NO-ENVY AND EFFICIENCY IN AN OVERLAPPING GENERATION MODEL

Abstract. We address the issue of existence of an allocation rule that satisfies both Pareto efficiency and a suitable notion of no-envy in the overlapping generations' economy formulated by Samuelson (1958).

Recent contributions to dynamic social welfare analysis have insisted on intergenerational equity by comparing infinite horizon utility streams (see among others Fleurbaey and Michel 2003; Basu and Mitra 2003; Bossert, Sprumont and Suzumura 2007); in contrast to this literature, we will not assume any preference comparability nor the existence of an infinite living representative agent, our approach aims at finding and analysing extensions of static allocation rules to a simple dynamic overlapping generation model.

1. Introduction

Is it necessary to emphasize the importance of intergenerational concerns in the public debate ? Issues such that global warming, fishery management, forest management, sustainable development are clearly linked to intergenerational equity. These problematics are also in vivid debates of economic research. For instance Arrow et Al. 1996, Dasgupta et Al. 1999 or, for more theoretical approaches, Chichilnisky 1996, Fleurbaey and Michel 2003, Basu and Mitra 2003; Bossert, Sprumont and Suzumura 2007.

This contribution belongs to the *theoretical* literature and analyzes the possibility to avoid what could be view as a *disgusting* characteristic, i.e. assuming a representative agent or the comparability of preferences.

We aim at answering to two questions. Is it possible to give some policy indications about equity in a dynamic framework without assuming preference comparability? If yes, to which extent?

In the last four decades social welfare economists have developed many different axioms of equity that an allocation rule should satisfy to be socially defensible. Foley (1967), Kolm (1972) and Varian (1974) have insisted that a central concept of distributional equity is "no-envy": no person should prefer the consumption bundle of any other person to his/her own.

In a static economy with fixed resources to be distributed by a social planner, a pareto efficient and no-envy allocation rule is always possible for any number of agents and goods as long as convexity of preferences is assumed. We will show that, in an overlapping generation framework and with the least restrictive assumptions, distributing agents over a larger time horizon leads to some impossibility results, in the sense that it is impossible to determine an allocation rule for any possible preferences of the agents that satisfies efficiency and equity axioms, and to some weak possibily result.

The next paragraph introduces the OLG framework with the corresponding notation and states the axioms of efficiency and equity we are interested in. The third paragraph collects the possibility and impossibility theorems ordered by the assumptions on the time horizon.

2. Framework

This model describes the economy with a 2-period living overlapping agents. There is one agent per generation so that in each period there will be one young and one old agent. We will consider three possible time horizons for this model: finite time horizon, the Lionel model, with $t \in \left[\underline{t}, \bar{t}\right]$; a right infinite time horizon, the Adam model, $t \in [\underline{t}, +\infty)$; and an infinite time horizon with $t \in (-\infty, +\infty)$.

Assuming the existence of an initial time period or of a final time period will introduce in the model a special agent: Adam will be the first old agent that arises by introducing an initial time, since the model will describe his behavior in time $\underline{t}$, when he is old, and not at time $\underline{t} - 1$, when he is young; Lionel[1], conversely, will live only when young in this model, at time $\bar{t}$.

The preferences of each agent will be represented by a rational preference relation $\succsim_t$ for every agent such that $(c_t, d_t) \succsim_t \left(\bar{c}_t, \bar{d}_t\right)$ means that the consumption vector (c_t, d_t) is preferred by agent t to the consumption vector $\left(\bar{c}_t, \bar{d}_t\right)$: with t we will identify the generation that is young at time t[2] ; c_t will represent consumption of generation t when young (so their consumption at time t) and d_t is consumption of generation t when old (so their consumption at time $t+1$). From the preference relation $\succsim_t$, we can define the strict preference relation $\succ_t$ and the indifference relation $\sim_t$.[3]

The problem of the social planner that we address is to distribute among the agents the resources of this economy which amount to 1 unit of the unique commodity (eventually a composite good) per period of time: resources will be identified by a vector $\phi = (..., \phi_1, \phi_2, ..., \phi_t, ...)$ where $\phi_t = 1 \; \forall t$. The problem of the social planner is therefore to fix the consumption vectors of the agents: $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$.

Let's now introduce the axioms and some definitions we require the allocations to satisfy in this overlapping generation context.

Definition 2.1. (Feasibility) An allocation $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ is feasible if $d_{t-1} + c_t \leq \phi_t \; \forall t$.

