and forward to the novel view of growing economic systems

Luigi L. Pasinetti, Nadia Garbellini

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Abstract

The type of analysis carried out on the Sraffa system allows us to present a general version of the much discussed problem of the 'transformation of values into prices of production', which Marx sets out in Volume III of *Das Kapital* (Marx, 1959, chapter IX). Moreover, the present contribution aims at showing how the introduction of 'Keynesian' income distribution theory within a multisectoral framework—along the lines of Pasinetti (1962a, 1981, 1988)—allows to reformulate a perfectly general theory of value in such a way as to uncover the properties of the labour values in terms of long-run growth of the economic system.

1. Foreword

The participants to the present workshop have been asked a simple but specific question: What have we learnt on Classical political economy from, and since, Sraffa?

The authors of the present paper are convinced that we have learnt a lot. In this paper, they will concentrate on the novelties that have energed in two directions: first by looking back to the much discussed problem of the transformation of Marxian values into prices of production; and secondly, by looking forward through a new radical way of considering the problems of growth and income distribution, that, starting from Kaldor's so called 'Keynesian' theory of income distribution, has achieved a real breakthrough by penetrating into the macroeconomic relations through the structural dynamics analysis of vertical integration.[1]

[1] In what follows, the basic references are Pasinetti (1962a, 1977, 1988), of which the present paper collects a mixture of resumptions, updating and concluding synthesis.

2. An ‘ideal’ price system

Consider an economic system with single-product industries. The technique of the system is represented by a matrix of interindustry coefficients $\mathbf{A}$, assumed to be viable (i.e., to have a maximum eigenvalue $\lambda_m < 1$), and by a vector of (direct) labour coefficients $\mathbf{a}_n$.[2]

In this case, the price system might be written as:

$$\mathbf{p}^T = w\mathbf{a}_n^T + \mathbf{p}^T\mathbf{A}(1+\pi)$$

if wages are paid at the end of the production process, or:

$$\mathbf{p}^T = (w\mathbf{a}_n^T + \mathbf{p}^T\mathbf{A})(1+\pi)$$

if they are paid in advance.

In an ‘ideal’ system of prices, as understood, for example, by the ‘Ricardian socialists’[3] (who had claimed, at the beginning of the XIX century that the whole net product of an economic system, ought to go to the workers), the wage rate corresponds the per-worker net product: we shall call it the ‘complete’ wage rate and denote it by w^*. By introducing such a wage rate into the above expression, and choosing it as the *numéraire* of the price system—i.e. setting $w = w^* = 1$ and $\pi = 0$—we obtain:

$$\mathbf{v}^T = w^*\mathbf{a}_n^T + \mathbf{v}^T\mathbf{A} = \mathbf{a}_n^T + \mathbf{v}^T\mathbf{A} \tag{1}$$

The exchange ratios now become exactly *equal* to the physical quantities of embodied, direct and indirect, labour.

It is precisely these quantities of embodied labour that Marx called ‘values’ and, looking at them from another point of view, a century later, Pasinetti (1973)—in his mathematical formulation of Sraffa’s framework—has called *vertically integrated* labour coefficients.

It should be noted that system (1) has the property of making ‘labour embodied’ coincide with ‘labour commanded’. Each commodity i, evaluated at the corresponding v_i, $i = 1, \ldots, n-1$, can purchase in the economic system a

[2]In what follows, boldface capital letters will denote matrices, boldface small capitals vectors. Vectors are to be intended as column vectors unless explicitly transposed. A vector with a hat will denote a diagonal matrix with the elements of the corresponding vector in the main diagonal.

[3]See for example Lowenthal (1911).

quantity of labour ('labour commanded') exactly equal to the labour embodied in it.

3. The Marxian 'value system'

Let us begin with Marx. He carries out his analysis in terms of 'values', defined here by (1). But he focuses his attention on what is for him a fundamental distortion that occurs in a capitalist society. The owners of the means of production (the 'capitalists') find themselves in a privileged position, which allows them not to pay the 'complete' wage rate w^*.

Suppose that we are able to define in physical terms, as a bundle of heterogeneous commodities, a set of h quantities $d_1, d_2, \ldots, d_h$, which together constitute the subsistence real wage $\mathbf{d}$:

$$\mathbf{d}^T = \begin{bmatrix} d_1 & d_2 & \cdots & d_h & 0 & \cdots & 0 \end{bmatrix} \tag{2}$$

(It will be assumed that the matrix $\mathbf{A}$ is so ordered that its first h rows and h columns refer to the h subsistence commodities and industries.)

Of course, $h < n - 1$.[4] Marx argues that

$$\mathbf{v}^T\mathbf{d} = \delta w^*, \qquad \text{where} \qquad \delta < 1. \tag{3}$$

That is, the capitalists pay the workers only the subsistence wage $\mathbf{v}^T\mathbf{d}$, which represents a fraction $\delta < 1$ of the 'complete' wage w^*. What remains, i.e., the fraction $(1 - \delta)$ of w^*, represents the 'unpaid wage' or 'surplus value', in Marx's terminology, which is appropriated by the capitalists.

Using (1), in which $w^* = 1$, and substituting (3) therein, we can write:

$$\mathbf{v}^T\mathbf{A} + \frac{1}{\delta}\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T = \mathbf{v}^T \tag{4}$$

or

$$\mathbf{v}^T\mathbf{A} + \mathbf{v}^T\mathbf{d}\mathbf{a}_n^T + \frac{1-\delta}{\delta}\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T = \mathbf{v}^T \tag{5}$$

[4] Notice that, as a consequence of the hypothesis that all commodities require labour, directly or indirectly, to be produced, wage goods are necessarily basic commodities.

