Marx, the Production Function and the Old Neoclassical Equilibrium: Workable under the Same Assumptions?

Contribution to the Conference

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1. Long-period positions: a stochastic approach

The critique of capital advanced by the Cambridge economists affected both neoclassical and Marxian theories. The aim of this paper is to analyse what remains of the critique if the physical data of the theory are regarded as stochastic magnitudes. The question has already been pursued for the aggregate production function in Schefold (2013) and for Marx in Schefold (2014). Here, the critique is extended to what we call the old neoclassical equilibrium. The conception must first be differentiated from other neoclassical models.

Most economists of the 19th century and beyond shared the conviction that not only prices of homogeneous commodities and of homogeneous non-produced factors of production tend to get uniform, if the process of competition is not disrupted, but also the rates of profit on the cost of capital advanced. They analysed economic development, which is an evolutionary process in which the data like natural resources, technical knowledge and mental attitudes change slowly, by taking these data for so-called long-period positions as given and fixed. The art of the approach consisted in a periodization such that the time for convergence towards uniform prices was sufficiently long to lend credibility to the assumption that the tendency had become reality; change then was analysed on the assumption that the forces engendering change could be understood by comparing different long-period positions. Various authors, in particular Petri (2004), following in the footsteps of Garegnani (1960) and of Sraffa (1960), have shown how this method was used both by the classical economists, who started from a determination of distribution by means of a given real wage, *and* by neoclassical economists who determined distribution by means of supply and demand for labour and capital, where capital had to be conceived as given in terms of an amount of value, and the amounts of capital goods used in the several lines of production were determined endogenously; their total value corresponded to the value of the capital endowment.

This 'old neoclassical equilibrium' approach contrasts with that of modern intertemporal general equilibrium theory, where the endowments are given as quantities of capital goods and non-produced means of production, with the consequence that the rate of profit can become uniform only as a tendency over many periods (so-called turnpike theorem by Dorfman, Samuelson and Solow 1958, see Schefold 1957, pp. 425–501). Among the older neoclassical economists only Walras had a vector of endowments of capital goods inherited from the past and of non-produced factors at the beginning of a long-period position which he associated with normal prices. That this was a mistake has been pointed out by Gareganani (1960). It has been analysed in more detail by Eatwell (1987) and Petri (2004), among others. The Walrasian model of capital formation was formalised by Morishima with inequalities. Morishima showed that an equilibrium existed under rather general conditions, but this could also involve degenerate solutions, involving no reproduction of capital at all (Schefold 2015). The Walras-Morishima model therefore is not successful as a representation of how economies can reach steady reproduction, starting from arbitrary initial conditions, but it remains interesting as a reference case.

The solution of the old neoclassical economists of taking the quantity of capital as given in the form of a value magnitude has survived until today in the form of the production function which is introduced as a one-sector model, but then applied to the economy as a whole. The Cambridge debate attempted to show that the aggregation of capital, with the aim of reproducing the results of the one-sector model, was in general impossible because of paradoxes in the valuation of capital, in particular reverse capital deepening. The critique of the old neoclassical equilibrium model and that of the production function must be similar because both use the concept of aggregate capital, as will be confirmed in this paper.

Meanwhile, it has been found that the paradoxes of capital seem to occur only rarely in empirical investigations (Han and Schefold 2006), while many neoclassicals and Marxist authors disregard this critique often without knowing it. This reluctance to face objections is, to say the least, not always reasonable, but it also has rational causes. There are exceptions to the critique. It is generally accepted that it does not apply to a one-commodity world. Why should the economy as a whole not behave by and large as a one-commodity world? A more accurate investigation seems to be needed to determine the conditions under which the paradoxes can appear or not appear. It turns out that a stochastic approach changes the picture, as I have shown in three publications: Linear wage curves as in one-commodity models result, if prices are expressed in terms of the standard commodity which is a left-hand side eigenvector of the input matrix or if the labour theory of value holds, with equal organic compositions of capital, which is the case if the labour vector happens to be in the right-hand side eigenvector. It had not been shown that simple wage curves could also result from special forms of the input matrix itself. In brief: If the matrices have random properties and if the numéraire vector and the labour vector also stand in certain random relationships to each other, the wage curves tend to be straight lines. In Schefold (2013), sufficient conditions are given for the construction of approximate surrogate production functions, extending the realm in which this is possible from one-commodity world to more realistic conditions. Schefold (2013a) shows that only a few of a multiplicity of linear wage curves will appear on the envelope, if the position of the wage curve is random, and Schefold (2014) shows that a central proposition of Marx in his transformation of values into prices of production holds under analogous conditions: profits equal total surplus value.

I shall here summarise these results, adding a clarification of the method, and showing that a similar argument can be made to provide a partial justification of the old neoclassical general equilibrium model, which used the idea of capital as an endowment of a quantity of value. It will be seen that the conditions to establish the Marxian result are somewhat less restrictive than those required for the old neoclassical general equilibrium model and for the production function. By contrast, the implications for the critique of general intertemporal equilibrium theory remain to be investigated in the future.

This is not primarily a contribution to applied economics, as some thought, when they first saw the three papers mentioned above. It is primarily an exercise in pure theory. Just as Sraffa shows that the wage curve will be linear under a stated assumption (if the numéraire is equal to the standard commodity), it is here shown that the wage curve tends to be linear with probability one, if certain stochastic conditions are met. To show this with full mathematical rigour would require a very complicated paper. Like Sraffa I shall not use the most advanced mathematical methods to present the argument. As in Sraffa and as in classical and old neoclassical economics, long-period positions are assumed, without providing explicit models for the gravitation of market prices to normal prices. As John Bates Clark puts it: “In the midst of all changes there are at work forces that fix rates to which, at any one moment, wages and interest tend conform … What would be the rate of wages, if labour and capital were to remain fixed in quantity, if improvements in the mode of production were to stop, if the consolidating of capital were to cease and if the wants of consumers were never to alter?” (Clark 1899, p. vi). Assumptions in economic theory are always to some extent counterfactual. We not only work with models in which we assume that the uniformity of the rates of remuneration has been obtained, but, also here, with single product systems (although I spent so many years on joint production). Here we have the randomness assumption concerning the input matrix and various covariance-assumptions concerning the numéraire vector, the labour vector and the vector for the composition of output or the surplus. Like the assumptions about the uniformity of prices, the assumptions about the stochastic properties cannot be expected to be fulfilled perfectly in actual reality. As we can observe different rates of profits in different sectors, we can observe that indicators of the stochastic properties of the system are not proof of a perfect fulfilment of the conditions, but of a tendency towards such fulfilment. Research on this has only begun. Indicators of such fulfilment are, among others, the variance of the input-output coefficients in actual input-output tables, considered at various levels of aggregation and the spectrum of the eigenvalues.[1]

The problem of empirical application can only be touched upon in this paper, but I briefly try to reply to some objections which have been made to the earlier papers in letters and discussions. Some of these objections concerned the use of input-output tables as a proxy to represent the spectrum of techniques used in the theory. They are in fact more aggregated than the theory ideally would require. On the other hand, the theory is not

[1] The spectrum of the eigenvalues has been examined for the input-output tables of a number of economies by Mariolis and Tsoulfidis, see in particular Mariolis and Tsoulfidis (2014). I owe special gratitude to Anwar Shaikh for a provisional analysis of disaggregation on the variance of the input-output coefficients and for discussions concerning the theory of prices in random systems, see his forthcoming book (Shaikh 2015). I have received special advice on the mathematical properties of random matrices by Professors Joachim Weidmann and Götz Kersting, both in the Mathematical Faculty of my University.

concerned with the individual commodities, as produced in handicraft by individual producers and bought by individual consumers. But then one would be concerned with market prices; to begin at a certain intermediate level of aggregation is inherent in the conception of natural prices. It has also been objected that input-output tables do not describe individual methods of production; they concern average techniques. In this, they are closer to Marx with his conception of socially necessary labour time than to the dominant, cost minimising techniques considered in classical theory and also in most interpretations of Sraffa (Schefold 1988). Here we are concerned with their stochastic properties. Schefold (2013) bases the argument on a theorem by Goldstein and Neumann which states that the subdominant eigenvalues of a semi-positive matrix, the input coefficients of which are independently and identically distributed with a mean characteristic for each industry, will, if certain conditions on the variance of the coefficients in each row and on the covariance in comparison between rows (the rows stand for the industries) are met, tend to zero with probability one, as the number of the sectors tends to infinity, with the dominant eigenvalue being kept constant. Loosely speaking: All eigenvalues except the dominant eigenvalue will be small in modulus for large matrices, if the coefficients in each industry are random, with a mean specific for the industry.

