Statistics and Probability Letters 214 (2024) 110185

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

The log $GARCH$ stochastic volatility model

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ARTICLE INFO

Keywords:
Log*GARCH* model
Stochastic volatility model
Stationarity
Higher order moments
Particle filtering
EM algorithm

ABSTRACT

This article introduces a new class of stochastic volatility models called log $GARCH$ Stochastic Volatility models (log $GARCH$-SV). We establish the strict stationarity and second-order stationarity properties of this model class. Additionally, we provide conditions for the existence of higher-order moments. To estimate the parameters of the proposed model, we utilize a sequential Monte Carlo method. Finally, we assess the performance of the suggested estimation method through a simulation study.

1. Introduction

Over the last three decades, there has been increasing interest in modeling the dynamic variation in volatility and behavior of high-frequency financial time series. The stochastic volatility model proposed by Taylor (1982) has been instrumental in this field. Extensions of the autoregressive stochastic volatility model (AR-SV) have been proposed, including incorporating the leverage effect (e.g., Black, 1986; Boussaha et al., 2023; Jacquier et al., 2004) and modeling volatility as a long-memory process or exhibiting long-range persistence (e.g., Ding et al., 1993; Harvey, 2007). Multivariate stochastic volatility models (e.g., Harvey et al., 1994; Pitt and Shephard, 1999) have also been introduced to capture time-varying volatilities and interdependencies in financial time series.

Stochastic volatility (SV) models provide an alternative to Generalized Autoregressive Conditional Heteroscedasticity ($GARCH$) models for analyzing time-dependent variances. SV models capture the stochastic nature of volatility, allowing for a wider range of volatility dynamics compared to the deterministic nature of $GARCH$-type models. The flexibility of SV models to incorporate an innovation term in the latent volatility process leads to a more realistic representation of real-world financial data, as highlighted in studies such as Carnero et al. (2004). Various estimation methods, including generalized method of moments (Melino and Turnbull, 1990), Bayesian approaches with Markov chain Monte Carlo methods (see Chib et al., 2002; Doucet et al., 2000), and maximum likelihood methods with particle filters (Danielson, 1994), have been developed for parameter estimation in AR-SV models.

In this paper, our aim is to introduce a new formulation of SV models that extends the classical AR-SV one. This new formulation is known as the log-$GARCH$ Stochastic Volatility models (log $GARCH$-SV). This new model allows for a connection between the observed process and its previous values indirectly through the dynamics of log-volatility. Specifically, the latent volatility is modeled by incorporating a regression trend. This trend is represented by the log-squared past observation and is added to the classical AR-SV model. By including this regression trend, we enhance the flexibility and explanatory power of the SV model in capturing the dynamics of volatility. For more details on the impact of integrating this regression trend, one can refer, for instance, to Francq and Sucarrat (2018), Sucarrat (2019), Sucarrat and Escribano (2018) and the references therein.

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https://doi.org/10.1016/j.spl.2024.110185
Received 19 July 2023; Received in revised form 28 March 2024; Accepted 11 June 2024
Available online 22 June 2024

The rest of the paper is structured as follows. In Section 2, we present the definition of the model under study. In Section 3, we provide some probabilistic properties of the proposed model, specifically focusing on the strict and second order stationarity and the existence of higher-order moments. In Section 4, estimation of the parameters of the model using a sequential Monte Carlo method is undertaken. Finally, we present a simulation study to evaluate the performance of the proposed estimation method in the last section.

2. The $\log GARCH\text{-}SV$ model

A stochastic process $\{\varepsilon_t; t \in \mathbb{Z}\}$ has a $\log GARCH\text{-}SV$ representation if it satisfies the stochastic difference equation:

$$\varepsilon_t = \eta_t \exp\left\{x_t/2\right\} \text{ and } x_t = \mu + \alpha \log \varepsilon_{t-1}^2 + \beta x_{t-1} + \delta e_t,\ t \in \mathbb{Z}, \tag{1}$$

where ε_t is typically interpreted as a financial return (which may be adjusted for mean) or the residual term in a regression analysis. (η_t) and (e_t) represent two independent sequences of independent and identically distributed (*i.i.d.*) zero-mean and unit-variance random variables. Additionally, we make the assumption that $\mathbb{P}(\eta_t = 0) = 0$. Finally, the set of model parameters includes μ, α, β and δ. Under conditions ensuring the existence of a solution, the variable x_t is referred to as the log-volatility of ε_t. As in $\log GARCH$ and SV models, the magnitude of ε_t is proportional to $\exp\left\{\frac{1}{2}x_t\right\}$. The connection between ε_t and its previous values is established indirectly through the log-volatility dynamics. Specifically, the log-volatility at time t can be expressed in an autoregressive form that involves three elements: the lagged log-volatility x_{t-1}, the drift term γ, and the centered random variable w_t. The drift term γ can be obtained by multiplying the expected value of the logarithm of η_t^2 by α and adding the parameter μ to the product. Additionally, w_t can be defined as α multiplied by the difference between the logarithm of η_t^2 and its expected value, plus δe_t. This equation can be represented as:

$$x_t = \gamma + (\alpha + \beta) x_{t-1} + w_t, \tag{2}$$

where $\gamma = \mu + \alpha \mathbb{E}\left(\log \eta_{t-1}^2\right)$ and $w_t = \alpha\left(\log \eta_{t-1}^2 - \mathbb{E}\left(\log \eta_{t-1}^2\right)\right) + \delta e_t$. It should be noted that (w_t) is a sequence of *i.i.d.* random variables with zero mean and variance $var(w_t) := \sigma_w^2 = \alpha^2 var\left(\log \eta_1^2\right) + \delta^2$. The parameter γ acts as a scaling factor for the volatility, whereas σ_w^2 represents the volatility of the log-volatility. If η_t is normally distributed, the variance of w_t is given by $\sigma_w^2 = \alpha^2 \frac{\pi^2}{2} + \delta^2$. Furthermore, if e_t is also normally distributed, the moment generating function of w_t is determined as follows: $M_w(z) = \mathbb{E}\left(\exp\left\{z w_t\right\}\right) = \exp\left\{\frac{z^2\delta^2}{2} - z\alpha \mathbb{E}\left(\log \eta_{t-1}^2\right)\right\} \frac{\Gamma\left(z\alpha + \frac{1}{2}\right)}{2^{-z\alpha}\sqrt{\pi}}$, provided that $z\alpha > -1/2$, where $\mathbb{E}\left(\log \eta_{t-1}^2\right)$ can be approximated by -1.2749. It becomes apparent that in this Gaussian scenario, a negative α amplifies the significance of the scaling factor compared to a positive α. As a result, models with a negative α can effectively represent returns and volatilities with significant magnitudes. The model (1) is referred to as the $\log GARCH\text{-}SV$ model since it specifies the volatility as a $\log GARCH(1,1)$ model. It should be noted that this model is more flexible than the $\log GARCH$ model due to the presence of an additional innovation term in the log-volatility dynamics. Moreover, when the stochastic component of volatility, e_t, is absent, the model reduces to $\log GARCH$, which highlights a substantial difference in modeling variance between $\log GARCH$ and $\log GARCH\text{-}SV$ approaches. It is worth noting that unlike $GARCH$ -type models, the volatility x_t cannot be interpreted as the conditional variance of ε_t in $\log GARCH\text{-}SV$ model, since the conditional variance is not explicitly stated. Another way in which our model differs from $GARCH$ specifications is that instead of specifying $AR(1)$ dynamics on the volatility, we specify it on the log -volatility, as shown in (2). As a result, the parameters μ, α and β are not necessarily subject to positivity constraints. The parameter $\alpha + \beta$ represents the persistence coefficient in our model. When $\alpha + \beta$ is close to 1, a positive shock to the log-volatility (as a result of a large positive value of w_t) has a persistent effect on $\exp\{x_t\}$: the volatility will remain high for several periods. Similarly, a negative shock to the log-volatility, indicating a small level for the volatility $\exp\{x_t\}$, is also persistent when $\alpha + \beta$ is close to 1. If $\alpha + \beta$ is close to 0, the effect of large shocks dissipates rapidly. When $\alpha + \beta$ is close to -1, a positive shock leads to an instantaneously large volatility. However, in the subsequent period, this volatility decreases and subsequently oscillates between large and small values, provided there are no other significant shocks. A negative shock produces similar alternating effects. The occurrence of such behaviors in financial series is uncommon, making negative values of $\alpha + \beta$ impractical or unrealistic in practice.

3. Stationarity, moments and dynamic structure

The dependence between noise (η_t) and log-volatility sequence (x_t) in the $\log GARCH\text{-}SV$ model (1) introduces a much more complex probabilistic structure compared to a basic SV model which lacks explicit feedback of current log-volatility with previous observations. However, this structure remains relatively simpler than that of a $GARCH$ process. In fact, the first equation of (1) indicates that the existence of a causal strict stationary solution to (1) is the same as the existence of a causal strict stationary solution to the AR model defined in (2).

Theorem 1. *The $\log GARCH\text{-}SV$ model, as defined in (1), admits a unique strictly stationary and ergodic solution which is non-anticipative, given by*

$$\varepsilon_t = \eta_t \exp\left\{\frac{\gamma}{2(1-\alpha-\beta)} + \frac{1}{2}\sum_{l\geq 0}(\alpha+\beta)^l w_{t-l}\right\},\ t \in \mathbb{Z}, \tag{3}$$

where the series in (3) converges almost surely, if and only if, $|\alpha + \beta| < 1$. Moreover, if (e_t) and (η_t) are normally distributed, and $\min\{(\alpha+\beta)\alpha, \alpha\} > -\frac{1}{2}$, the solution (3) is also second-order stationary.

Proof. Please refer to the supplementary material.

Remark 1. In the general case, and in contrast to the standard Gaussian assumption made on the distributions of (e_t) and (η_t), the condition $|\alpha+\beta| < 1$ alone is not sufficient to ensure the second-order stationarity of (1). Additionally, it is necessary to satisfy the following condition: $\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right) < \infty$. To demonstrate the finiteness of $\mathbb{E}\left(\varepsilon^2_{s+\tau S}\right)$, it is enough to prove that $\mathbb{E}\left(\prod_{l\geq 0}\exp\left\{(\alpha+\beta)^l w_{t-l}\right\}\right) < \infty$. This is true according to Fatou's lemma, as $\left(U_{t,L}\right)_{L\geq 0}$, where $U_{t,L} = \prod_{l=0}^{L}\exp\left\{\frac{1}{2}(\alpha+\beta)^l w_{t-l}\right\}$, is a sequence of positive integrable random variables that converges almost surely.

When $\alpha = 0$, the previous theorem aligns with the findings of Francq and Zakoïan (Francq and Zakoian, 2019, Theorem 12.1 and Theorem 12.2) concerning the strict and second-order stationarity of AR-SV models. In addition, if e_t follows a standard normal distribution, our result coincides with the statement made in Francq and Zakoïan (Francq and Zakoian, 2019, Remarks 12.1). Furthermore, the explicit expression for the variance of ε_t, given by $var\left(\varepsilon_t\right) = \mathbb{E}\left(\varepsilon_t^2\right) = \exp\left\{\frac{\mu}{1-\beta} + \frac{\delta^2}{2(1-\beta^2)}\right\}$, sets it apart from our current case. Specifically, when $\alpha \neq 0$, establishing a more explicit expression requires determining the infinite product $\prod_{l\geq 0}\Gamma\left(\alpha(\alpha+\beta)^l + \frac{1}{2}\right)\Big/ 2^{-\alpha(\alpha+\beta)^l}\sqrt{\pi}$, in closed form, using the model's parameters. However, this task proves to be challenging.