It may now be noted that the ratio $(1-\delta)/\delta$, which we shall call σ, so that

$$\sigma = \frac{1-\delta}{\delta} = \frac{1}{\delta} - 1 \tag{6}$$

represents what Marx calls the 'rate of surplus value', i.e., the unpaid part of w^*, expressed as a percentage fraction of the part which is paid. Or, as it is also said (all the quantities involved being expressed in terms of physical quantities of embodied labour), σ represents the ratio of 'surplus labour' to 'necessary labour'. Or again, $(1-\delta)$ also represents the fraction of the working day (or of the 'working year' if a 'year' is the unit of time) in which the worker works for the capitalist, and δ represents the fraction in which he works for himself. This was Marx's reason for calling their ratio σ not only the 'rate of surplus value' but also the 'rate of exploitation'.

It may also be noted that, since the subsistence wage (3) is assumed to be the same for all workers, the rate of surplus value (or of exploitation) σ proves to be the same in all industries. In fact, (5) may be rewritten

$$\mathbf{v}^T\mathbf{A} + \mathbf{v}^T\mathbf{d}\mathbf{a}_n^T(1+\sigma) = \mathbf{v}^T \tag{7}$$

or

$$\mathbf{v}^T[\mathbf{I} - \mathbf{A} - (1+\sigma)\mathbf{d}\mathbf{a}_n^T] = \mathbf{0}^T \tag{8}$$

where σ is indeed the same in all industries. The solutions for $\mathbf{v}^T$ and σ (and δ) are, of course, still those obtained from (1), together with (2), (3), (6).

The same solutions for $\mathbf{v}^T$ and σ can, however, be obtained in an alternative way directly from (8) by adding the equation

$$\mathbf{v}^T\mathbf{d}(1+\sigma) = w^* = 1, \tag{9}$$

which—as in (1)—defines the value of a unit of embodied labour to be the *numéraire* for the system of 'values'. Expression (8) is a system of linear and homogeneous equations in the 'values'. A necessary condition for the existence of non-zero solutions is that:

$$\det[\mathbf{I} - \mathbf{A} - (1+\sigma)\mathbf{d}\mathbf{a}_n^T] = 0. \tag{10}$$

Solving this equation, we obtain the value of σ, which, when substituted into (9), enables us to determine the $n-1$ elements of $\mathbf{v}^T$ (from $n-1$ independent equations).

As emerges clearly from (5), the 'value' of each commodity may be regarded as the sum of three components: (i) replacement of the means of production, or 'constant capital' (each element of the vector $\mathbf{v}^T\mathbf{A}$); (ii) replacement of the capital advanced by the capitalists as wages, or 'variable capital' (each element of the vector $\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T$); and (iii) 'surplus value' (each element of the vector $\sigma\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T$). In the same way, if $\mathbf{q}$ is the vector of the physical quantities of all commodities produced in a given economic system, the 'value' of total gross product $\mathbf{v}^T\widehat{\mathbf{q}}$ may be expressed as the sum of three components:

$$\mathbf{v}^T\mathbf{A}\widehat{\mathbf{q}} + \mathbf{v}^T\mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}} + \sigma\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}} = \mathbf{v}^T\widehat{\mathbf{q}}, \tag{11}$$

which represent the 'constant capital', the 'variable capital', and the total 'surplus value', respectively, for the whole economy. All the physical quantities, being multiplied by $\mathbf{v}^T$, come to be expressed in terms of 'values' or quantities of embodied labour.

Notice that the rate of surplus value can also be derived as the ratio between the two aggregates which are shown as the third and second terms, respectively, in the sum on the left side of (11). On the other hand, contrary to what was thought by Marx,[5] the rate of profit for the system as a whole, as we shall see in a moment, cannot be obtained, in general, as the ratio between the third term (total 'surplus value') and the sum of the first two terms (total capital) of (11). To obtain the rate of profit, one must turn to a different system of equations, referring not to the 'Marxian values', but to the 'prices of production'.

4. The 'price-of-production system'

The system of equations of the previous section, which may be called 'the Marxian value system', cannot be observed in the practice of a capitalist economy, because that part of the net product which is not paid in wages is generally distributed in the form of profits. And profits are distributed in proportion to total capital, not in proportion to total wages (as, by definition, surplus value would be).

This implies that the exchange ratios, or 'prices of production', which can be observed in a capitalist system, will differ from the 'Marxian values'. Now, if we assume that wages are advanced by the capitalists at the beginning of the

[5] See for example Marx (1959, p. 42).

production period, and hence that the wages fund is itself part of capital, the 'prices of production' will be given by the system of equations

$$(\mathbf{p}^T\mathbf{A} + w\mathbf{a}_n^T)(1+\pi) = \mathbf{p}^T \tag{12}$$

Moreover, if we assume, as in the previous section, that the wage rate can be specified in physical terms as $\mathbf{d}$, defined by (2), and thus that

$$w = \mathbf{p}^T\mathbf{d} \tag{13}$$

we obtain, after substitution in (12),

$$(\mathbf{p}^T\mathbf{A} + \mathbf{p}^T\mathbf{d}\mathbf{a}_n^T)(1+\pi) = \mathbf{p}^T \tag{14}$$

that is:

$$\mathbf{p}^T[\mathbf{I} - (1+\pi)(\mathbf{A} + \mathbf{d}\mathbf{a}_n^T)] = \mathbf{0}^T \tag{15}$$