Input-output matrices are in fact fairly large; hence the effect of randomness on the spectrum of eigenvalues should be visible, and it is. In order to see how, assume that the matrix is diagonalisable. We then get, for a matrix of order n, n eigenvalues $\mu_1,...,\mu_n$, which we can assume to be ordered according to modulus, such that $\mu_1 > |\mu_2| \geq ... \geq |\mu_n| \geq 0$, where we also assume that the matrix is imprimitive and $\mu_1 = dom\,\mathbf{A}$, $A \geq 0$ and $\mathbf{A}$ indecomposable. The difference $\mu_1 - |\mu_2|$ is often called the spectral gap. We measure it in percentage terms: $(\mu_1 - |\mu_2|)/\mu_1$. The theorem by Goldstein and Neumann therefore says that the spectral gap for what they define as random matrices will tend to 100 % as $n \to \infty$. The empirical analysis by Mariolis and Tsoulfidis and by Anwar Shaikh do not indicate that the spectral gap rapidly tends to 100 % for empirical input-output matrices. Rather, one finds that the gap is less then 50 %, but it seems to increase with disaggregation according to an example by Anwar Shaikh and, what is more important, the remaining eigenvalues tend to zero quite rapidly after a handful of the first few which tend to zero slowly. The implication of this finding has been discussed in Schefold (2013): This small number of eigenvalues with significant modulus can give rise to wiggles of wage curves which otherwise turn out to be stretched hyperbolas approximating linearity.

The following is a mathematical result for the spectral gap which I have not yet seen quoted in the economic literature (Haveliwala and Kamvar 2003): *If* $\mathbf{A} = \gamma\mathbf{P} + (1-\gamma)\mathbf{ce}$*, where* $\mathbf{P}$ *is a stochastic matrix* $\mathbf{P} \geq 0, \mathbf{eP} = 0$*, where* $\mathbf{c}$ *is a positive column vector and* $\mathbf{e} = (1,...,1)$ *a row vector with* $\mathbf{ec} = 1$*, the modulus of the second eigenvalue is* $|\mu_2| \leq \gamma$. $\mathbf{P}$

can here be interpreted as a probability matrix and $\mathbf{c}$ as a probability distribution. We therefore get a result which is related to that of Goldstein and Neumann. The coefficients on each row are all equal and therefore equal to their mean; instead of having this mean given and the coefficients being different with a certain variance, as in Goldstein and Neumann, they are here equal but the coefficients of the input matrix $\mathbf{A}$, which interests us, are each augmented by a matrix of perturbations. These perturbations are here all positive and represent a probability distribution.

This confirms an assertion made in Schefold (2013): our $\mathbf{A}$ matrices can be interpreted as determinate matrices of a very simple structure, namely $\mathbf{A}$ equals $\mathbf{ce}$ where $\mathbf{c} > 0$ and $\mathbf{ec} = \mu_1$ is the dominant root of $\mathbf{A}$, but this simple structure is disturbed by a matrix of perturbations which, in the case of Haveliwala and Kamvar, is semi-positive. This latter property represents a drawback, for we are interested in input-output structures where some, perhaps many, inputs are zero. The variance condition by Goldstein and Neumann allows this to happen, but not the formulation by Haveliwala and Kamvar. It would be desirable to modify their theorem accordingly. Moreover, the considerations in Schefold (2013a) suggest that a generalisation must be possible. The matrix $\mathbf{ce}$ which is being disturbed can be replaced by a matrix $\mathbf{cf}$, with any $\mathbf{f} > 0$, representing a different distribution from that indicated by $\mathbf{e}$, and $\mathbf{f}$ was interpreted as a leading industry in Schefold (2013a). These are therefore future extensions. In this paper, we stick to matrices with coefficients which are distributed independently and identically on each row.

I often hear the objection that input coefficients are not random. “Cars have four wheels!”, it is said. But cars can have six wheels, if they are lorries, and whether such interdependences show is, up to a point, a question of aggregation. The tallness of people is regarded as a random variable in a population, although there are twins.

It is also objected that random matrices of the type used by Goldstein and Neumann (henceforth to be called random matrices of type $\mathbf{ce}$) would have to show a very even distribution of input coefficients along each row (for each industry), and this seems not to be realistic. But the variance in the theorem is large enough to allow zeros in the matrix. It cannot be a sparse matrix, but it can have many zeros, if other coefficients are correspondingly larger. This is a loose formulation, but perhaps necessary to quell doubts based on a misleading intuition. I want to present a simple mathematical argument, which I have not yet seen in the literature, to show why random matrices of type $\mathbf{ce}$, of given order n, can exhibit a considerable variance of the coefficients on each row, although all non-dominant eigenvalues are small.

Consider an input matrix $\mathbf{A}$ of order n, fulfilling the same assumptions as above (semi-positive, indecomposable and diagonalisable). Hence there is an invertible matrix $\mathbf{T}$, $\mathbf{T}^{-1} = \mathbf{G}$, such that

$$\mathbf{TAT}^{-1} = \mathbf{D} = diag\{\mu_1, \mu_2, ..., \mu_n\} ,$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues on the diagonal and $\mu_1 = dom\,\mathbf{A}$. Hence also $\mathbf{AT}^{-1} = \mathbf{T}^{-1}\mathbf{D}$ and $\mathbf{T}^{-1}\mathbf{DT} = \mathbf{A}$. The rows of $\mathbf{T}$, $\mathbf{t}_i$, and the columns of $\mathbf{G}$, $\mathbf{g}^j$, are orthogonal and $\mathbf{t}_i\mathbf{g}^j = \delta_{ij}$, that is: the scalar product of the rows of $\mathbf{T}$ and the columns of $\mathbf{G}$ are zero, if row and column belong to different eigenvalues, and equal to one, if they belong to the same eigenvalue. Thus we can write

$$\begin{aligned}\mu_1\mathbf{g}^1\mathbf{t}_1 + ... + \mu_n\mathbf{g}^n\mathbf{t}_n &= \mathbf{GT}[\mu_1\mathbf{g}^1\mathbf{t}_1 + ... + \mu_n\mathbf{g}^n\mathbf{t}_n] \\ &= \mathbf{G}\begin{bmatrix}\mu_1\mathbf{t}_1 \\ \vdots \\ \mu_n\mathbf{t}_n\end{bmatrix} = \mathbf{GDT} = \mathbf{A} \\ &= \mu_1\mathbf{M}_1 + ... + \mu_n\mathbf{M}_n.\end{aligned}$$

where $\mathbf{M}_i = \mathbf{g}^i\mathbf{t}_i$ are idempotent matrices of rank 1 and orthogonal in the following sense:

$$\mathbf{M}_i\mathbf{M}_i = (\mathbf{g}^i\mathbf{t}_i)(\mathbf{g}^i\mathbf{t}_i) = \mathbf{g}^i(\mathbf{t}_i\mathbf{g}^i)\mathbf{t}_i = \mathbf{g}^i\mathbf{t}_i = \mathbf{M}_i;\ \mathbf{M}_i\mathbf{M}_j = 0\,(i \neq j).$$

Matrix $\mathbf{A}$ therefore is a linear combination of n matrices, each of rank 1, with the eigenvalues as coefficients: $\mathbf{A} = \mu_1\mathbf{M}_1 + ... + \mu_n\mathbf{M}_n$. The converse is also true, as one can show easily: If a system of matrices $\mathbf{M}_i$ of rank 1 is given, $\mathbf{M}_i$ can easily be shown to be of the form of the form $\mathbf{g}^i\mathbf{t}_i$ with $\mathbf{g}^i$ and $\mathbf{t}_i$ being a column and a row vector in n dimensional complex space, respectively. If the matrices fulfil the orthogonality conditions, the vectors $\mathbf{t}_i$ and $\mathbf{g}^i$ turn out to be the eigenvectors pertaining to given complex eigenvalues $\mu_1,...,\mu_n$.