Definition 2.2. (Pareto efficiency) A feasible allocation $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ is Pareto efficient if there is no other feasible allocation $\left(..., d_0^{'}, c_1^{'}, d_1^{'}, c_2^{'}, ..., d_t^{'}, c_{t+1}^{'}, ...\right)$ such that no agent is worse off and at least one agent is strictly better off; that is if $\neg\exists \left(..., d_0^{'}, c_1^{'}, d_1^{'}, c_2^{'}, ..., d_t^{'}, c_{t+1}^{'}, ...\right)$ s.t. $\forall t \; (c_t, d_t) \succsim_t \left(c_t^{'}, d_t^{'}\right)$ and $(c_\tau, d_\tau) \succ_\tau \left(c_\tau^{'}, d_\tau^{'}\right)$ for some τ.

[1] While Adam is notoriously the first man, Lionel is the last man on earth in the apocalyptic science fiction novel by Mary Shelley "The last man", first published in 1826.

[2] we can use time as the index of the agents because of the assumption of only one young generation per period.

[3] Notice that the special agents, Adam and Lionel, will have preferences depending only on one variable to be distributed, but to allow for comparisons between bundles with different generations we will assume that, in the period not modelized, they get a quantity of good between zero and one to be opportunely fixed.

Definition 2.3. (Stationarity) An allocation $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ is stationary if every agent obtains the same lifetime allocation; that is if $\forall t, \tau$ $c_t = c_\tau = c_{\overline{t}}$, $d_t = d_\tau = d_{\underline{t}-1}$.

The no-envy allocation has to be redefined to fit this framework. Suzumura (2002) proposed two distinct concepts for equity starting from no-envy: no-envy in overlapping consumptions, no agent should prefer the current consumption bundle of another agent to his own; no-envy in lifetime consumptions, no agent should prefer the lifetime consumption bundle of any other agent for any period to the own one.

We believe that the first concept does not fit this model: is it socially relevant that a young agent envies the contemporary old agent because of a different consumption level? to do so will impede a different distribution of wealth over time for the two agents: the agent that does not save during his first period of life is allowed, in the second period, to envy another agent that wisely saved a higher amount of wealth. The second concept of no-envy compares lifetime consumption bundles and is, we believe, a good indicator for intergenerational equity.

Definition 2.4. (No-envy) An allocation $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ is envy free if no agent strictly prefers the consumption bundle of some other agent to the own; that is if $\forall t \in [\underline{t} - 1, \overline{t}]$ it is true that $(c_t, d_t) \succsim_t (c_\tau, d_\tau) \forall \tau \neq t$, where $c_{\underline{t}-1} = d_{\overline{t}} = 1$ if Adam or Lionel are comparing other allocations to the own one, $c_{\underline{t}-1} = d_{\overline{t}} = 0$ if someone is comparing their allocation to Adam's or Lionel's one.

This assumption about Adam and Lionel are necessary to allow for some equity comparison with these special generations giving them the weaker possible position: the highest level of consumption when comparing other allocations to the own, and the minimum possible level when other people compare with this agent's allocations.

Similarly we define the concept of no domination in this framework.

Definition 2.5. (No domination) An allocation $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ satisfies no domination if no agent is assigned a consumption bundle which is strictly less then the consumption bundle of some other agent; that is if $\forall t \in [\underline{t} - 1, \overline{t}]$ it is never verified that $(c_\tau, d_\tau) \gg (c_t, d_t)$ $\forall \tau \neq t$, where $c_{\underline{t}-1} = d_{\overline{t}} = 1$ if Adam or Lionel are comparing other allocations to the own one, $c_{\underline{t}-1} = d_{\overline{t}} = 0$ if someone is comparing their allocation to Adam's or Lionel's one.

Definition 2.6. (Equal treatment of equals) An allocation $(..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ satisfies equal treatment of equals if any two agents that share the same preferences are indifferent between the allocation they are assigned; that is if for some $\tau, \theta \in [\underline{t}, \overline{t} - 1]$ $\succsim_\tau \equiv \succsim_\theta \equiv \succsim^*$ then $(c_\tau, d_\tau) \sim^* (c_\theta, d_\theta)$.