Theorem 2. *Let $\left\{\varepsilon_t; t\in\mathbb{Z}\right\}$ be a strict stationary solution of (1), with $\mathbb{E}\left(\left|\eta_t\right|^r\right) < \infty$, for any $r \in \mathbb{N}^*$. Then a sufficient condition for $\mathbb{E}\left(\varepsilon_t^r\right)$ to be finite is that $\prod_{l\geq 0} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right) < \infty$. Furthermore, the closed-form of the rth moment of ε_t, is given by: $\mu_{\varepsilon^r} := \mathbb{E}\left(\varepsilon_t^r\right) = \mathbb{E}\left(\eta_t^r\right)\exp\left\{\frac{r\gamma}{2(1-\alpha-\beta)}\right\}\prod_{l\geq 0} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)$.*

Proof. Please refer to the supplementary material.

Remark 2. From (3), the conditions necessary for the existence of a fourth-order moment for $\left(\varepsilon_t\right)$ are as follows: $|\alpha+\beta| < 1$, $\prod_{l\geq 0} M_w\left(2(\alpha+\beta)^l\right) < \infty$ and $\mathbb{E}\left(\eta_t^4\right) < \infty$. Under these conditions, we have $\kappa_\varepsilon := \mathbb{E}\left(\varepsilon_t^4\right)\Big/\left[\mathbb{E}\left(\varepsilon_t^2\right)\right]^2 = \kappa_\eta \prod_{l\geq 0}\frac{\mathbb{E}\left(\exp\left\{2(\alpha+\beta)^l w_{t-l}\right\}\right)}{\left(\mathbb{E}\left(\exp\left\{(\alpha+\beta)^l w_{t-l}\right\}\right)\right)^2} = \kappa_\eta\prod_{l\geq 0}\kappa^{(l)}$, where κ_η represents the kurtosis coefficient of $\left(\eta_t\right)$, while $\kappa^{(l)}$ represents the kurtosis of $\exp\left\{\frac{1}{2}(\alpha+\beta)^l w_t\right\}$. As the distribution of $\left(w_t\right)$ is not degenerate, $\kappa^{(l)}$ is greater than 1. Consequently, the kurtosis of the distribution of $\left(\varepsilon_t\right)$ is greater than that of $\left(\eta_t\right)$. If $\left(\eta_t\right)$ follows a normal distribution, it is evident that the minimum value of κ_ε is determined by the kurtosis coefficient of $\left(\eta_t\right)$, which equals 3. Moreover, if $\left(e_t\right)$ is also normally distributed, we have $\kappa_\varepsilon = 3\exp\left\{\frac{\delta^2}{1-(\alpha+\beta)^2}\right\}\prod_{l\geq 0}\frac{\Gamma\left(2a_l+\frac{1}{2}\right)\sqrt{\pi}}{\left(\Gamma\left(a_l+\frac{1}{2}\right)\right)^2} > 3$, provided that $\min\left\{(\alpha+\beta)\alpha, \alpha\right\} > -\frac{1}{4}$.

Proposition 1. *Let $\left\{\varepsilon_t; t\in\mathbb{Z}\right\}$ be a stationary solution of (1), satisfying $\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right) < \infty$. Then $\left\{\varepsilon_t;\ t\in\mathbb{Z}\right\}$ is a weak white noise process with variance $\sigma_\varepsilon^2 := var\left(\varepsilon_t\right) = \exp\left\{\frac{\gamma}{(1-\alpha-\beta)}\right\}\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right)$.*

Proof. Please refer to the supplementary material.

4. Estimating the $\log GARCH$-SV model

The estimation of parameters in the $\log GARCH$-SV model (1) is challenging due to the unobservability of the volatility, which requires techniques for handling missing data. One such method is the EM algorithm (Dempster et al., 1977), a popular iterative method for computing maximum likelihood estimates in incomplete data problems. The EM algorithm consists of two steps in each iteration: the E-step and the M-step, where the expected likelihood is calculated and the parameter estimates are updated. However, for complex problems like $\log GARCH$-SV, the expected likelihood is often intractable, and Monte Carlo methods such as particle filters are more efficient. Particle filters are sequential Monte Carlo methods that can solve optimal estimation problems in nonlinear and non-Gaussian state-space models (see, e.g. Doucet and Johansen, 2011), making them a powerful tool in the $\log GARCH$-SV framework. Consider the linearized state-space representation obtained by applying the logarithm to the squared observed process ε_t^2. This yields a state-space representation of the model given by:

$$y_t = x_t + d + u_t \text{ and } x_t = \mu + \alpha y_{t-1} + \beta x_{t-1} + \delta e_t, t\in\mathbb{Z}. \tag{4}$$

This state-space representation involves three components: the log-transformed observation $y_t = \log\left(\varepsilon_t^2\right)$, the drift term d, and the centered random variable u_t. We can obtain the drift term d by taking the expected value of the logarithm of η_t^2. Finally, we define u_t as the difference between the logarithm of η_t^2 and its expected value. Note that the random variable u_t has a centered $\log\chi^2$ distribution with one degree of freedom when η_t is assumed to be Gaussian, which is a common assumption in the literature. Let $\underline{Y} = (x_0,\ x_1,\dots,x_n, y_1, y_2,\dots,y_n)$ and $\underline{X} = \left(x_0,\ x_1,\dots,x_n\right)$ denote the vector containing, respectively, the complete data and the log-volatility data. For a given realization $y = \left(y_1, y_2,\dots,y_n\right)$ of stationary model (4), the twice-complete log-conditional likelihood function (given initial value y_0) of the parameter vector $\theta = (\mu,\alpha,\beta,\delta)'$, can be expressed as follows:

$$2\mathbf{L}(\theta;\underline{Y}) = -\log\sigma_x^2 - \frac{\left(x_0-\mu_x\right)^2}{\sigma_x^2} - n\log\delta^2 - \sum_{t=1}^n\frac{\left(x_t-\mu-\alpha y_{t-1}-\beta x_{t-1}\right)^2}{\delta^2}$$
$$-\sum_{t=1}^n\exp\left(y_t-x_t\right) + \sum_{t=1}^n\left(y_t-x_t\right) - (2n+1)\log(2\pi).$$

Table 1
Results of the first and second simulation study for the $\log GARCH\text{-}SV$ model.