We now can see how examples of random matrices of type $\mathbf{ce}$ with considerable variance can be generated. The difficulty is to make sure that they are non-negative. For instance, let the dominant root be a given magnitude, hence μ_1 is positive and given, and $\mathbf{t}_1 = \mathbf{e}$. Now, $n^2 - 1$ parameters can be chosen, namely the elements of the vectors $\mathbf{t}_2,...,\mathbf{t}_n$ and the remaining eigenvalues $\mu_2,...,\mu_n$. $\mathbf{G} = \mathbf{T}^{-1}$ then is determined. Even if the chosen coefficients are all positive, the resulting $\mathbf{A}$ will not necessarily be non-negative, however, since $\mathbf{G}$ will contain negative elements. Hence it is better to test the variability of the coefficients by starting from a given matrix $\mathbf{A} = \mathbf{ce} > 0$, with all non-dominant eigenvalues being zero, to combine eigenvectors of $\mathbf{A}$ and to choose a linear combination of the $\mathbf{M}_i$

by choosing small non-negative eigenvalues so as to modify $\mathbf{A}$. Here is an example for $n = 2$: Let $dom\,\mathbf{A} = 1$ be given. We choose $c_1 = 1/3$, hence $\mathbf{A}$ can be written as

$$\tilde{\mathbf{A}} = \mu_1 \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} + \mu_2 \mathbf{M}_2 ,$$

where $\mu_1 = 1$ is given. Next we choose $\mu_2 = 1/5$, so that the pre-assigned spectral gap is 80 %. We have $\mathbf{M}_2 = \mathbf{g}^2 \mathbf{t}_2$ and choose $t_{21} = 1$. Having chosen $n^2 - 1 = 3$ parameters, we obtain with $\mathbf{t}_1 = \mathbf{e}$ and $\mathbf{g}^1 = \mathbf{c}$ from the orthogonality relationships $\mathbf{t}_2 = (1, -1/2)$ and $\mathbf{g}^2 = (2/3, -2/3)^{\mathbf{T}}$. One thus finds a modified

$$\mathbf{A} = \begin{bmatrix} 7/15 & 4/15 \\ 8/15 & 11/15 \end{bmatrix};$$

it is not at all obvious that this is a random matrix of type $\mathbf{ce}$.

I hope that this example helps to understand that matrices with small non-dominant eigenvalues illustrate the property of the random matrices of the Goldstein-Neumann theorem. Other objections like the reasons for abstracting from fixed capital have been discussed in the papers referred to.

To consider the coefficients of I/O tables as random numbers is unusual and seems not to have been done before in the Sraffa tradition, but it is clear that each coefficient a_{ij} is subject to manifold accidental influences. Each is a statistical construct, based (in principle) on the observation of many firms in industry $\mathbf{a}_i$ which uses commodity j under different circumstances (local variations of the weather, affecting different coefficients in different ways, local variations in the supply of commodity j to the firms in industry i, working conditions). Marx therefore spoke of ‘averages’ and of ‘socially necessary’ techniques. The coefficients may vary and are up to a point uncertain in consequence. Dry years may mean more expense for water and less labour for the harvest. The multiplicity of the influences justifies the consideration of the coefficients a_{ij} of each industry i as independent. It could also be argued, however, that the coefficients in the columns are independent, since the commodities required by different industries depend on influences, some which are industry-specific, like armaments on wars, building on money rates of interest etc. It remains to be seen whether a variant of the Goldstein-Neumann theorem can be proved, where the distributions on the columns of $\mathbf{A}$ are independent and identical on the rows.

To consider the coefficients on the rows as identically distributed seems inappropriate at first sight, since each industry appears to have a specific group of suppliers: the food industry depends on agricultural products, the steel industry has a specific supplier: coal. If industry i delivers a unique input to industry $i+1$; $i=1,...,n-1$; and industry n delivers to industry 1, as in the following example for $n=3$

$$\mathbf{A}=\begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix},$$

or, more generally, if $a_{ij}>0$ for $(i,j)=(1,n)$ and for $i=j+1,\ \ j=1,...,n-1$ and if $a_{ij}=0$ otherwise, we have a circular system, $\mathbf{A}$ is imprimitive and the eigenvalues of $\mathbf{A}$ all have equal moduli; they are $n-th$ roots of $a_{1n}a_{21}\bullet...\bullet a_{n,n-1}$. As soon as we deviate from the circular pattern, zero eigenvalues can appear. In the following example, where industry 1 produces for industries 2 and 3 and these produce for industry 1,

$$\mathbf{A}=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

the eigenvalues are $\sqrt{2},\ \ -\sqrt{2},0$.

The habit of economists to order industries according to the destination of the products (criticised by Sraffa in 1925, Sraffa 1925) causes them to imagine a nearly circular structure where the inputs in each industry form narrow groups of positive entries and the other a_{ij} are supposed to be zero. This is less absurd at an extreme level of disaggregation, up to the point where each commodity produced and exchanged is an individual object, but the output of an industry consists of a class of goods and the inputs are many.

The mistaken conclusion can be called the fallacy of mistaken arrangements. It is known that the throws of a die are i.i.d., even if the die is loaded. Let the numbers on the die be $0,1,...,5$ instead of $1,...,6$. Imagine that there are as many such dice as there are industries, and each die is loaded in a manner specific for the industry. Let n such dice be thrown n times. The matrix of the results is analogous to an input matrix $\mathbf{A}$. The random sequence in which the numbers 0 to 5 appear in row 1 can be replaced by an ordered sequence by renumbering the columns in such a way that the first $a_{1j}=0$, the next a_{1j} equal one etc. up to the last $a_{1j}=5$. Let the first industry represent food-processing, let the zeros correspond to no input, the fives to agricultural inputs, the other numbers to other inputs and let the first die be loaded so that there is a bias in favour of the fives: then we have an analogue for ordered arrangements of the inputs to the food-processing industry, which primarily needs foodstuff as inputs, and some others like lorries and buildings as well.

The fallacy now consists in the belief that the possibility of an ordered arrangement of the inputs in one industry contradicts the hypothesis of an identical distribution of the coefficients a_{1j} of industry one and of the other industries. Identical means: almost each sample $\{a_{i,j1},...,a_{i,js}\}$ exhibits the same distribution for large s and n; $1 \le j_1 < ... < j_s \le n$. The fallacy is double. First, the ordered arrangement is found only *ex post*; it is only one ordering among $n!$ permutations of the of the a_{1j}; identical distribution then means that the same distribution is generically encountered in almost all other permutations. Second, even if a clustering of inputs can be made to appear in an input-output table by suitable renumbering of the industries and commodities, this renumbering in one industry will immediately destroy any ordering that existed in the other industries. Otherwise, production would be nearly circular.

The intuition for the assumption, on which the paper is built, therefore is as follows: the output of each industry and commodity (or class of commodities) is one unit which represents total annual production of the commodity. Each a_{ij} represents a share of this annual production, needed for reproduction. Each industry has a certain weight in the economy. Say, the car industry i is large and represents 5 % of total output, measured as $\mathbf{a}_i\bar{\mathbf{p}}/\bar{\mathbf{p}}_i$ (in terms of the prices developed in the following sections. The a_{1j} then must be roughly equal to 5 % on average. If $\mathbf{A} = \mathbf{ce} + \mathbf{P}$ as above, where $\mathbf{P}$ is a perturbation matrix with positive and negative entries and $\mathbf{A} \ge 0$, this means $c_i = 5$ %. If some commodity j is not used in the car industry, $a_{ij} = 0$ and some other a_{il} will be larger than 5 %. In this manner, 5 % is interpreted as the expected value of the coefficients a_{1j} of industry i by virtue of the weak law of large numbers. Even if the coefficients of empirical input-output matrices turn out only to approximately be independent and identically distributed, the hypothesis can legitimately be made as a strict assumption in the theory. Those who do not accept this as an explanation for the empirical distribution of the eigenvalues of input-output tables should propose another.