The matrix $\mathbf{A} + \mathbf{d}\mathbf{a}_n^T$, which may be called $\mathbf{A}^{(+)}$, that is,

$$\mathbf{A}^{(+)} = \mathbf{A} + \mathbf{d}\mathbf{a}_n^T \tag{16}$$

is a matrix of inter-industry coefficients which is, so to speak, 'augmented' by the consumption coefficients needed for the maintenance of the workers. Using (16), we can re-write (15) as

$$\mathbf{p}^T[\mathbf{I} - (1+\pi)\mathbf{A}^{(+)}] = \mathbf{0}^T \tag{17}$$

We thus obtain a system of linear and homogeneous equations of a kind which we have already considered several times. We know that a necessary condition for the system to have economically meaningful solutions is that

$$\det[\mathbf{I} - (1+\pi)\mathbf{A}^{(+)}] = 0 \tag{18}$$

or

$$\det[\mathbf{I} - (1+\pi)(\mathbf{A} + \mathbf{d}\mathbf{a}_n^T)] = 0 \tag{19}$$

Furthermore, we know that the rate of profit satisfying the characteristic equation (18), to be denoted by $\mathbf{\Pi}^{(A+)}$, is non-negative if the maximum eigenvalue of $\mathbf{A}^{(+)}$ is less than or equal to unity, which we shall of course assume. In symbols, we assume that

$$\lambda_m^{(A+)} \leq 1, \qquad \text{which implies} \qquad \mathbf{\Pi}^{(A+)} \geq 0, \tag{20}$$

After substituting this rate of profit into (17) and adding whatever further equation we choose in order to define a *numéraire* for the price system, we can determine all the 'prices of production' in terms of the chosen *numéraire*.

5. A comparison

It is not difficult at this point to understand why the Marxian values (8) differs from the 'prices of production' (15). The fundamental diversity lies in the different ways in which the part of the net product which is not paid as wages is distributed among the various industries. In the Marxian values it is distributed (as 'surplus value') in proportion to the wages advanced to the workers (or 'variable capital'); in the 'prices of production system' it is distributed (as profits) in proportion to the sum of 'variable capital' and 'constant capital'. The rate of profit will therefore be *less than* (or, in the limiting case in which there is no constant capital, equal to) the rate of surplus value, simply because the same physical quantities, of which both surplus value and profits are made up, must in the case of profits be spread over a larger amount of capital (total capital, and not merely variable capital).

Rigorous confirmation of this intuitive argument can be found in the fact that the rate of surplus value and the rate of profit are the solutions of the characteristic equations (10) and (19), respectively. Straightforward comparison of these two equations provides immediate confirmation of Marx's three well known[6] propositions, namely, that: (i) $\pi > 0$ if and only if $\sigma > 0$; (ii) $\pi < \sigma$, except in the limiting case of $\mathbf{A} = \mathbf{O}$, when π and σ become equal; (iii) π is a monotonically increasing function of σ and *vice versa*.

It may be of interest to note that, from a Sraffian standpoint, the Marxian values and the 'prices of production' formally correspond to two extreme (opposite) cases considered for exogenous determination of the distributive variable, but with respect to two different coefficients matrices. Marxian values are derived by setting the rate of profit equal to zero in a system involving the matrix

[6] For details see Pasinetti (1977, §4. 2, pp. 20-22).

$\mathbf{A}$, while 'prices of production' are derived by setting the (surplus) wage rate equal to zero in a system involving the matrix $\mathbf{A}^{(+)}$. These being two different systems of equations, it is hardly surprising that they should have different solutions. Hence, in general, $\mathbf{v}^T \neq \mathbf{p}^T$.

It follows, for example, that for any total output, given in physical terms by the (column) vector $\widehat{\mathbf{q}}$, the evaluation of this output will differ according as it is done in terms of Marxian values or in terms of 'prices of production'; i. e., in general,

$$\mathbf{v}^T\widehat{\mathbf{q}} \neq \mathbf{p}^T\widehat{\mathbf{q}}. \tag{21}$$

Moreover, the ratio of total surplus value to total capital, expressed in Marxian values, will differ from the rate of profit (more specifically from the ratio of total profits to total capital) for the system as a whole; i.e., in general,

$$\frac{\sigma\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}}}{\mathbf{v}^T\mathbf{A} + \mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}}} \neq \frac{\mathbf{\Pi}^{(A+)}\mathbf{p}^T\mathbf{A} + \mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}}}{\mathbf{p}^T\mathbf{A} + \mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}}}. \tag{22}$$

Thirdly, total surplus value will be different from total profits; i.e., in general,

$$\sigma\mathbf{v}^T\mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}} \neq \mathbf{\Pi}^{(A+)}\mathbf{p}^T\mathbf{A} + \mathbf{d}\mathbf{a}_n^T\widehat{\mathbf{q}}. \tag{23}$$

As is well known, Marx thought (incorrectly) that these three expressions would all be equalities[7]. That cannot be so, except in special cases.[8]

6. Sraffa's 'standard system' as a special case

Before proceeding, it may be useful to consider the special case of a hypothetical economic system which happens to realise the proportions of Sraffa's standard system, defined by:[9]

$$[\mathbf{I} - (1+R)\mathbf{A}]\mathbf{q}^* = \mathbf{0} \tag{24}$$

$$\mathbf{a}_n^T\mathbf{q}^* = 1 \tag{25}$$

[7]See, for example, Marx (1959, pp. 42, 157, 164.)

[8]Of course, one can always take advantage of the degree of freedom in the price system and explicitly choose the *numéraire* in such a way as to reduce at least one of these three inequalities to an equality.