		First simulation				Second simulation			
		μ	α	β	δ^2	μ	α	β	δ^2
	TV	1.2000	0.3000	0.6000	1.0000	0.9000	−0.1200	0.9000	1.6000
$n = 500$	$Mean$	1.2871	0.3015	0.5878	1.0158	1.0931	−0.0810	0.8308	1.7301
	Std	0.2326	0.0436	0.0546	0.1814	0.2568	0.0590	0.0838	0.2958
$n = 1000$	$Mean$	1.2576	0.3044	0.5892	1.0182	1.0556	−0.0823	0.8397	1.7257
	Std	0.1619	0.0321	0.0397	0.1266	0.1842	0.0413	0.0584	0.2150
$n = 1500$	$Mean$	1.2536	0.3085	0.5864	1.0140	1.0550	−0.0815	0.8396	1.7224
	Std	0.1277	0.0264	0.0322	0.1033	0.1443	0.0351	0.0477	0.1663
$n = 2000$	$Mean$	1.2199	0.3026	0.5962	1.0122	0.9655	−0.1013	0.8701	1.6527
	Std	0.0922	0.0155	0.0209	0.0789	0.1015	0.0264	0.0348	0.1362

To initiate the ith iteration of the EM algorithm, we use the estimates $\widehat{\theta}^{(i-1)}$ from the previous iteration and define the Q function in the E-step

$$\begin{aligned} Q\left(\theta, \widehat{\theta}^{(i-1)}\right) &:= \mathbb{E}\left(2\mathbf{L}(\theta; Y)|Y, \widehat{\theta}^{(i-1)}\right) \\ &= -(2n+1)\log(2\pi) - \log\sigma_x^2 - \frac{\left(x_0^{(n)}-\mu_x\right)^2+P_0^{(n)}}{\sigma_x^2} - n\log\delta^2 \\ &- \sum_{t=1}^n \frac{\left(x_t^{(n)}-\mu-\alpha y_{t-1}-\beta x_{t-1}^{(n)}\right)^2+P_t^{(n)}+\beta^2 P_{t-1}^{(n)}-2\beta P_{t,t-1}^{(n)}}{\delta^2} \\ &- \sum_{t=1}^n \mathbb{E}\left[\exp\left(y_t - x_t\right)\middle| Y, \widehat{\theta}^{(i-1)}\right] + \sum_{t=1}^n \left(y_t - x_t^{(n)}\right), \end{aligned} \tag{5}$$

where $x_t^{(n)} = \mathbb{E}\left(x_t|Y, \widehat{\theta}^{(i-1)}\right)$, $P_t^{(n)} = \mathbb{E}\left(\left(x_t - x_t^{(n)}\right)^2\middle| Y, \widehat{\theta}^{(i-1)}\right)$ and $P_{t,t-1}^{(n)} = \mathbb{E}\left(\left(x_t - x_t^{(n)}\right)\left(x_{t-1} - x_{t-1}^{(n)}\right)\middle| Y, \widehat{\theta}^{(i-1)}\right)$. By maximizing (5), we get the maximum likelihood (ML) estimate $\widehat{\theta}^{(i)}$ of θ as follows:

$$\begin{cases} \left(\widehat{\mu}^{(i)}, \widehat{\alpha}^{(i)}, \widehat{\beta}^{(i)}\right)' = A^{-1}b, \\ \widehat{\delta}^{(i)} = \left(\frac{1}{n}\sum_{t=1}^n \left\{\left(\widehat{z}_t^{(n)}\right)^2 + P_t^{(n)} + \left(\widehat{\beta}^{(i)}\right)^2 P_{t-1}^{(n)} - 2\widehat{\beta}^{(i)} P_{t,t-1}^{(n)}\right\}\right)^{1/2}, \end{cases} \tag{6}$$

where $b = \left(\sum_{t=1}^n \left(P_{t,t-1}^{(n)} + x_t^{(n)}x_{t-1}^{(n)}\right), \sum_{t=1}^n x_t^{(n)}y_{t-1}, \sum_{t=1}^n x_t^{(n)}\right)'$,

$$A = \begin{pmatrix} \sum_{t=1}^n x_{t-1}^{(n)} & \sum_{t=1}^n y_{t-1}x_{t-1}^{(n)} & \sum_{t=1}^n \left(\left(x_{t-1}^{(n)}\right)^2 + P_{t-1}^{(n)}\right) \\ \sum_{t=1}^n y_{t-1} & \sum_{t=1}^n y_{t-1}^2 & \sum_{t=1}^n x_{t-1}^{(n)}y_{t-1} \\ n & \sum_{t=1}^n y_{t-1} & \sum_{t=1}^n x_{t-1}^{(n)} \end{pmatrix},$$

and $\widehat{z}_t^{(n)} = x_t^{(n)} - \widehat{\mu}^{(i)} - \widehat{\alpha}^{(i)}y_{t-1} - \widehat{\beta}^{(i)}x_{t-1}^{(n)}$. The explicit expression of $\widehat{\theta}^{(i)}$ given by (6) depends on the values of $x_t^{(n)}$, $P_t^{(n)}$ and $P_{t,t-1}^{(n)}$, making it impossible to obtain exact values due to the complex dependence structure of the $\log GARCH\text{-}SV$ model. Particle algorithms can be used to approximate these quantities using the algorithm's output (e.g., Boussaha and Hamdi, 2018; Boussaha et al., 2023). The supplementary material presents a filtering algorithm (Algorithm 1) for generating filtered particles and a smoothing algorithm (Algorithm 2) to approximate (6) using the smoothed particles.