2. An Application to the Marxian Transformation Problem

Of course, there is much more to the Marxian theory of value than the transformation problem which has captured the attention of academic economists, but it is interesting not only because of the controversies surrounding it, but also because it is directly linked to the Marxian proposition that profits are nothing but redistributed surplus value. Since one such equality could trivially be established by choosing the surplus as the numéraire, a test of the proposition is whether the rates of profit measured in value terms and in price terms coincide and

$$r_{(value)} = \frac{M}{C+V} = \frac{P}{K+W} = r_{(prices)} \qquad (1)$$

holds. M here stands for surplus value (*Mehrwert*); otherwise, the notation is standard. The quantitative quality of surplus value M and profits P is based on the assumption, used by Marx himself, that the total product serves as numéraire, hence that $C+V+M=K+W+P$ by definition. $P=M$ then is not a trivial equality. It does, in fact, not hold in general, but only in special cases, but we want to argue that it holds in an important special case in a stochastic setting. Marx and Engels spoke of averages, with somewhat different interpretations of the term (see Schefold 2014). Since the interpretation has been discussed elsewhere, we here only focus on the quantitative relationship. To show that this exists is more than critics of Marx have admitted so far, but less than what Marx intended to show. It is one thing to demonstrate that the quantitative equality results, when it is derived from the input-output structure of the economy and another to insist that value is created by labour, that surplus value results from exploitation and that profits also qualitatively are surplus value, only distributed in proportion to capital advanced after having arisen in proportion to labour performed; the problem of the formal redundance of labour values remains.

Marx assumes that wages are advanced, and this assumption is used in Schefold (2014), but the proposition we are interested in follows also if wages are paid *ex post*, and we shall here adopt this convention so as to be able to use the same formulas afterwards for the analysis of the old neoclassical equilibrium.

Hence we have Sraffa prices

$$(1+r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p} \qquad (2)$$

in standard notation. Marx considers one technique at any one time, which evolves in the process of accumulation; it is, as already stated, not necessarily a dominant, but a socially necessary technique.

The vector of gross output $\mathbf{y}$, equal to the vector of activity levels, can be written as, if $\mathbf{b}$ is the vector of the commodities consumed by the workers and $\mathbf{s}$ the surplus in the Marxian sense, the vector of the commodities consumed by the capitalists in the stationary state (no net investment)

$$\mathbf{y} = \mathbf{yA} + \mathbf{b} + \mathbf{s}, \quad \mathbf{y} = (\mathbf{b} + \mathbf{s})(\mathbf{I} - \mathbf{A})^{-1}. \tag{3}$$

Profits $P = \mathbf{sp}(r)$, capital $K = \mathbf{yAp}(r)$ and wages $W = w\mathbf{yl}$ can be measured in the prices resulting from (2), as soon as a numéraire has been chosen, so that a wage curve $w = w(r)$ is given. The rate of profit can here be varied hypothetically, and the measurement is in prices proportional to labour values if $r = 0$, but there is an actual rate of profit r^* which is consistent with the distribution of the commodities produced as expressed in (3). As one shows easily, there is exactly one actual rate of profit r^* consistent with this physical distribution

$$r^* = \frac{P}{K + W} = \frac{\mathbf{sp}(r^*)}{\mathbf{yAp}(r^*) + w(r^*)\mathbf{yl}}. \tag{4}$$

The numéraire is denoted by $\mathbf{d}$; Marx in effect puts $\mathbf{d} = \mathbf{y}$. We now use the assumptions introduced in the first section. There are therefore n linearly independent eigenvectors (rows) such that $\mathbf{q}_i\mathbf{A} = \mu_i\mathbf{q}_i$ and linearly independent eigenvectors $\mathbf{x}^i$ (columns) such that $\mathbf{Ax}^i = \mu_i\mathbf{x}^i; i = 1,...,n;$ which allow to represent the numéraire vector $\mathbf{d}$ and the labour vector $\mathbf{l}$ as linear combinations. It is possible to normalise the eigenvectors (so-called strong normalisation) such that

$$\mathbf{d} = \mathbf{q}_1 + ... + \mathbf{q}_n; \ \mathbf{l} = \mathbf{x}^1 + ... + \mathbf{x}^n, \tag{5}$$

as one proves easily. As we had ordered the eigenvalues according to their modulus, $\mu_1 = dom\,\mathbf{A}$ and $\mathbf{q}_1$ is proportional to the standard commodity; we call it the Sraffa vector. As is well known, the labour theory of value holds if and only if the labour vector is the right-hand side Frobenius eigenvector of the input matrix. Hence $\mathbf{x}^1$ is a positive vector which, if it was the labour vector, would lead to prices being equal to values; we call it the Marx vector. The deviations $\mathbf{m}$ of the numéraire vector from the Sraffa vector and the deviations $\mathbf{v}$ of the labour vector from the Marx vector are of interest:

$$\mathbf{m} = \mathbf{d} - \mathbf{q}_1 = \mathbf{q}_2 + ... + \mathbf{q}_n; \ \mathbf{v} = \mathbf{l} - \mathbf{x}^1 = \mathbf{x}^2 + ... + \mathbf{x}^n. \tag{6}$$

If $\mathbf{A}$ is random of type $\mathbf{ce}$, $\mathbf{e}$ tends to be the left-hand side eigenvector, for $\mathbf{eA}$ tends to $\mathbf{e}(\mathbf{ce}) = (\mathbf{ec})\mathbf{e}$; therefore $\mathbf{ec} = \mu_1$ and $\mathbf{e} = \mathbf{q}_1$. The standard commodity of such a system is proportional to the summation vector $\mathbf{e}$, but it is here *not* the numéraire and the economy is not in standard proportions, and yet we shall get $P = M$!

This can be shown by means of the following formulas which are explained in more detail in the paper referred to. Using the abbreviation $\rho = 1 + r$ and the formula $(\mathbf{I} - \rho\mathbf{A})\mathbf{x}^i = (1 - \rho\mu_i)\mathbf{x}^i$, we get a representation of prices

$$\mathbf{p} = w(\mathbf{I} - (1+r)\mathbf{A})^{-1}\mathbf{l} = w\sum_{i=1}^{n}\frac{\mathbf{x}^i}{1 - \rho\mu_i} \tag{7}$$

which is a very general formula of prices; prices in terms of the wage rate p / w are a sum of hyperbolas. If the spectral gap is large enough (if the conditions of the theorem by Goldstein and Neumann are fulfilled), prices can be approximated by setting $\mu_2 = ... = \mu_n = 0$ and (7) is transformed into

$$\mathbf{p} = w\left[\frac{\mathbf{x}^1}{1 - \rho\mu_1} + \mathbf{v}\right]. \tag{8}$$

We now use that gross outputs or activity levels $\mathbf{y}$ are the numéraire $\mathbf{d}$ and the orthogonality $\mathbf{q}_i\mathbf{x}^j = 0$, $i \neq j$, to get

$$1 = \mathbf{yp} = w\left[\frac{\mathbf{q}_1\mathbf{x}^1}{1 - \rho\mu_1} + \sum_{i,j=2}^{n}\mathbf{q}_i\mathbf{x}^j\right] = w\left[\frac{\mathbf{q}_1\mathbf{x}^1}{1 - \rho\mu_1} + \mathbf{mv}\right]; \tag{9}$$

the wage curve is a hyperbola. We have so far used only the standard assumptions for Sraffa systems and the assumptions for a large spectral gap. We can invert our proposition and say: *Wage curves of Sraffa systems are simple hyperbolas except for wiggles due to non-dominant eigenvalues which are not zero*.