[9]For details see Pasinetti (1977, Chapter V).

and take the standard net product $\mathbf{y}^* = (\mathbf{I} - \mathbf{A})\mathbf{q}^*$ as the *numéraire* commodity, i.e. set

$$\mathbf{p}^T\mathbf{y}^* = \mathbf{p}^T(\mathbf{I} - \mathbf{A})\mathbf{q}^* = 1 \tag{26}$$

Before proceeding, notice that expression (25) can be written as:

$$\mathbf{a}_n^T(\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}^* = \mathbf{v}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}^* = 1$$

Hence:

$$\mathbf{v}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}^* = \mathbf{p}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}^* \tag{27}$$

In order to establish an equivalence with the Marxian framework, we have to slightly modify the price system for the case where wages are paid in advance—whereas Sraffa assumed they were paid at the end of the production process:

$$\mathbf{p}^T = w\mathbf{a}_n^T(1 + \pi) + \mathbf{p}^T\mathbf{A}(1 + \pi) \tag{28}$$

and hence:

$$\mathbf{p}^T\mathbf{q}^* = w\mathbf{a}_n^T\mathbf{q}^*(1 + \pi) + \mathbf{p}^T\mathbf{A}\mathbf{q}^* + \mathbf{p}^T\mathbf{A}\mathbf{q}^*\pi \tag{29}$$

or, using (25) and (26):

$$\mathbf{p}^T(\mathbf{I} - \mathbf{A})\mathbf{q}^* = 1 = w(1 + \pi) + \mathbf{p}^T\mathbf{A}\mathbf{q}^*\pi \tag{30}$$

Moreover, from (24):

$$\mathbf{p}^T\mathbf{A}\mathbf{q}^* = \frac{1}{R} \tag{31}$$

and hence:

$$w(1 + \pi) + \frac{\pi}{R} = 1 \tag{32}$$

from which we get:

$$\pi = \frac{R(1-w)}{1+wR} \tag{33}$$

Notice that the quantity of labour embodied in the net product of the system is equal to unity. Thus, owing to the normalisation adopted, the net product comes to be equal to unity, whether it is evaluated at prices of production or at 'Marxian values'. It also follows that thinking of fractions of the *standard* net product in terms of prices always comes to be the same as thinking of them in terms of 'Marxian values' (i.e. of quantities of embodied labour). Hence, if wages were actually paid in standard commodity, i.e. in fractions of the standard net product, as we have here assumed, the wage rate w, which in the normalised standard system also represents the fraction of the net product which goes to wages, comes to represent also the fraction of the total labour which is embodied in wages (or the fraction of the working 'year' in which the worker works for himself), which we previously called δ (section 3). This implies that w can be replaced by δ in relation (33) derived for the 'modified' standard system:

$$\pi = \frac{R(1-\delta)}{1+\delta R} \tag{34}$$

from which, using (6), one can also derive the relation between the rate of profit and the rate of surplus value:

$$\pi = R\frac{\sigma}{1+\sigma+R} \tag{35}$$

$$\sigma = \pi\frac{1+R}{R-\pi} \tag{36}$$

These two expressions, incidentally, provide a confirmation of the three propositions concerning the relations between π and σ stated in the previous section.

It is now interesting to check what happens to the three inequalities (21), (22), (23). First of all, total gross output, given by the vector $\mathbf{q}^*$, is a scalar multiple of the standard net product,[10] i.e., of the vector $(\mathbf{I}-\mathbf{A})\mathbf{q}^*$. And, since $\mathbf{p}^T(\mathbf{I}-\mathbf{A})\mathbf{q}^* = \mathbf{v}^T(\mathbf{I}-\mathbf{A})\mathbf{q}^*$, it must also be true that $\mathbf{p}^T\mathbf{q}^* = \mathbf{v}^T\mathbf{q}^*$. Secondly, with wages being paid in standard commodity, all that remains after the payment of wages—surplus value, if evaluated at 'Marxian values'; or profits, if

[10]More precisely, in Sraffa's standard system $\mathbf{y}^* = R/(1+R)\mathbf{q}^*$. See Pasinetti (1977, p. 98).

evaluated at prices—will also consist of standard commodity and will therefore be a scalar multiple of the net product. This means that total surplus value will be equal to total profits. Finally, the rate of profit, by being a ratio between two scalar multiples of the same standard net product, will remain the same, whether evaluated at 'values' or at prices. In other words, all three inequalities (21), (22), (23) become equalities in the present case.

But let us now consider the standard system defined with respect to the matrix $\mathbf{A}^{(+)} = \mathbf{A} + \mathbf{d}\mathbf{a}_n^T$. Its solution will here be denoted by $\mathbf{q}^{**}$. In formal terms, this new standard system is represented by the same equations as considered above—(24), (25)—except that the matrix $\mathbf{A}$ must be replaced by the matrix $\mathbf{A}^{(+)}$. However, we have to distinguish two cases: (i) the case in which the real wage $\mathbf{d}$ is of the same composition as the standard commodity, so that $\mathbf{d} = \mathbf{d}^*$, and (ii) the case in which the real wage $\mathbf{d}$ has a composition different from that of the standard commodity. In case (i), it will be clear that $\mathbf{q}^* = \mathbf{q}^{**}$, and hence that one returns to the standard commodity defined with respect to the matrix $\mathbf{A}$ (i.e., the two standard systems come to coincide). In case (ii), by contrast, $\mathbf{q}^* \neq \mathbf{q}^{**}$. The standard system defined with respect to the matrix $\mathbf{A}$ differs from that defined with respect to matrix $\mathbf{A}^{(+)}$. With this new standard system it is no longer possible to use the normalisation of the price system suggested by Sraffa. For (25), which now becomes $\mathbf{a}_n^T\mathbf{q}^{**} = 1$, always implies that the net product of the system, wages plus profits, is equal to unity when evaluated at 'Marxian values'. But the expression $\mathbf{p}^T(\mathbf{I} - \mathbf{A}^{(+)})\mathbf{q}^{**}$ no longer represents the net product of the system but rather a 'net product' which does not include wages; i.e., it represents only profits. (This is the Classical notion of 'net product'.) If one wishes to maintain direct comparability between 'Marxian values' and 'prices of production', this 'net product', evaluated at 'prices of production', must be set equal to the quantity of labour embodied in it; hence the price system must be closed, not by equation $\mathbf{p}^T(\mathbf{I} - \mathbf{A}^{(+)})\mathbf{q}^{**} = 1$, but rather by equation

$$\mathbf{p}^T(\mathbf{I} - \mathbf{A}^{(+)})\mathbf{q}^{**} = \mathbf{v}^T(\mathbf{I} - \mathbf{A}^{(+)})\mathbf{q}^{**}, \tag{37}$$

in which both sides will, of course, be *less than* unity. (It also follows that the sum of wages and profits, which has a 'value' equal to unity, will no longer necessarily be equal to unity if evaluated at 'prices of production'.)