5. Simulation study

To assess the effectiveness of our proposed ML estimation procedure, we conducted several Monte Carlo experiments and report two simulation studies in this section, each using different sample lengths. The sample sizes, n, considered in these simulation studies were 500, 1000, 1500, and 2000. In each case, we employed a sequence of $M = 200$ particles. It is important to note that in our experiments, the corresponding parameter values were chosen such that they satisfy the stationary condition $|\alpha + \beta| < 1$.

For every set of observations, we computed the ML estimate of the parameter vector θ. This process was repeated 1000 times, and we summarized the finite sample properties of the estimators in Table 1. The table provides the true values (TV) of the parameters for each $\log GARCH\text{-}SV$ model under consideration, along with the empirical mean ($Mean$) and empirical standard deviation (Std) of the estimators.

Table 1 demonstrates that the mean values of all parameters are in close proximity to their true values. Additionally, as the value of n increases, the standard deviations become smaller. These findings affirm the desirable consistency property of the ML estimators, indicating that the proposed estimation procedure yields satisfactory results. Note that we conducted two additional simulations in which we adjusted the distribution of observation and log-volatility errors. Specifically, we replaced the Gaussian distribution with a Student-t distribution with 7 degrees of freedom (refer to the $3rd$ and $4th$ simulations provided in the supplementary material). This adjustment enables us to assess the estimator's performance in scenarios where the normality assumption may not be applicable. Our results demonstrate that our approach produces empirically robust results, confirming its potential applicability, especially in scenarios with a substantial number of observations. These findings prompted us to use this estimation approach in modeling real data with the $\log GARCH\text{-}SV$ class and to compare the estimated models with competing classes documented in the literature (for more details, refer to Section 5 of the supplementary material).

Data availability

Data will be made available on request.

Acknowledgments

We are grateful to the Editor-in-Chief, Elton P. Hsu, the Associate Editor, and two anonymous reviewers for useful comments and suggestions. The work has received partial support from Ministry of Higher Education and Scientific Research, Algeria through the two contracts PRFU/C00L03UN160420190005 and PRFU/C00L03UN160420200016.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2024.110185.

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Supplementary material for "The $\log GARCH$ Stochastic Volatility Model"

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June 14, 2024

Abstract

This supplementary material contains proofs of Theorems 1-2 and Proposition 1, additional illustrations of the $\log GARCH$-SV model, and further details on estimating model parameters using a sequential Monte Carlo method. It includes the filtering algorithm (Algorithm 1) for generating filtered particles and the smoothing algorithm (Algorithm 2) for approximating the explicit expression of ML estimates. It also incorporates an empirical demonstration of the utility of $\log GARCH$-SV in modeling financial time series.

1 The $\log GARCH$-SV model

A stochastic process $\{\varepsilon_t; t \in \mathbb{Z}\}$ has a $\log GARCH$-SV representation if it satisfies the stochastic difference equation:

$$\left\{ \begin{array}{l} \varepsilon_t = \eta_t \exp\left\{\frac{1}{2}x_t\right\}, \\ x_t = \mu + \alpha \log \varepsilon_{t-1}^2 + \beta x_{t-1} + \delta e_t, \end{array} \right. \quad t \in \mathbb{Z}, \tag{1}$$

where ε_t is typically interpreted as a financial return (which may be adjusted for mean) or the residual term in a regression analysis. (η_t) and (e_t) represent two independent sequences of independent and identically distributed ($i.i.d.$) zero-mean and unit-variance random variables. Additionally, we make the assumption that $\mathbb{P}(\eta_t = 0) = 0$. Finally, the set of model parameters includes μ, α, β and δ.

Under conditions ensuring the existence of a solution, the variable x_t is referred to as the log-volatility of ε_t. As in $\log GARCH$ and SV models, the magnitude of ε_t is proportional to $\exp\left\{\frac{1}{2}x_t\right\}$. The connection between ε_t and its previous values is established indirectly through the log-volatility dynamics. Specifically, the log-volatility at time t can be expressed in an autoregressive form that involves three elements: the lagged log-volatility x_{t-1}, the drift term γ, and the centered random variable w_t. The drift term γ can be obtained by multiplying the expected value of the logarithm of η_t^2 by α and adding the parameter μ to the product. Additionally, w_t can be defined as α multiplied by the difference between the logarithm of η_t^2 and its expected value, plus δe_t. This equation can be represented as:

$$x_t = \gamma + (\alpha + \beta) x_{t-1} + w_t, \tag{2}$$

where $\gamma = \mu + \alpha \mathbb{E}\left(\log \eta_{t-1}^2\right)$ and $w_t = \alpha\left(\log \eta_{t-1}^2 - \mathbb{E}\left(\log \eta_{t-1}^2\right)\right) + \delta e_t$. It should be noted that (w_t) is a sequence of $i.i.d.$ random variables with zero mean and variance $var(w_t) := \sigma_w^2 = \alpha^2 var\left(\log \eta_1^2\right) + \delta^2$. The parameter γ acts as a scaling factor for the volatility, whereas σ_w^2 represents the volatility of the log-volatility. If η_t is normally distributed, the variance of w_t is given by $\sigma_w^2 = \alpha^2 \frac{\pi^2}{2} + \delta^2$. Furthermore, if e_t is also normally distributed, the moment generating function of w_t is determined as follows

$$M_w(z) = \mathbb{E}(\exp\{zw_t\}) = \exp\left\{\frac{z^2\delta^2}{2} - z\alpha\mathbb{E}\left(\log \eta_{t-1}^2\right)\right\} \frac{\Gamma\left(z\alpha + \frac{1}{2}\right)}{2^{-z\alpha}\sqrt{\pi}}, \text{ provided that } z\alpha > -\frac{1}{2},$$

where $\mathbb{E}\left(\log \eta_{t-1}^2\right)$ can be accurately approximated by -1.2749. It becomes apparent that in this Gaussian scenario, a negative α amplifies the significance of the scaling factor compared to a positive α. As a result, models with a negative α can effectively represent returns and volatilities with significant magnitudes.