But here we focus on random systems which allow to go a step further. We consider the components of the deviation vectors $\mathbf{m}$ and $\mathbf{v}$ as random variables which we may assume to be uncorrelated, for the composition of output depends on factors such as the taste of consumers and the labour vector represents technology. Making therefore our second assumption $\text{cov}(\mathbf{m}, \mathbf{v}) = 0$, one gets from the standard formula for the co-variance $\mathbf{mv} = n\bar{m}\bar{v}$, where $\bar{m}$ and $\bar{v}$ are averages $\bar{m} = \mathbf{em} / n$ and $\bar{v} = \mathbf{ev} / n$. But the summation vector $\mathbf{e}$ happens to be in the limit, the Frobenius eigenvector of $\mathbf{A}$ and we can use the orthogonality relationships

$$n\bar{\mathbf{v}} = n\mathbf{ev} / n = \mathbf{e}(\mathbf{l} - \mathbf{x}^1) = \mathbf{e}(\mathbf{x}^2 + ... + \mathbf{x}^n) \to 0\ . \tag{10}$$

It follows from this second assumption that the expression $\mathbf{mv}$ in (9) can be replaced by $n\bar{m}\bar{v} = 0$, so that (9) yields a *linear wage curve*

$$\bar{w} = \frac{1-\rho\mu_1}{\mathbf{q}_1\mathbf{x}^1}; \quad (11)$$

we can insert it into the price equations (8) and get

$$\bar{\mathbf{p}} = \frac{\mathbf{x}^1}{\mathbf{q}_1\mathbf{x}^1} + (1-\rho\mu_1)\frac{\mathbf{v}}{\mathbf{q}_1\mathbf{x}^1}; \quad (12)$$

the prices and the wage rate in (12) and in (11), $\bar{\mathbf{p}}$ and $\bar{w}$, are now expressed in terms of the numéraire. We have therefore obtained a linear wage curve, although we have neither used the standard commodity nor the condition that prices are equal to values. *Prices are here a linear function of the rate of profit*, while the price vectors at n different rates of profit are linearly independent in the general case. A third result is the following. We can speak of an average of prices or values, given the normalisation. The average of prices is $\mathbf{e}\bar{\mathbf{p}}/n$. Since $\mathbf{ev}=0$, the average of prices is independent of the rate of profit:

$$(1/n)\mathbf{e}\bar{\mathbf{p}}(r) = (1/n)\frac{\mathbf{ex}^1}{\mathbf{q}_1\mathbf{x}^1} = 1/n; \quad (13)$$

this means that, for a given system and given n, *prices and values are equal on average*. That the average price or value must tend to zero, if normalised and for $n \to \infty$, is obvious. The Marxian proposition that aggregates measured in prices and values must be the same on average in the economy at large has here found a precise theoretical expression, as a proposition which is true in the limit. But the individual prices are not equal to values.

However, a third assumption is required to apply the statement to the assertion about profits and surplus value. The physical surplus is $\mathbf{s}$. We assume, for analogous reasons as above, that $\mathrm{cov}(\mathbf{s},\mathbf{v})=0$. Total profits are equal to

$$P = \mathbf{s}\bar{\mathbf{p}} = \frac{\mathbf{sx}^1}{\mathbf{q}_1\mathbf{x}^1} = \frac{\mathbf{sx}^1}{\mathbf{ex}^1}, \quad (14)$$

because $\mathbf{sv} = n\overline{sv}$ and $n\bar{v}=0$. *The amount to profits* remains constant with a virtual variation of the rate of profit and *is* therefore *equal to* the value of the surplus or *surplus value*, as Marx postulated. To interpret formula (14), one can imagine a modification of the actual system in which the actual labour vector is replaced by the Marx vector. Prices are proportional to labour values in this modified economy and profit in the actual economy is equal to profit in the modified economy, as formula (14) shows, where the values are normalised by dividing by the sum of the components of the Marx vector. The main result, however, is that *the rate of profit in value terms and in price terms* coincide, hence that (1) is fulfilled.

A general equilibrium model according to the old neoclassical theory with K given

Modern general equilibrium theory (intertemporal) is marred by its main success. The existence of equilibria is proved under conditions which are so general that we do not really know how the solutions look like, in particular, whether they are stable and unique or what explains the failure if there is underemployment of resources and how the time path of the solutions evolves over time. Another important problem is caused by degenerate solutions where one of the distributional variables is zero. We do not worry if the rent of desert land is zero, but what does equilibrium mean if the wage is zero? It is no wonder that economists often turn to the production function, if they want to derive definite results for the theory of growth, but the production function implies that one deals with long-period positions, be it in the form of comparisons or a slow transformation of the data as in Solovian growth models. It is implicitly assumed that the solutions are normal in that there is a unique rate of interest. Full employment follows from marginal productivity, with clear exceptions due to limited possibilities of substitution or the imposition of disequilibrium prices by imperfect competition or state intervention. The conditions under which the production function works can be clarified by formulating the model of the old neoclassical equilibrium explicitly, and it will now be shown that the conditions for its functioning are basically the same as those required for the existence of an approximate surrogate production function, and these in turn are similar to the conditions which we encountered in our discussion of Marx, with complications mainly due to the problem of representing technical substitution in a general equilibrium framework. We thus concentrate on the existence of normal solutions.

We assume constant returns to scale, for the reasons advanced by Sraffa in 1925 which are here appropriate, and I propose to interpret the spectrum of techniques as in my papers on the surrogate production function referred to above, where the techniques are those represented by input-output tables with n sectors and the number of tables corresponds to the number of countries h. Even the most enthusiastic defender of liberalism cannot postulate that entrepreneurs are omniscient. I like to think in this stylized model that entrepreneurs in h countries, say $h=10$, know the techniques employed in their sector (the techniques in the industries in which they are active themselves) and that they have some knowledge of what their rivals are doing in the other countries, while they have only vague ideas of what happens in other sectors than their own. The knowledge about technology thus is decentralised. The level of aggregation is assumed to be such that there are no significant links between the sectors so that, in principle, each of h

methods employed in one sector can be combined with any of h methods employed in any other sector. It is therefore possible to select h^n combinations of methods from the h input-output tables. The number of combinations is, for instance, 10^{100}, if $h=10$ and $n=100$. In all h economies, entrepreneurs strive to find and to employ the best method. The theory normally deals with the ideal solution, which is, given the rate profit, the cost minimising technique; it is then the same for all countries. But reality never quite achieves this, and success is different in different countries and industries. Hence we assume that competition has resulted in different techniques in different countries. We can leave it open whether the input-output tables in different countries reflect a different average for each industry or whether only one method is used uniformly in each industry in each country. This is how I like to think about the matter in order to get a satisfactory representation of the theory in a field where realism is very difficult to approximate and realism often is claimed for quite daring intellectual constructions.

But the formal results are based on more conventional assumptions. For what follows, it suffice to assume that there are s techniques, where $s = h^n$ in the illustration just discussed. The techniques σ; $\sigma = 1,...,s$; are denoted $\mathbf{A}^\sigma, \mathbf{l}^\sigma$. For purposes of general equilibrium theory, it is convenient to arrange all methods of production in one rectangular matrix, of hn rows and n columns, if we stick to the illustration. $\mathbf{A}$ then consists of a column of h input-output tables. It is not excluded that some countries use the same method for the production of a particular commodity. The matrix is associated with a corresponding labour vector and an output matrix which repeats the unit matrix $\mathbf{I}$ h times, arranged in a column to form output matrix $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^1 \\ \vdots \\ \mathbf{A}^h \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} \mathbf{l}^1 \\ \vdots \\ \mathbf{l}^n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix}$$

$\mathbf{A}^i, \mathbf{l}^i$ are therefore the input-output tables with associated labour vectors; $i = 1,...,h;$ and $\mathbf{A}^\sigma, \mathbf{l}^\sigma$ the techniques which can be formed by combining methods from these tables. The activity level vector is $\mathbf{y}$, if we refer to the spectrum of techniques ($\mathbf{A},\mathbf{l}$), and we write $\mathbf{y}^\sigma$ when we speak of gross outputs and the activities of any given – possibly cost minimising – technique σ.