Notice that the last two definitions imply a weaker concept of equitable allocation compared to no-envy: equal treatment of equals requires that no envy is applied not to all agents, but only to agents that share the same preferences, which makes it a weaker concept; wheras no domination is weaker because we do not allow allocations that, whatever the preferences of the agents[4], will give envy: that is when the own allocation is in both components lower then the allocation given to another agent.

3. Possibility and impossibility results

The first theorem is valid for every assumption about the time horizon of the model; in the case of finite time horizon we just need four complete agents, so at least five periods[5], so $\bar{t} - \underline{t} \geq 4$.

Theorem 3.1. *If the domain of the preference relations is restricted to satisfy monotonicity, it is not possible to define an allocation rule that satisfies both pareto efficiency and no domination (resp. equal treatment of equals).*

Proof. Take agent $t-1$ and t's preferences to care only about consumption when old $(c_t, d_t) \succsim_t (\bar{c}_t, \bar{d}_t)$ if and only if $d_t \geq \bar{d}_t$ and agent $t+1$ and $t+2$'s preferences to care only about consumption when young $(c_{t+1}, d_{t+1}) \succsim_{t+1} (\bar{c}_{t+1}, \bar{d}_{t+1})$ if and only if $c_{t+1} \geq \bar{c}_{t+1}$; by efficiency $c_t = d_{t+1} = 0$ and $d_{t-1} = c_{t+2} = 1$. By no dominance (resp. by equal treatment of equals) agent t should get $d_t = 1$ (since $d_{t-1} = 1$); for the same reason agent $t+1$ should get $c_{t+1} = 1$ (since $c_{t+2} = 1$). By the resource constraint this is not possible, leading to a contradiction. □

Even if this theorem looks very restrictive, the strenghtening of the assumption on preference relations to satisfy strict monotonicity will lead to a possibility results for the Lionel model and for the Adam model [6].

Lionel model. The time horizon is therefor fixed in $t \in [1, \bar{t}]$.

Theorem 3.2. *If the domain of the preference relations is restricted to satisfy strict monotonicity, every feasible stationary allocation satisfy both Pareto efficiency and no-envy.*

Proof. Take a stationary allocation $a = (d_0, c_1, d_1, c_2, ..., d_{\bar{t}-1}, c_{\bar{t}}) = (1-\bar{c}, \bar{c}, 1-\bar{c}, \bar{c}, ..., 1-\bar{c}, \bar{c})$. Since it is a stationary allocation it will be envy free: no agent will prefer the allocation of somebody else, since the allocations are the same for every agent; so what remains to proof is Pareto efficiency.

An allocation a' will differ from the previous one for at least an agent $t \in [0, \bar{t}]$: suppose consumption when young of agent t is greater; this difference will necessarily imply, by the feasibility constraint, a lower consumption when old of the previous generation. In order to compensate this agent $t-1$ it is necessary to increase his consumption when young, but this will reduce consumption when old of generation $t-2$ that should be compensated by increasing again consumption when young. This process will stop with the first agent, Adam, that can not be compensated with an increase of consumption when young since he lives only one period as old agent making this new allocation not a Pareto improvement. The same argument applies if for some agent t consumption when old is higher, as agent

[4] We need but to assume no satiation, strict monotonic preferences.

[5] Actually, two complete agents and three periods is sufficient for the same result where equal treatment of equals is substituted with no domination.

[6] From now on initial time will be fixed for notational semplicity: $\underline{t} = 1$.

Lionel can not be compensated for the decrease in consumption when young. These two impossibilities prove the result. □

This theorem shows the effect of the assumption about the time horizon: the two special agents, Adam and Lionel, will never concordate a different stationary allocation, so all the other agents in-between result to be "victim" of their crucial role. In fact, the preferences of the agents have no importance at all, the allocations are chosen by Adam and Lionel that play the part of the two ancient rome consuls (the analysis of the dictatorial position of one agent, or the oligocracy that seems to characterize this context are of great importance, not only because of the Arrowian impossibility theorem, but also because dictatorialship was found to be a necessary condition also for intergenerational equity analysis, see among other Bossert, Sprumont and Suzumura 2008).