The model, as defined in (1), is referred to as the $\log GARCH\text{-}SV$ model since it specifies the volatility as a $\log GARCH(1,1)$ model. It should be noted that this model is more flexible than the $\log GARCH$ model due to the presence of an additional innovation term in the log-volatility dynamics. Moreover, when the stochastic component of volatility, e_t, is absent, the model reduces to $\log GARCH$, which highlights a substantial difference in modeling variance between $\log GARCH$ and $\log GARCH\text{-}SV$ approaches.

It is worth noting that unlike $GARCH$-type models, the volatility x_t cannot be interpreted as the conditional variance of ε_t in $\log GARCH\text{-}SV$ model, since the conditional variance is not explicitly stated. Another way in which our model differs from $GARCH$ specifications is that instead of specifying $AR(1)$ dynamics on the volatility, we specify it on the log-volatility, as shown in (2). As a result, the parameters μ, α and β are not necessarily subject to positivity constraints. The parameter $\alpha+\beta$ represents the persistence coefficient in our model. When $\alpha+\beta$ is close to 1, a positive shock to the log-volatility (as a result of a large positive value of w_t) has a persistent effect on $\exp\{x_t\}$: the volatility will remain high for several periods. Similarly, a negative shock to the log-volatility, indicating a small level for the volatility $\exp\{x_t\}$, is also persistent when $\alpha+\beta$ is close to 1. If $\alpha+\beta$ is close to 0, the effect of large shocks dissipates rapidly. When $\alpha+\beta$ is close to -1, a positive shock leads to an instantaneously large volatility. However, in the subsequent period, this volatility decreases and subsequently oscillates between large and small values, provided there are no other significant shocks. A negative shock produces similar alternating effects. The occurrence of such behaviors in financial series is uncommon, making negative values of $\alpha+\beta$ impractical or unrealistic in practice (Francq and Zakoïan, 2019).

2 Stationarity, moments and dynamic structure

The dependence between noise (η_t) and log-volatility sequence (x_t) in the $\log GARCH\text{-}SV$ model (1) introduces a much more complex probabilistic structure compared to a basic SV model which lacks explicit feedback of current log-volatility with previous observations. However, this structure remains relatively simpler than that of a $GARCH$ process. In fact, the first equation of (1) indicates that the existence of a causal strict stationary solution to (1) is the same as the existence of a causal strict stationary solution to the AR model defined in (2).

Theorem 1. The $\log GARCH\text{-}SV$ model, as defined in (1), admits a unique strictly stationary and ergodic solution which is non-anticipative, given by

$$\varepsilon_t = \eta_t \exp\left\{\frac{\gamma}{2(1-\alpha-\beta)} + \frac{1}{2}\sum_{l\geq 0} (\alpha+\beta)^l w_{t-l}\right\}, \ t \in \mathbb{Z}, \tag{3}$$

where the series in (3) converges almost surely, if and only if, $|\alpha+\beta| < 1$. Moreover, if (e_t) and (η_t) are normally distributed, and $\min\{(\alpha+\beta)\alpha, \alpha\} > -\frac{1}{2}$, the solution (3) is also second-order stationary.

Proof. From the multiplicative form of the first equation in model (1), it becomes evident that the existence of a causal strict stationary solution for (1) is equivalent to the existence of a causal strict stationary solution for the AR model described by (2), where $|\alpha+\beta| < 1$. In fact, when this condition is satisfied, the log-volatility x_t can be represented causally as follows:

$$x_t = \mu_x + \sum_{l\geq 0} (\alpha+\beta)^l \, w_{t-l}.$$

The mean $\mu_x := \mathbb{E}(x_t)$ and the variance $\sigma_x^2 := var(x_t)$ are given by

$$\mu_x = \frac{\gamma}{1-\alpha-\beta} \text{ and } \sigma_x^2 = \frac{\alpha^2 var(\log \eta_t^2) + \delta^2}{1-(\alpha+\beta)^2}.$$

Therefore, we obtain

$$\mathbb{E}\left(\varepsilon_t^2\right) = \exp\left\{\frac{\gamma}{1-\alpha-\beta}\right\} \mathbb{E}\left(\prod_{l\geq 0} \exp\left\{(\alpha+\beta)^l \, w_{t-l}\right\}\right).$$