Nevertheless, even with this normalisation, the three inequalities (21), (22), (23) become equalities in the new standard system. Total surplus value and total profits are equal to each other owing to (37). Moreover, $\mathbf{p}^T\mathbf{q}^{**} = \mathbf{v}^T\mathbf{q}^{**}$, because each side of this equality is the same scalar multiple of the corresponding term in (37). Finally, as in all standard systems, the rate of profit is a physical notion and is therefore independent of both ‘values’ and prices.

Sraffa’s concept of ‘standard system’ (and ‘standard commodity’) therefore yields further contributions also in the field of ‘Marxian value’ analysis. It confirms that, even in the case in which the wage rate is paid at the beginning of the production period, the distribution of income (i.e. the relation between π and w) remains independent of the price system. By simply being converted, from a straight, to a hyperbolic, line, it shows absolutely no dependence on prices. Moreover, the three inequalities specified at the end of the previous section are all transformed into equalities. Of course, all this is for the special case in which production is actually carried out in the proportions of the Sraffa ‘standard system’.

7. The linear operator which effects the ‘transformation’

But what happens in the general case? Which is the linear operator that, in general, ‘transforms’ the vector of Marxian values $\mathbf{v}^T$ into the vector of ‘prices of production’ $\mathbf{p}^T$?

There are many alternative ways of expressing such linear operator, depending on the convention chosen to close the degree of freedom of each of our equation systems. For our purposes, it may be useful to consider at least two of them.

The way that emerges as the most natural one, from the point of view of the present analysis, is that of closing the ‘prices of production’ with the equation

$$\mathbf{p}^T\mathbf{d}(1+\sigma) = 1, \tag{38}$$

which is symmetrical to the equation (9), with which we had earlier (in section 3) closed the Marxian values. This amounts to expressing both the ‘prices of production’ and the Marxian values in terms of the per-worker net product, as composed by the wage goods, specified in vector $\mathbf{d}$—see (2). Thanks to (38), the price of each commodity is being expressed in terms of what we called the ‘ideal’, or ‘complete’, wage rate, a wage rate which—if realised—*could* purchase (i.e. *command*) a quantity of labour exactly equal to the labour embodied in it.

In this case, by adding (38), as the equation defining the *numéraire*, into the equations system (12)-(15), we obtain the new expression:

$$\mathbf{p}^T = \mathbf{v}^T[\mathbf{I} - \pi\mathbf{A}(\mathbf{I}-\mathbf{A})^{-1}]^{-1}\frac{1+\pi}{1+\sigma}, \tag{39}$$

Expression (39) now gives us explicitly the linear operator which we have been looking for—and hence an explicit solution to the Marxian problem of 'transformation of values into prices of production'.

There is another (alternative) way of looking at the same 'transformation', which is generally considered to be more congenial to Marxian economists. It consists in keeping (9) as the *numéraire* of the Marxian values, but adopting, as *numéraire* of the price system, not the 'ideal', or 'complete', wage, but the wage rate which is actually paid:

$$w = \mathbf{p}^T \mathbf{d} = 1, \tag{40}$$

In this case, all prices come to be expressed in terms of the quantity of labour which any particular commodity can purchase in the economic system (i.e. in terms of 'labour commanded').

After a few algebraic manipulations, we get:

$$\mathbf{p}^T = \mathbf{v}^T [\mathbf{I} - \pi \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}]^{-1}(1 + \pi). \tag{41}$$

Again our 'transformation' problem is solved.

Notice that matrix $\mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{H}$, appearing in both expressions (39) and (41), is the matrix of *vertically integrated* productive capacity defined by Pasinetti (1973).

What are the different implications—we may ask—of these two *numéraire*s, which may alternatively be chosen for the same 'prices of production'?

In (41), as may be verified by inspection, the 'prices of production' will *always* turn out to be greater than the corresponding 'values', except in the limiting case in which $\pi = 0$, and thus $\mathbf{p}^T = \mathbf{v}^T$. The reason for this is that, whereas the Marxian values are by definition expressed in terms of 'embodied labour', the 'prices of production' are expressed here—owing to (40)—in terms of 'labour commanded'. These two types of quantities of labour only coincide in the limiting case in which the rate of profit is zero. But, as long as the rate of profit is positive, each commodity will be able to purchase, in the economic system, a quantity of labour ('labour commanded') which is greater than the quantity of labour embodied in it. This may also explain why this way of fixing the *numéraire* is generally the one considered by Marxian economists: it evinces explicitly the gap between each 'price' and its labour content, owing to the presence of a profit component, which obviously disappears only when the profit rate is squeezed to zero. Analytically, this procedure has the drawback of having the Marxian values and the 'prices of production' expressed in terms of two different kinds of quantities of labour, and therefore of making not immediate the comparison between the two (having to be made via the relation between 'labour commanded' and 'labour embodied').

This drawback is avoided by the first *numéraire*, which relies on exactly the same physical (composite) commodity for both Marxian values and 'prices of production'. By contrast with (41), prices (39) are not, unilaterally, greater than the corresponding 'values'. Some prices, as we have seen, will prove to be greater than, while others will prove to be smaller than, the corresponding 'values', the borderline case being provided by the composite commodity $\mathbf{d}(1 + \sigma)$, which, *ex hypothesis*, has been set with both its 'price' and its 'value' equal to unity.