Adam case. Moving to the Adam model, the following theorem will show how the role of Adam remains untouched, but more interestingly the Lionel's decision power will not vanish or is devoluted in favour of Adam or some future generation; we should rather say that Lionel is substituted by a "new" kind of agent: the least impatient agent. We will define him as follows:

Definition 3.3. (Least impatient agent). The least impatent agent, Lia, is the agent that preferes the stationary feasible allocation with lowest consumption when young; that is $c_{Lia} \leq c_t$ $\forall t \in [1, +\infty)$ with $(c_t, 1-c_t) \succsim_t (c, 1-c)$ $\forall c \in [0,1]$; if the minimum is not reached, Lia will be a fictitious agent such that $c_{Lia} = inf\left[\{c_t\}_{t\in[1,+\infty)}\right]$ with $(c_t, 1-c_t) \succsim_t (c, 1-c)$ $\forall c \in [0,1]$.

Theorem 3.4. *If the domain of the preference relations is restricted to satisfy strict monotonicity, all the stationary allocations* $(1-\lambda c_{Lia}, \lambda c_{Lia}, ..., 1-\lambda c_{Lia}, \lambda c_{Lia}, ...)$ *with* $\lambda \in [0,1]$ *satisfy both Pareto efficiency and no-envy.*

Proof. Since no-envy is a byproduct of all stationary allocations, it remains to proof efficiency in Pareto's sense. Assume the allocation $(d_0, c_1, d_1, c_2, ..., d_{t-1}, c_t, ...) = (1-\lambda c_{Lia}, \lambda c_{Lia}, 1-\lambda c_{Lia}, \lambda c_{Lia}, ..., 1-\lambda c_{Lia}, \lambda c_{Lia}, ...)$ is not Pareto efficient; there will exist another allocation $\left(d_0^{'}, c_1^{'}, d_1^{'}, c_2^{'}, ..., d_t^{'}, c_{t+1}^{'}, ...\right)$ that will differ for at least one element from the previous one. A greater consumption when young c_t for any agent $t \in [1, \infty)$ will lead to a reduction of consumption of the first old, Adam, after compensatig the agents in-between and is therefore not a Pareto improvement; conversely, a higher consumption when old d_t for any agent $t \in [0, \infty)$ will make compensation for the next agents not feasible: the increase in consumption when old necessary to compensate for the welfare loss is higher than the decrease in consumption when young because of the definition of least impatient allocation; in fact $\forall \tau \in [1, \infty)$ $(c_{Lia}, 1-c_{Lia}) \succ_\tau (\lambda c_{Lia}, 1-\lambda c_{Lia})$ for $\lambda \in [0,1)$ so compensation should be more than proportional and becomes explosive per iteration to infinite while resources are bounded. These contradictions prove the theorem. □

From a point of view of intergenerational equity we do prefer the $(c_{Lia}, 1-c_{Lia})$ allocation to the allocation that gives all the resources to the old generation: but it should be clear that choosing $\lambda = 1$ is not equivalent to eliminating the semi-dictatorial role of the first old. As the next paragraph will show, if you assume an infinite time horizon, Lia vanishes together with Adam and no allocation rule satisfying Pareto efficiency and no-evny can be found.

Infinite case. Let's now assume time $t \in (-\infty, +\infty)$: in this case the previous impossibility result is restored as the following theorem proves.

Theorem 3.5. *If the domain of the preference relations is restricted to satisfy strict monotonicity, it is not possible to define an allocation rule that satisfies both pareto efficiency and equal treatment of equals.*

Proof. Take agent t's preferences described by the following functions:

$$U_t(c_t, d_t) = \begin{cases} U_\theta(c_t, d_t) & if \quad t \leq 0 \\ U_\tau(c_t, d_t) & if \quad t > 0 \end{cases}$$

such that the preferred stationary allocation for agent θ (so under the constraint that $c_t + d_t = 1$) is $(c_\theta, d_\theta) = (1, 0)$ while the equivalent allocation for agent τ is $(c_\tau, d_\tau) = (0, 1)$.

Assume $\exists \bar{t} > 0$ such that $(c_{\bar{t}}, d_{\bar{t}}) \succ_{\bar{t}} (0, 1)$; by ETE we would need $(c_t, d_t) \succ_t (0, 1)\ \forall t \geq \bar{t}$, but this allocation would not be sustainable since there is no stationary allocation on that indifference curve ($c_t + d_t > 1\ \forall t \geq \bar{t}$, so moving on that indifference curve is not possible by feasibility constraint). Since $(c_t, d_t) = (0, 1)\ \forall t > 0$ is feasible we can also exclude that $\exists \tilde{t} > 0$ such that $(c_{\tilde{t}}, d_{\tilde{t}}) \prec_{\tilde{t}} (0, 1)$. Hence, $(c_t, d_t) \sim_t (0, 1)\ \forall t > 0$. If $\exists t' > 0$ such that $(0, 1) \neq (c_{t'}, d_{t'})$ and $(c_t, d_t) \sim_t (0, 1)\ \forall t \geq t'$ the allocation is not PE since it is dominated by the allocation that gives $(c_{t'}, 1) \succ_{t'} (0, 1)$ to agent t' and the indifferent allocation $(0, 1)$ to all later agents. So by PE and ETE it must be that $(c_t, d_t) = (0, 1)\ \forall t > 0$.