There are n commodities which are both capital goods and consumption goods (for a similar model where consumption goods and capital goods are separate, see the Walras-Morishima model discussed in Schefold (2015). These commodities must be available in definite proportion to guarantee the stationary reproduction which we want to represent; the appropriate composition which must be available as the stocks for production is denoted by $\mathbf{f}$. The composition of $\mathbf{f}$ remains to be determined. There is labour of one kind, available in quantity L, and there is, apart from the capital goods $\mathbf{f}$, also capital as the value magnitude K. John Bates Clark (1899) insisted that capital K is mobile, while capital goods are not. The modern interpretation of this is to say that capital is "malleable", but Clark thought that the capital goods, with total value K, could individually be sold and replaced by other goods. It may sound shocking, but this conception is not so different from the Marxian one when he spoke of the circulation of capital: in the Second Volume capital is advanced in monetary form, this money capital changes form, in that the capital is used by means of production and labour power, production takes place, value is added, the produced commodities are sold and surplus value is realised, if total production is purchased. The proceeds in the form of money are distributed as revenue, but in part turned into money capital for reproduction. The reproduction of this stock of value is precarious in Marx because the realisation may fail, but Clark was confident that capital could be preserved in the abstract, even when it was a matter of changing techniques in the face of obsolescence. One example he gives is the switch from the catch of whales to the manufacturing of cloth: "As the vessels (of the whale catchers – BS) were worn out, the part of their earnings that might have been used to build more vessels was actually used to build mills. The nautical *form* (! BS) of the capital perished; but the capital survived and, as it were, migrated from the one set of material bodies to the other" (Clark 1965 [1869], p.118). It is a major problem of the approach to distinguish monetary wealth that will never be transformed into physical capital from money capital that in the process of circulation is transformed into means of production and labour power. Marx at least conceptually solves this problem by means of his theory of forms of value.

The idea that capital is physically malleable is, of course, a metaphor. There only can be realism in the idea of a lasting stock of capital as value, taking the form of ever changing capital goods, if one pursues these transformations and describes how capital goods are bought and sold, ever changing the form, while the substance remains the same – not with the invariability of a natural law such as that of the preservation of energy or mass, but with the precariousness encountered at all stages of the process, but in particular at the stage of "realisation".

Equilibrium equations, here reproduced in a modernised form as inequalities, do not capture the precarious (but not utopian) nature of the process of circulation. Who will buy the horses, when the farmers switch to tractors? But the transition is possible gradually if, as in Clark's example, amortisation funds or the replacing of old horses are used for a gradual modernisation. The argument of asset-specificity should neither be dismissed nor exaggerated.

We arrive at the following modernised inequalities and equations for the old equilibrium

$$(1+r)\mathbf{Ap} + w\mathbf{l} \geqq \mathbf{Bp} \qquad (15.1)$$

$$\mathbf{yA} \leqq \mathbf{f} \qquad (15.2)$$

$$\mathbf{yl} \leqq L \qquad (15.3)$$

$$\mathbf{y}\left[(1+r)\mathbf{Ap} + w\mathbf{l} - \mathbf{Bp}\right] = 0 \qquad (15.4)$$

$$\left[\mathbf{f} - \mathbf{yA}\right]\mathbf{p} = 0 \qquad (15.5)$$

$$\left[L - \mathbf{yl}\right]w = 0 \qquad (15.6)$$

$$\mathbf{fp} \leqq K \qquad (15.7)$$

$$\left[K - \mathbf{fp}\right]r = 0 \qquad (15.8)$$

The equations mean successively that prices are competitive, that production is bounded by the available capital goods, that employment is bounded by available labour, that only processes covering costs are activated, that underutilised means of production have price zero, that unemployment leads to a zero wage, that the value of the capital goods employed does not exceed the value of capital available and that the rate of interest is zero, if the value of capital is not fully employed. Demand for consumption goods result from the maximisation of utility. Since we are considering a system in stationary reproduction, we do not consider net savings (growth will be considered in a future paper) and saving for reproduction is implicit. Hence we have a vector of demand for consumption goods and Walras' law (15.10).

$$\mathbf{z} = \mathbf{z}(\mathbf{p}, w, r) \qquad (15.9)$$

$$rK + wL = \mathbf{zp} \qquad (15.10)$$

Capital must be reproduced. Gross investment is given by vector $\mathbf{f}^*$. $\mathbf{f}^*$ cannot exceed total production minus what is taken away for consumption and stationarity requires that the composition of capital remain the same:

$$\mathbf{z} + \mathbf{f}^* \leq \mathbf{yB} \qquad (15.11)$$

$$\mathbf{f}^* = \mathbf{f} \qquad (15.12)$$

This last of our 12 equations is formally the simplest, but conceptually the most difficult. All others result from the theory of demand and supply, broadly speaking: if demand equals supply, transactions take place at prices equal to normal costs. If demand falls short of supply, prices fall. It is not so clear how the adjustment takes place if demand exceeds supply, since prices exceeding costs are simply ruled out by implicit reference to competition. It is, however, the advantage of the old neoclassical model that normal solutions can be constructed, and their stability properties are relatively transparent. This holds, by and large, except for equation (15.12) which formulates the condition for a stationary long-period position. It should be established by gravitation. It looks like an investment function for static entrepreneurs without the ability or the will to grow and to innovate as in a Schumpeterian stationary state.

We proceed to solve equations (15). Normal prices result, given a rate of interest, from cost minimisation and the choice of a numéraire $\mathbf{d}$. We have, except at a switch-point which we may at first ignore, a unique cost minimising technique σ and a wage curve $\bar{w}_\sigma(r)$ and prices $\bar{\mathbf{p}}_\sigma(r)$ von r as above (equations 11 and 12), if the same assumptions as in the Marxian case are made regarding the techniques ($\mathbf{A}^\sigma \geq 0$ productive, indecomposable, diagonalisable, $\mathbf{l}^\sigma > 0$). It may seem a more drastic assumption (to be introduced later) to demand that the matrices be random and that the covariance conditions be fulfilled, if there is a large spectrum of techniques, but in the Marxian case the assumption was made for any technique, which really amounts to the same. To each technique there is a consumption demand $\mathbf{z}_\sigma(r)$ which we assume to be positive as soon as $(\bar{w}, r) \geq 0$. Since we are in stationary production, we have, with $\mathbf{y}_\sigma$ as the vector of activity levels for the technique chosen, $\mathbf{y}_\sigma = \mathbf{z}_\sigma + \mathbf{y}_\sigma \mathbf{A}^\sigma$, therefore

$$\mathbf{y}_\sigma = \mathbf{z}_\sigma (\mathbf{I} - \mathbf{A}^\sigma)^{-1} > 0 . \qquad (16)$$

The point is that $\mathbf{z}_\sigma$ is determined, once prices are given, and since prices depend only on the rate of interest, $\mathbf{z}_\sigma$ is a function of r and so is $\mathbf{y}_\sigma$. The augmented activity level vector $\mathbf{y}$ has the same components as $\mathbf{y}_\sigma$, where applied to the corresponding activities, and zeros otherwise. With this, equations (15.1) and (15.4) are fulfilled. One obtains the unknowns $\mathbf{f} = \mathbf{yA}$ and $\mathbf{f}^* = \mathbf{f}$, fulfilling (15.2), (15.5), (15.11) and (15.12). (15.9) and (15.10) are true identically. We still need to determine (15.3), (15.6), (15.7) and (15.8). From 2 in the form $(\mathbf{I} - \mathbf{A}^\sigma)\bar{\mathbf{p}}_\sigma = r\mathbf{A}^\sigma \bar{\mathbf{p}} + \bar{w}_\sigma \mathbf{l}^\sigma$ and using (16), we get Say's law:

$$rK + w_\sigma L = \mathbf{z}_\sigma \bar{\mathbf{p}}_\sigma = \mathbf{y}_\sigma (\mathbf{I} - \mathbf{A}^\sigma)\bar{\mathbf{p}}_\sigma = r\mathbf{y}_\sigma \mathbf{A}^\sigma \bar{\mathbf{p}} + \bar{w}\mathbf{y}_\sigma \mathbf{l}^\sigma ; \qquad (17)$$

The revenues on the left are adequate to by total output on the right. It follows that equilibrium on one market, for instance the labour market with $\mathbf{y}_\sigma = L$ (equations 15.3 and 15.6), implies equilibrium on the aggregate capital market (equations 15.7 and 15.8, using 15.2 and 15.5). Hence the famous idea that one variable which has still not been determined, distribution, so far given in the form of an arbitrary rate of interest, can be varied to clear one of these markets in order to have cleared the other. The task looks deceptively simple: fixing the wage rate to clear the labour market seems to be the obvious solution. Petri (2004) confirms that this was the approach of the old neoclassicals.