8. The rate of surplus value inside the price system

Expressions (39) and (41) are worth being looked into a little more closely. Many of their properties are by now well known to us.

In the particular case in which the rate of profit (and thus the rate of surplus value) is zero, prices (39) and prices (41) coincide with each other. Moreover, both of them coincide with Marxian values (1). Yet, as soon as the rate of profit is positive (i.e. $\pi > 0$ and thus also $\sigma > 0$), they all become different from one another. 'Prices' and 'values' will in general differ in a very complicated way, as we have seen. On the other hand, within the price system, the two alternative formulations (39) and (41)—since they express exactly the same relative prices (the same price structure), their only difference being the different chosen *numéraire*-–will obviously differ only by a scalar multiple. The interesting thing to note is that this scalar multiple (by which they differ) happens to be $(1 + \sigma)$, i.e., the same scalar multiple by which 'necessary labour' differs from total labour (the sum of 'necessary labour' plus 'surplus labour').[11] This circumstance carries a crucial consequence.

Prices (41), by being in terms of the actual wage rate, represent labour commanded as we know. Prices (39), on the other hand, are in terms of the 'ideal' wage rate, which coincides with the per-worker net product. From the way we have constructed prices (39), the assumption underlying their *numéraire* is that of a composition equal to that of the actual wage. Considered singly, these prices do not, of course, represent labour embodied in the corresponding commodities (for, by definition, it is the Marxian values that represent labour embodied). However we know that, in the aggregate, the amount of labour embodied into the *total* net product of an economic system is always the same, irrespective of its

[11] See p. 4 above.

composition. This allows us to start directly from expression (38), defining the *numéraire*, which, for prices (39), implies the remarkable relation:

$$w(1+\sigma) = w^* = 1, \tag{42}$$

or

$$\sigma = \frac{1}{w} - 1. \tag{43}$$

Notice now that this expression reproduces exactly the same relation between σ and δ given by (6), i.e. a relation in terms of physical quantities of embodied labour, although w is here the wage rate that emerges from the 'prices of production'. In other words, the particular *numéraire* (38) is allowing us, in this context, to obtain the relation between rate of surplus value and labour embodied in the real wage (or 'necessary labour') by simply using the wage rate as it emerges from the 'prices of production'. In this price system, w and δ have become interchangeable, precisely as they are in Sraffa's standard system (see section 6, p. 10, above.) But the crucial difference, with respect to the Sraffa's case (and the analysis of section 6) is that in Sraffa's it was necessary to assume a real wage $\mathbf{d}^*$ having the same composition as the standard commodity. Nothing of this sort is required here. Expression (43) is a relation which is valid in general. How can this be?

What generates this remarkable result is that the rate of surplus value is in terms of labour. As may now be realised, it does not therefore depend on the composition of that part of the net product that goes to the capitalists. It only depends on the physical composition, and on the magnitude, of the real wage. This is what confers upon the composite commodity (2), i.e. on the actual real wage, a relevance that goes beyond, and has nothing to do with, the circumstance of whether it is or it is not an 'average' commodity. Relation (43) does indeed hold for the general case![12]

To conclude, in any actual economic system (and not only in an economic system in which the net product is all expended on wage goods), it is the particular composition of the real wage—kept constant in the process of blowing it up to the point of absorbing the whole net product—that confers on the per-worker *net product* the property, if used as the *numéraire* of the price system, to make the wage rate w acquire the meaning of fraction of 'necessary labour' out of to-

[12]To be more specific, just in the same way as in the Sraffa's case the above relation depends on the wage rate being of exactly the standard composition $\mathbf{d}^*$, here too the same relation depends on the wage rate being of the composition of vector $\mathbf{d}$. However here it is precisely the composition of vector $\mathbf{d}$ that determines the magnitude of the rate of surplus.

tal labour. This was denoted by δ in section 3, but here we have a context in which it coincides with w. It follows that, in such a price system, the Marxian rate of surplus value can be shown directly from the 'prices of production' as the reciprocal of the wage rate minus one, as is revealed by expression (43).

9. Beyond Marx and beyond Sraffa. *Bouleversement* of a causal relation, considered in an extended 'normative' framework

We may stop here, in our attempt at taking advantage of Sraffa's analytical framework to improve an understanding of the Marxian efforts to penetrate into the inner characteristics of a capitalistic economic system.

If the readers think that in this way the Sraffian analysis of the Marxian framework has been pushed to its extreme possibilities, we may well even accept this attitude, provided that we do not preclude that it could be pushed further still. In this case, not only beyond Marx, but beyond Sraffa himself.

We may recall that all the Classical economists (Marx included) carried on their investigations on the assumption of wages at the subsistence level, with all the surplus of the economic system going to capitalists in the form of profits. It was argued,[13] that this assumption created problems both to Ricardo and to Marx: they were both thinking in terms of a labour theory of value, within a conceptual framework actually leading to the opposite result, i.e. to a *pure capital* theory of value.

A century later, the Kaldor-denominated 'Keynesian' theory of income distribution came to reverse this chain of reasoning—namely the idea of wages determined by subsistence necessity and profits absorbing the whole surplus: Kaldor (1955), and more substantially Pasinetti (1962b), starting from Ricardo's original formulation, proceeded in such a way as to determine *first* the 'equilibrium' profits, and then attributing the whole of the residual surplus to workers in the form of wages. As is well known, this is what then became known as the 'Kaldor-Pasinetti' theory of income distribution. Such 'equilibrium' profits, being those profits guaranteeing the amount of investment necessary for the *extended* reproduction of the economic system, could be regarded as the 'extended reproduction' profits.