Let $U_{t,L} = \prod_{l=0}^{L} \exp\left\{\frac{1}{2}(\alpha+\beta)^l \, w_{t-l}\right\}$. It can be demonstrated that $(U_{t,L})_{L\geq 0}$ is a quadratically integrable and non-negative submartingale for all $t \in \mathbb{Z}$. Consider now $S_{t,L} = \sup_{k\leq L} U_{t,k}$ and $S_{t,\infty} = \sup_k U_{t,k}$, we have from Doob's martingale inequality and from Fatou's lemma

$$\mathbb{E}(S_{t,\infty}) \leq \exp\left\{\frac{\delta^2}{2\left(1-(\alpha+\beta)^2\right)} - \frac{\alpha \mathbb{E}\left(\log \eta_{t-1}^2\right)}{1-\alpha-\beta}\right\} \prod_{l\geq 0} \frac{\Gamma\left(\frac{a_l+1}{2}\right)}{2^{-a_l}\sqrt{\pi}}, \text{ provided that } a_l > -1,$$

where $a_l = \alpha(\alpha+\beta)^l$, for all $l \geq 0$. The finiteness of the second term in this inequality is guaranteed by the condition $|\alpha+\beta| < 1$, which ensures the existence of an integer l_0 such that for all $l \geq l_0$, $|a_l| < 1$. Additionally, by utilizing the Legendre duplication formula, we can demonstrate that:

$$\frac{\Gamma\left(\frac{a_l+1}{2}\right)}{2^{-a_l}\sqrt{\pi}} = \frac{2\Gamma(a_l)}{\Gamma\left(\frac{a_l}{2}\right)} = \frac{2\Gamma(a_l+1)}{a_l} \frac{\frac{a_l}{2}}{\Gamma\left(\frac{a_l}{2}+1\right)} = \frac{\Gamma(a_l+1)}{\Gamma\left(\frac{a_l}{2}+1\right)}, \text{ for all } l \geq l_0.$$

Finally, from the continuity of the gamma function over the interval $]0, +\infty[$

$$\frac{\Gamma(a_l+1)}{\Gamma\left(\frac{a_l}{2}+1\right)} \to 1 \text{ when } l \to \infty,$$

which proves that $\prod_{l\geq 0} \Gamma\left(\frac{a_l+1}{2}\right) / 2^{-a_l}\sqrt{\pi} < \infty$. So, as $\mathbb{E}(S_{t,\infty}) < \infty$, we can use the dominated convergence theorem, and thus

$$\begin{aligned} \mathbb{E}\left(\varepsilon_t^2\right) &= \exp\left\{\frac{\gamma}{1-\alpha-\beta}\right\} \prod_{l\geq 0} \mathbb{E}\left(\exp\left\{(\alpha+\beta)^l \, w_{t-l}\right\}\right) \\ &= \exp\left\{\frac{\mu}{1-\alpha-\beta} + \frac{\delta^2}{2\left(1-(\alpha+\beta)^2\right)}\right\} \prod_{l\geq 0} \frac{\Gamma\left(a_l+\frac{1}{2}\right)}{2^{-a_l}\sqrt{\pi}} < \infty, \end{aligned}$$

provided that $a_l + \frac{1}{2} > 0$. Finally, it is evident that this last constraint is equivalent to $\min\{(\alpha+\beta)\alpha, \alpha\} > -\frac{1}{2}$ under the strict stationarity condition. □

Remark 1.

In the general case, and in contrast to the standard Gaussian assumption made on the distributions of (e_t) and (η_t), the condition $|\alpha+\beta| < 1$ alone is not sufficient to ensure the second-order stationarity of (1). Additionally, it is necessary to satisfy the following condition:

$$\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right) < \infty.$$

To demonstrate the finiteness of $\mathbb{E}\left(\varepsilon_{s+\tau S}^2\right)$, it is enough to prove that $\mathbb{E}\left(\prod_{l\geq 0}\exp\left\{(\alpha+\beta)^l w_{t-l}\right\}\right)<\infty$. This is true according to Fatou's lemma, as $\left(U_{t,L}\right)_{L\geq 0}$, where $U_{t,L}=\prod_{l=0}^{L}\exp\left\{\frac{1}{2}(\alpha+\beta)^l w_{t-l}\right\}$, is a sequence of positive integrable random variables that converges almost surely.

When $\alpha=0$, the previous theorem aligns with the findings of Francq and Zakoïan (2019, Theorem 12.1 and Theorem 12.2) concerning the strict and second-order stationarity of *AR-SV* models. In addition, if e_t follows a standard normal distribution, our result coincides with the statement made in Francq and Zakoïan (2019, Remark 12.1). Furthermore, the explicit expression for the variance of ε_t, given by

$$var(\varepsilon_t)=\mathbb{E}\left(\varepsilon_t^2\right)=\exp\left\{\frac{\mu}{1-\beta}+\frac{\delta^2}{2\left(1-\beta^2\right)}\right\},$$

sets it apart from our current case. Specifically, when $\alpha\neq 0$, establishing a more explicit expression requires determining the infinite product

$$\prod_{l\geq 0}\Gamma\left(\alpha(\alpha+\beta)^l+\frac{1}{2}\right)\Big/\,2^{-\alpha(\alpha+\beta)^l}\sqrt{\pi},$$

in closed form, using the model's parameters. However, this task proves to be challenging.

Theorem 2. Let $\{\varepsilon_t;t\in\mathbb{Z}\}$ be a strict stationary solution of (1), with $\mathbb{E}\left(|\eta_t|^r\right)<\infty$, for any positive integer r. Then a sufficient condition for $\mathbb{E}\left(\varepsilon_t^r\right)$ to be finite is that $\prod_{l\geq 0}M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)<\infty$. Furthermore, the closed-form of the r-th moment of ε_t, is given by

$$\mu_{\varepsilon^r}:=\mathbb{E}\left(\varepsilon_t^r\right)=\mathbb{E}\left(\eta_t^r\right)\exp\left\{\frac{r\gamma}{2(1-\alpha-\beta)}\right\}\prod_{l\geq 0}M_w\left(\frac{r}{2}(\alpha+\beta)^l\right).$$

Proof. It suffices to proceed exactly in the same way as in the proof of the previous theorem where r was set to 2, but this time the submartingale $\left(U_{t,L}\right)_{L\geq 0}$ is defined by $U_{t,L}=\prod_{l=0}^{L}Z_{t,l}$, where $Z_{t,l}=\exp\left\{\frac{r}{4}(\alpha+\beta)^l w_{t-l}\right\}$, for all $t\in\mathbb{Z}$ and $L\in\mathbb{N}$. $\square$