The argument, modernised, is as follows. We have a finite spectrum of techniques. There is a largest maximum rate of profit among all maximum rates of profit, say of a technique

τ, denoted R_τ, and the rate of profit, here called the rate of interest, varies between zero and this maximum rate. The wage varies accordingly and in a strictly monotonic fashion between its maximum at $r = 0$ and zero at R_τ. This is because prices change continuously at each switch of techniques. Hence $\mathbf{z}(r)$ is continuous and each $\mathbf{y}_\sigma(r)$, but activity levels change discontinuously at switch-points, where there is a transition between two techniques; generically in the form that all methods of production in all industries but one remain the same, and in this one industry the switch takes place from one method of production to another: in such a way that a linear combination of the two methods is also feasible. It follows that $\mathbf{y}(r)$ and hence labour demand $\mathbf{yl}$ is an upper semi-continuous correspondence, dependent on r or $\bar{w}$. Labour demand $L^D(r) = \mathbf{yl}$ will intersect the curve for the labour supply $L^S = L$ (which in our case is a constant) in one (case a) or several (case b) or $L^D > L^S$ for all $0 \le r \le R_\tau$ (case c) or supply exceeds demand everywhere (case d). It is clear how the fixed point theorem by Kakutani can be used to render this argument rigorous. In cases (a) and (b), the stability remains to be discussed. In case (c) the excess of the labour demand over the labour supply means that there is, according to (17), an excess of the capital supply over the demand for capital, according to (15.7), so that the rate of interest must be zero. It then follows from (17) that the labour market must be in equilibrium. Conversely in case (d): if the labour supply exceeds labour demand, the wage rate is zero and (17) implies equilibrium on the capital market.

But these extreme solutions (c) *and* (d) *are degenerate*. The wage should not only not be zero but reach a minimum. Debreu (1959) ensured a minimum standard of life by assuming that each consumer had enough resources to live by his own means, if necessary. We want at least to find the conditions which ensure normal solutions, and this leads us to the conditions required for the construction of the production function. The following proposition is crucial:

A solution to equations (15) *will be a normal solution with equilibrium both in the capital and in the labour market and with* $w > 0$, $r > 0$, *if and only if*

$$\frac{K}{L} = \frac{\mathbf{y}_\sigma \mathbf{A}^\sigma \bar{\mathbf{p}}_\sigma}{\mathbf{y}_\sigma \mathbf{l}^\sigma}, \qquad (18)$$

where σ *denotes the cost minimising technique at the given level of distribution.* We denote $\mathbf{y}_\sigma \mathbf{A}^\sigma \bar{\mathbf{p}}_\sigma / \mathbf{y}_\sigma \mathbf{l}^\sigma = k$. Condition (18) obviously is necessary: it follows from (15.8), (15.6) and (15.5), if $w > 0, r > 0$ It is also sufficient. For if the supply of capital is not equal to the demand, without loss of generality exceeding it, it follows from (18) that $K / (\mathbf{y}_\sigma \mathbf{A}^\sigma \bar{\mathbf{p}}_\sigma) = L / (\mathbf{y}_\sigma \mathbf{l}^\sigma) > 1$, but (17) implies, if both $w > 0$ and $r > 0$, that an excess supply in one market means an excess demand in the other. Hence we have a contradiction; demand and supply must be equal, if $w > 0$ and $r > 0$.

The equilibrium solution is not stable, if the optimal technique σ at each given r is, in the neighbourhood of equilibrium, not such that $\mathbf{y}_\sigma \mathbf{A}^\sigma \overline{\mathbf{p}}_\sigma / \mathbf{y}_\sigma \mathbf{l}^\sigma = k(r)$, the intensity of capital, falls as r rises. This will be true even if there is only one technique and the intensity of capital changes because of Wicksell effects. For if the amount of capital is given and the rate of profit rises and the wage rate falls, a rise of the capital intensity would mean that the demand for labour would fall so that unemployment would be increasing, causing the wage to fall further and to move away from the equilibrium value. Such an instability would also follow from reswitching and reverse capital deepening.

We thus need the inverse relationship of the intensity of capital and of the rate of profit. As the rate of profit rises from zero to R_τ, the intensity of capital should fall in principle from infinity to zero, since the given ratio K / L can *a priori* be anything. This is why the Inada conditions are postulated for production functions. If they are fulfilled, a normal equilibrium is assured. It seems that we have found conditions which are both necessary and sufficient for normal solutions.

But this is not accurate. The spectrum of techniques is inherently finite. It can only approximate the extremes of the capital-labour ratios, in that there is always a finite positive minimum and maximum for k. To avoid degenerate solutions and to fulfil (18), K and L must be assumed to lie within certain bounds which depend on the technology. And it must now be made clear how the individual techniques have to be characterised so that a monotonic fall of the capital labour ratios can be observed on the envelope of the wage curves, so as to avoid the instabilities due to capital reversals.

It is not necessary for the existence of a normal solution, but it is an assumption known for facilitating a clear analysis, if we postulate that the composition of the numéraire is equal to the composition of output. The value of output per head y then is equal to the wage rate, if $r = 0$, and the intensity of capital can be represented geometrically by means of the formula $(y - w) / r = k$. If the wage curves are sufficiently numerous to approximate a continuum, the slope of the envelope of the wage curves is equal to the slope of the individual linear wage curve tangent to it so that $k(r) = -\hat{w}'(r)$, where $\hat{w}$ is the envelope. Schefold (2013) discusses the problem which arises if wage curves are not exactly straight, so that this calculation of the intensity of capital and that by means of the formula $(y - w) / r = k$ differ, y being the output per head of the individual technique chosen at r. Two different, contradictory measures for output per head then are obtained; this difference has been called declination. It is certain to disappear only if wage curves are straight. The assumption of straight wage curves therefore is sufficient, but not generally necessary for the existence of an old neoclassical equilibrium. However, the assumption of straight wage curves is convenient if one wants to be sure to avoid declination and the instability due to reswitching and reverse capital deepening.

With these assumptions, not only the existence of an old neoclassical equilibrium but also that of the production function in per capita terms is implied, for we have, after approximating the envelope by smooth function, on the one hand for output per head,

$$y(r) = \hat{w}(r) + rk = \hat{w} - r\hat{w}' , \qquad (19)$$

on the other

$$k(r) = -\hat{w}'(r) . \qquad (20)$$

(19) and (20) are a parametric representation of the per capita production function $y = f(r)$; the marginal productivity condition is fulfilled because (19) and (20) give

$$\frac{dy}{dk} = \frac{dy/dr}{dk/dr} = \frac{\hat{w}' - \hat{w}' - r\hat{w}''}{-\hat{w}''} = r . \qquad (21)$$