In this context—and within a normative framework—the novel idea that emerged was that of a complete reversal of the causation chain, in which the macroeconomic magnitudes of any industrial economic system are to be newly

[13]This issue is treated in detail in Pasinetti (1977, Chapter V).

conceived. It is what Pasinetti calls the 'natural' rate of profit, that is an equilibrium rate of profit that would take the central and primary place, in order to perform the 'social function' of ensuring full employment and satisfactory growth of the whole economic system.

Then, what is left over from the process of technical progress would become available, and be reversed on the workers wages. This would thus appear to be the aim and purpose of the whole production process. In this context, and within a 'normative' framework, the whole growing net national income would have to be directed towards that 'ideal' framework that had appeared so obvious to the Ricardian socialists at the beginning of the industrialisation era: profits to be addressed to investment and growth, so that what remains—a surplus in reverse—would be attributed to the workers. As will be realised, this is a complete *bouleversement* of the Smith-Ricardo-Marx stand.

10. The crucial role of the growing (or hyper-integrated) subsystems

Here is where we can proceed not only beyond Marx, but also beyond Sraffa. When growth is introduced into the picture, the very Sraffa notion of *subsystem* needs to be extended. Sraffa's idea was that of "[c]onsider[ing] a system of industries (each producing a different commodity) which is in a *self-replacing state*. [...] Such a system can be subdivided into as many parts as there are commodities in its net product, in such a way that each part forms a *smaller self-replacing system* the *net product* of which consists of only one kind of commodity. These parts we shall call 'sub-systems'." (Sraffa, 1960, p. 89, emphases added)

This could indeed be argued—with Sraffa—in a self-replacing system, where there are no new investments. But once replacements are included into the means of production—and hence into the *productive capacity* of each subsystem—the net product consists of consumption commodities only, and each subsystem is in a perfectly self-replacing state, without any necessity of exchanging anything with the other subsystems. But in the presence of growth, productive capacity needs to be expanded. The net product, in this case, would therefore consists of the whole set of commodities which are used as intermediate means of production. In other words, sybsystems' net product would not consist of a single commodity anymore. Generalisation requires a modification of the concept of net output itself, stemming from the consideration that new investments—as well as all commodities directly and indirectly needed for their (re)production—are also part of the means of production, since they are going to re-enter the circular flow in the following period as means of production, as opposed to consumption goods.

Moreover, for the system to be in equilibrium—in the sense of full employment of the labour force and full utilisation of productive capacity—new investments must be proportional, in each subsystem, to the growth of final demand for the corresponding net product. It follows as a logical consequence that such a redefinition also implies a modification of the concept of *embodied labour*, which is now constituted not only by direct and indirect, but also by what has been called *hyper-indirect* labour, i.e. the quantity of labour directly and indirectly needed for the expansion of the productive capacity.[14]

Let us resume, for these purposes, the latest developments. Consider an economic system of the type already considered in the previous sections, with the sole difference that final consumption demand of each commodity i grows at the rate $\gamma_i = g + r_i$, where g is the rate of population growth and r_i the rate of growth of per capita final demand. The corresponding *growing* subsystem is defined by:[15]

$$\mathbf{q}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} + \gamma_i \mathbf{A}\mathbf{q}^{(i)} + \mathbf{c}^{(i)} \tag{44}$$

and hence:

$$\mathbf{q}^{(i)} = (\mathbf{I} - (1+\gamma_i)\mathbf{A})^{-1}\mathbf{c}^{(i)} \tag{45}$$

or, equivalently:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \gamma_i(\mathbf{I}+\mathbf{H})\mathbf{A})^{-1}(\mathbf{I}+\mathbf{H})\mathbf{c}^{(i)} \tag{46}$$

Subsystems labour and productive capacity are given by the i-th element η_i and the i-th column $\mathbf{m}_i$, respectively, of:

$$L_i = \mathbf{a}_n^T \mathbf{q}^{(i)} = \mathbf{a}_n^T(\mathbf{I} - (1+\gamma_i)\mathbf{A})^{-1}\mathbf{c}^{(i)} = \boldsymbol{\eta}_i^T \mathbf{c}^{(i)} \tag{47}$$

$$S^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{A}(\mathbf{I} - \gamma_i(\mathbf{I}+\mathbf{H})\mathbf{A})^{-1}(\mathbf{I}+\mathbf{H})\mathbf{c}^{(i)} = \mathbf{M}^{(i)}\mathbf{c}^{(i)} \tag{48}$$

Where $\boldsymbol{\eta}_i^T$ is the vector of vertically hyper-integrated labour coefficients and $\mathbf{M}^{(i)}$

[14]See Pasinetti (1981) and, more extensively, Pasinetti (1988).

[15]In what follows, to keep comparability with the framework sketched above, we are going to consider a slightly modified setting with single production. For a treatment including joint production, see the original contribution (Pasinetti, 1988).

the matrix of vertically hyper-integrated productive capacity for subsystem i.[16]

From simple algebraic manipulation, we can moreover write:

$$(\mathbf{I}-(1+\gamma_i)\mathbf{A})^{-1}=(\mathbf{I}-\gamma_i(\mathbf{I}+\mathbf{H})\mathbf{A})^{-1}(\mathbf{I}+\mathbf{H})=\mathbf{I}+\mathbf{H}+\gamma_i\mathbf{M}^{(i)}$$

which allows to write subsystem's i total quantities, total labour and productive capacity as:

$$\mathbf{q}^{(i)}=\left(\mathbf{I}+\mathbf{H}+\gamma_i\mathbf{M}^{(i)}\right)\mathbf{c}^{(i)} \tag{49}$$

$$L_i=\mathbf{a}_n^T\left(\mathbf{I}+\mathbf{H}+\gamma_i\mathbf{M}^{(i)}\right)\mathbf{c}^{(i)} \tag{50}$$

$$\mathbf{S}^{(i)}=\mathbf{A}\left(\mathbf{I}+\mathbf{H}+\gamma_i\mathbf{M}^{(i)}\right)\mathbf{c}^{(i)} \tag{51}$$

from where the three components—direct, indirect, and hyper-indirect—emerge. The relevance of expression (50) with respect to the issue of the theory of labour is clearcut: in a growing system, embodied labour is greater than Marx's value, the difference being given precisely by hyper-indirect labour.