A similar argument proves $(c_t, d_t) = (1, 0)\ \forall t \leq 0$. But $(c_t, d_t) = (1, 0)\ \forall t \leq 0$ and $(c_t, d_t) = (0, 1)\ \forall t > 0$ is not Pareto efficient since there remains one unit of good in period $t = 1$ which is not allocated. □

The question that arises is: is it possible to weaken further the intergenerational equity requirment in order to have some possibility result? A weaker requirment, that seems very interesting from the point of view of sustainability, is the "one directional" no envy: we require that future generations do not envy previous ones. The idea is that an allocation is sustainable if future agents are not treated worse then previous ones.

Definition 3.6. (One directional no-envy) An allocation $a = (..., d_0, c_1, d_1, c_2, ..., d_t, c_{t+1}, ...)$ satisfies one directional no-envy if future generations prefer the own bundle to the boundle of previous generations; that is if $(c_\tau, d_\tau) \succsim_\tau (c_\theta, d_\theta)\ \forall \tau > \theta$.

Theorem 3.7. *If the domain of the preference relations is restricted to satisfy strict monotonicity, there exists no allocation rule that satisfies Pareto efficiency and one directional no-envy.*

Proof. The proofs follows immediately from the previous one. If future generations should be treated at least as good as previous ones it will still be the case that $(c_t, d_t) = (0, 1)\ \forall t > 0$. By efficiency agent 0 should receive $(c_0, 1)$ and by ODNE of future generations $c_0 = 0$; this requires, by iteration, that $(c_t, d_t) = (0, 1)\ \forall t \in (-\infty, +\infty)$ that was already shown to be inefficient. □

References

[1] Arrow, K. and W. R. Cline, K-G Maler, M. Munasinghe, R. Squintieri, J. E. Stiglitz "Intertemporal Equity, Discounting, and Economic Efficiency"in Climate Change 1995: Economic and Social Dimensions of Climate Change edited by J. P. Bruce et Al., pp. 125-144.

[2] Basu, K. and T. Mitra (2003) "Utilitarianism for Infinite Utility Streams with Intergererational Equity: the Impossibility of being Paretian", *Econometrica*, vol. 71, pp. 1557-1563.

[3] Bossert, W. and Y. Sprumont, K. Suzumura (2007) "Ordering Infinite Utility Streams", *Journal of Economic Theory*, vol. 135, pp. 579–589

[4] Chichilnisky, G. "An Axiomatic Approach to Sustainable Development", *Social Choice and Welfare*, vol. 13, pp. 231-257.

[5] Dasgupta, P. and K. G. Maeler, S. Barrett "Intergenerational Equity, Social Discounting and Global Warming"in Discounting and Intergenerational Equity edited by P. R. Portney and J. P. Weyant, pp. 51-78.

[6] Fleurbaey, M. and P. Michel (2003) "Intertemporal Equity and the Extension of the Ramsey Criterion", *Journal of Mathematical Economics*, vol. 39, pp. 777-802.

[7] Foley, D. (1967) "Resource Allocation and the Public Sector", *Yale Economic Essays*, vol. 7, pp. 45-98.

[8] Kolm, S-C. (1972) *Justice et Equité*, Paris, CNRS.

[9] Samuelson, P. A. (1958) "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money", *Journal of Political Economy*, vol. 66, pp. 467-482.

[10] Shinotsuka, T. and K. Suga, K. Suzumura, K. Tadenuma (2007) "Equity and Efficiency in Overlapping Generation Economies" in Intergenerational Equity and Sustainability (J. Roemer and K. Suzumura), *Palgrave Macmillan.*

[11] Suzumura, K. (2002) "On the Concept of Intergenerational Equity", mimeo.

[12] Varian, H. (1974) "Equity, envy and efficiency", *Journal of Economic Theory*, vol. 9, pp. 63-91.