The analysis of the old neoclassical equilibrium thus has led us back to the construction of the surrogate production function and to the conditions of its existence. Here the random matrices again come in. The wage curve will be approximately linear under the conditions derived in the first section of this paper. On the one hand, the subdominant eigenvalues must tend to vanish, and this will be the case in the limit for random matrices. On the other hand, we need the covariance condition for the numéraire so as to obtain equations (11) and (12). The condition must hold for all techniques whose wage curves appear on the envelope. A sacrifice had to be made in the transition from the old neoclassical equilibrium theory to the construction of the surrogate production function. The demand for consumption goods $\mathbf{z}(r)$ changes with distribution, and with it the composition of output $\mathbf{y}_\sigma(r)$, but a clear analysis of the technology seemed to require that $\mathbf{y}_\sigma(r)$ be rigidly given by the numéraire vector $\mathbf{d}$. The restriction need not be as strong in the analysis of an old neoclassical equilibrium, since such an equilibrium can be analysed locally. If an equilibrium has been determined, its stability properties depend primarily on the wage curve of the cost minimising technique. Its composition of output can be used as the numéraire and the neighbouring techniques will have a composition of output which is almost the same for reasons of continuity, so that it does not matter much, if this numéraire is kept constant for the local stability analysis. (Formally, output per head is in equilibrium for a technique with a random matrix of type $\mathbf{ce}$ in the limit equal to (compare equations 12 and 14)

$$y = \frac{\mathbf{z}_\sigma \bar{\mathbf{p}}_\sigma}{\mathbf{y}_\sigma \mathbf{l}^\sigma} = \frac{1}{\mathbf{y}_\sigma \mathbf{l}^\sigma} \frac{\mathbf{z}_\sigma \mathbf{x}_\sigma^1}{\mathbf{e}\mathbf{x}_\sigma^1} + (1 - \rho\mu_1^\sigma) \frac{\mathbf{z}_\sigma \mathbf{v}_\sigma}{\mathbf{e}\mathbf{x}_\sigma^1}. \quad (22)$$

Hence $\mathbf{z}_\sigma \bar{\mathbf{p}}_\sigma$ will be constant and independent of r , if $\mathrm{cov}(\mathbf{z}_\sigma, \mathbf{v}_\sigma) = 0$, since $n\bar{v}_\sigma = 0$ (10).)

Conclusions

It had been found that the paradoxes of capital are more rare than I at least had thought. A theoretical explanation of these findings has been proposed in Schefold (2010 and 2013). This paper confirms it and extends the applications. The paradoxes of capital are absent not only in one-commodity worlds, but also if the input matrices have random properties and the numéraire and the labour vector stand in a random relation to them. The findings can thus be explained in a stochastic setting. The critique of the general case remains valid, but its relevance is in question. A new critical argument has come up which is characteristic for the stochastic approach. If the wage curves do not deviate much from linearity and if their ordering is random, only a few wage curves will appear on the envelope, even if the number of potential techniques is very large. This supports an intuition held by many economists: if real wages are reduced in a close economy, no substantial gain in employment will result because of a substitution of labour for capital, because such substitution possibilities are quite limited. Of course, some such substitution possibilities exist. If real wages fall, more repair work will pay, more domestic labour will be employed, but there are irreversibilities; the plough will not be replaced by the spade. The argument is different for open economies. If one country lowers real wages and others do not, employment will be gained through increased exports.

It was the great merit of the debate about capital theory to question the generality of the substitution possibilities postulated in general equilibrium analysis and to shatter the belief indoctrinated by means of the production function. The other extreme consisted in the negation of normal levels for the distributional variables and the encouragement of trade union activism. The rise of wages is necessary to keep up demand when rationalisation leads to an increase in output per head, but if the rise is too fast, an induced substitution of capital for labour will set in.

I believe that a stochastic approach to the theory of production is justified. If this is accepted, it follows from this paper and the companion pieces that profits can be interpreted as redistributed surplus value in an approximation which is not a vague estimate but for which precise conditions are available. Under roughly the same conditions, the old neoclassical equilibrium is stable and determines a unique normal equilibrium which is essentially the same as that determined by means of an approximate surrogate production function. To me, this is a confirmation of the ambiguous nature of capitalism. Marx tried to prove that capitalism is antagonistic and that distribution reflects a power relationship; the neoclassical responded with marginal productivity in diverse variants. Neither side has won; political economy is a debate that goes on. The next step in the present research endeavour must be to find a better measure for the random character of input-output tables.

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References

John Bates Clark: *The Distribution of Wealth*, Reprint New York: Kelley 1965 [1899].

Gerald Debreu: *Theory of Value*, New York: Wiley 1959.

Robert Dorfman / Paul A. Samuelson / Robert Solow: *Linear Programming and Economic Analysis.* New York: McGraw Hill 1958.

John Eatwell: "Walras' theory of capital". *The New Palgrave*. A Dictionary of Economics, ed. by John Eatwell, Murray Milgate, Peter Newman; London: Macmillan 1987, vol. 4, pp. 868–872.

Pierangelo Garegnani: *Il capitale nelle teorie della distribuzione*. Milano: Giuffrè 1960.

Pierangelo Garegnani: "Savings, Investment and Capital in a System of General Intertemporal Equilibrium", in: *Sraffa and Modern Economics I*, edited by Roberto Ciccone, Christian Gehrke and Gary Mongiovi, London: Routledge 2011, pp. 13–73.

Zonghie Han / Bertram Schefold: "An Empirical Investigation of Paradoxes: Reswitching and Reverse Capital Deepening in Capital Theory", *Cambridge Journal of Economics* 30.5, 2006, pp. 737–765.

Taher Haveliwala / Sepandar D. Kamvar: The second eigenvalue of the google matrix, *Stanford University Technical Report* 2003

Theodore Mariolis / Lefteris Tsoulfidis: "On Bródy's conjecture: theory, facts and figures about instability of the US economy", in: *Economic Systems Research*, 2014 (preprint, 15 pp.).

Michio Morishima: "Existence of Solutions to the Walrasian System of Capital Formation and Credit", *Zeitschrift für Nationalökonomie*, vol. 20, 1960, pp. 238–243.

Michio Morishima: "Walras' Theory of Capital Formation", note in: *Equilibrium, Stability and Growth*, Oxford: Clarendon 1964, pp. 83–92.

Michio Morishima: *Walras' Economics. A Pure Theory of Capital and Money*, Cambridge: University Press 1977.

Fabio Petri: *General Equilibrium, Capital and Macroeconomics*. A Key to Recent Controversies in Equilibrium Theory, Cheltenham: Elgar 2004.

Bertram Schefold: The dominant technique in Joint Production Systems, *Cambridge Journal of Economics*, vol. 12 (1), 1988, pp. 97–123.

Bertram Schefold: *Normal Prices, Technical Change and Accumulation*, London: Macmillan 1997.

Bertram Schefold: “Savings, Investment and Capital in a System of General Intertemporal Equilibrium – an Extended Comment on Garegnani with a Note on Parinello“, in: *Sraffa or an Alternative Economics*, ed. by. Guglielmo Chiodi and Leonardo Ditta. Basingstoke: Macmillan 2008, pp. 127–184.

Bertram Schefold: “Families of strongly curved and of nearly linear wage curves: a contribution to the debate about the surrogate production function“, in: *Economic Theory and Economic Thought. Essays in Honour of Ian Steedman*. London and New York: Routledge 2010, pp. 117–137.

Bertram Schefold: Comment on Garegnani's "Savings, Investment and Capital in a System of General Intertemporal Equilibrium", in: *Sraffa and Modern Economics*, vol. I, ed. by Roberto Ciccone, Christian Gehrke and Gary Mongiovi. London: Routledge 2011, pp. 74–88.

Bertram Schefold: "Approximate surrogate production functions", in: *Cambridge Journal of Economics,* Oxford University Press 2013, 24 pages, doi: 10.1093/cje/bes056.

Bertram Schefold: "Only a Few Techniques Matter! On the Number of Curves on the Wage Frontier", in: Enrico Sergio Levrero, Antonella Palumbo and Antonella Stirati (eds.): *Sraffa and the Reconstruction of Economic Theory: Volume One*. Theories of Value and Distribution, Basingstoke: Palgrave Macmillan 2013a, pp. 46–69.

Bertram Schefold: “Profits Equal Surplus Value on Average and the significance of this result for the Marxian theory of accumulation”. Being a new Contribution to Engels' Prize Essay Competition, based on random matrices and on manuscripts recently published in the MEGA for the first time, accepted in principle by *Cambridge Journal of Economics*, 2014 (forthcoming).

Bertram Schefold: “Normal and Degenerate Solutions of the Walras-Morishima Model“. Contribution to the Festschrift for N.N. (forthcoming) 2015.

Anwar Shaikh: *Capitalism: Real Competition, Turbulent Dynamics and Global Crises*, Oxford: University Press 2015 (forthcoming).

Piero Sraffa: *Production of Commodities by Means of Commodities*. Cambridge: University Press 1960.