We can now proceed to analyse production prices, starting from exactly the same system as before and setting $w=1$:

$$\mathbf{p}^T=\mathbf{a}_n^T+\mathbf{p}^T\mathbf{A}(1+\pi)=w\mathbf{a}_n^T+\mathbf{p}^T\mathbf{A}(1+\gamma_i)+\mathbf{p}^T\mathbf{A}(\pi-\gamma_i)$$

i.e.:

$$\mathbf{p}^T=\boldsymbol{\eta}_i^T+\mathbf{p}^T(\pi-\gamma_i)\mathbf{M}^{(i)}$$

or:

$$\mathbf{p}^T=\boldsymbol{\eta}_i^T(\mathbf{I}-(\pi-\gamma_i)\mathbf{M}^{(i)})^{-1} \tag{52}$$

In other words, we can write $n-1$ price *equivalent* price systems, one for each ad every growing subsystem. In all of them, it is possible to identify the linear operator which effects the transformation from values to prices of production.

It is at this juncture that the idea of equilibrium profit, coming from Kaldor-Pasinetti income distribution theory, can be introduced. More specifically, as explained in detail in Pasinetti (1981), we can single out a set of $(n-1)$ *sectoral* rates of profit which guarantee an amount of purchasing power which exactly

[16]For the meaning of the other elements of vector $\boldsymbol{\eta}_i^T$ and of the other columns of matrix $\mathbf{M}^{(i)}$, see Pasinetti (1988).

equal, in each subsystem, as to provide the equilibrium amount of new investment, i.e. that amount of investment that permits to expand productive capacity in line with the evolution of final demand for consumtion commodities. These sectoral rates of profits, or *natural* rates of profit, are given by:

$$\pi_i = g + r_i = \gamma_i \tag{53}$$

When introduced into expression (52), they lead to:

$$\mathbf{p}^T = \boldsymbol{\eta}_i^T \tag{54}$$

When the particular set of rates of profit (53) is used to close the price system(s), the concept of 'values' takes on a well-defined meaning: values are those exchange ratios that provide exactly the amount of profits, give by equilibrium new investment, allowing each single subsystem i to be kept on a steady growth path. In other words, given the evolution of final demand and the technique in use in every single time period, ther does exist a specific set of equilibrium investment, one for each growing subsystem. Given these equilibrium investments, there does exist a corresponding set of equilibrium rates of profit which guarantee long run steady growth.

In this way, value do not simply express those exchange ratios in which embodied labour is equal to labour commanded, but also those exchange ratio which can guarantee stable, full employment growth to the economic system.

To keep comparability with the Marxian setting, the price system can also be written under the assumption of advanced wages, in which case the above expression would be:

$$\mathbf{p}^T = \boldsymbol{\eta}_i^T (1 + \pi)(\mathbf{I} - (\pi - \gamma_i)\mathbf{M}^{(i)})^{-1}$$

which, by introducing equilibrium profits (53), becomes:

$$\mathbf{p}^T = \boldsymbol{\eta}_i^T (1 + \gamma_i)(\mathbf{I} - (\pi - \gamma_i)\mathbf{M}^{(i)})^{-1}$$

In other words, prices are given not only by the quantity of labour embodied in net output, but also by the addition of the quantity of labour which would be embodied in the *increased* output of the following production period. To put it differently, these prices would coincide with values if we were to choose as the *numéraire* commodity not the current quantity of labour employed, but the increased quntity of labour which is possible to employ in the following period thanks to increased final demand. This means setting $w(1+\gamma) = 1$ rather than $w = 1$ — in other words, the wage rate is not unitary but given by $1/(1+\gamma_i)$.

11. Concluding remarks

The aim of the present paper was first to go back from Sraffa's work, and particularly from his concept of 'standard system', to Classical economic theory—in the particular version of the Marxian theory of labour values.

But we have seen that Sraffa's idea of subsystem can be expanded further, in an attempt to introduce 'Keynesian' income distribution theory within a Classical multisectoral framework. This implies explicitly acknowledging the importance of economic growth, and leads to a consistent redefinition of the concept of embodied labour and, thus, of labour values. After such redefinition, labour embodied comes to express not only the direct and indirect labour necessary for the production of the net product, but also the *hyper-indirect* labour, i.e. labour necessary for the equilibrium (steady growth) expansion of each growing subsystem's productive capacity.

Clearly, a proviso is necessary at this point: as emerges from the analysis of the previous sections, the 'natural' price system implies a unique price for each *consumption* commodity, but also a whole set of prices for *intermediate* commodities, according to the subsystem in which it is used. Analytically, the issue has been dealt with, by Pasinetti (1981), by using the device of measuring intermediate commodities as *units of productive capacity*. This is another novel conception that has emerged from the above mentioned (appropriately expanded) structural dynamics analysis. In this way, each subsystem is led back to use a single, sector-specific *composite* commodity for carrying on the production process.

This remark raises the second proviso that the *equilibrium relations* which have been obtained belong to a conception of the economic system, called 'natural', which possesses many normative properties that within our current institutional framework cannot be satisfied. In the aftermath of the present economic crisis, the theoretical framework presented in the previous sections paves the way to design the appropriate institutions for leading our societies towards a path of inclusive growth.

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