Remark 2. From (3), the conditions necessary for the existence of a fourth-order moment for (ε_t) are as follows: $|\alpha+\beta|<1$, $\prod_{l\geq 0}M_w\left(2(\alpha+\beta)^l\right)<\infty$ and $\mathbb{E}\left(\eta_t^4\right)<\infty$. Under these conditions, we have

$$\mathbb{E}\left(\varepsilon_t^4\right)=\mathbb{E}\left(\eta_t^4\right)\exp\left\{\frac{2\gamma}{1-\alpha-\beta}\right\}\prod_{l\geq 0}\mathbb{E}\left(\exp\left\{2(\alpha+\beta)^l w_{t-l}\right\}\right),$$

and the kurtosis coefficient of (ε_t) is given by

$$\kappa_\varepsilon:=\frac{\mathbb{E}\left(\varepsilon_t^4\right)}{\left[\mathbb{E}\left(\varepsilon_t^2\right)\right]^2}=\kappa_\eta\prod_{l\geq 0}\frac{\mathbb{E}\left(\exp\left\{2(\alpha+\beta)^l w_{t-l}\right\}\right)}{\left(\mathbb{E}\left(\exp\left\{(\alpha+\beta)^l w_{t-l}\right\}\right)\right)^2}=\kappa_\eta\prod_{l\geq 0}\kappa^{(l)},$$

where κ_η represents the kurtosis coefficient of (η_t), while $\kappa^{(l)}$ represents the kurtosis of $\exp\left\{\frac{1}{2}(\alpha+\beta)^l w_t\right\}$. As the distribution of (w_t) is not degenerate, $\kappa^{(l)}$ is greater than 1. Consequently, the kurtosis of the distribution of (ε_t) is greater than that of (η_t). If (η_t) follows a normal distribution, it is evident that the minimum value of κ_ε is determined by the kurtosis coefficient of (η_t), which equals 3. Moreover, if (e_t) is also normally distributed, we have

$$\kappa_\varepsilon=3\exp\left\{\frac{\delta^2}{1-(\alpha+\beta)^2}\right\}\prod_{l\geq 0}\frac{\Gamma\left(2a_l+\frac{1}{2}\right)\sqrt{\pi}}{\left(\Gamma\left(a_l+\frac{1}{2}\right)\right)^2}>3,$$

provided that $\min\{(\alpha+\beta)\alpha,\alpha\} > -\frac{1}{4}$. Hence, the tail distribution of a $\log GARCH-SV$ model is heavier compared to that of both a normal distribution and a basic SV model with the same persistence coefficient. In Figure 1, we use graphs for different values of α and β to illustrate the behavior of $\prod_{l\geq 0} A_l(\alpha,\beta)$ compared to 1, where $A_l(\alpha,\beta) = \Gamma(2a_l+\frac{1}{2})\sqrt{\pi}\Big/\left(\Gamma(a_l+\frac{1}{2})\right)^2$.

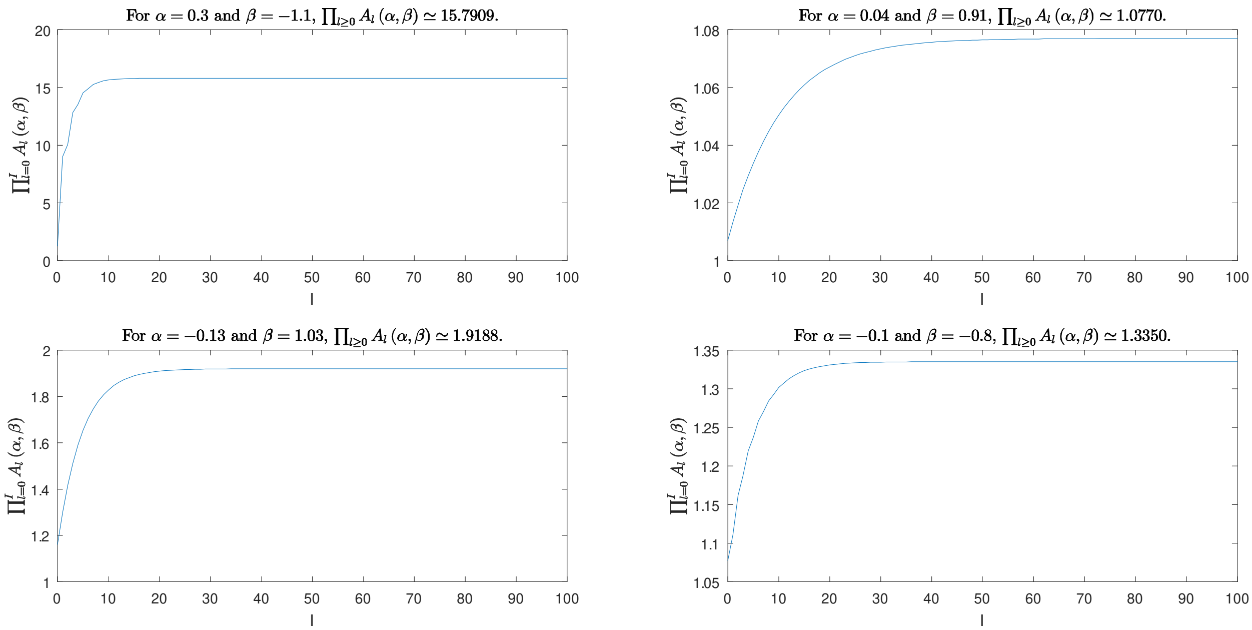

Figure 1. Illustration of $\prod_{l\geq 0} A_l(\alpha,\beta)$ behavior for various α and β.

On the other hand, for illustration, Figure 2 shows the regions of strict stationarity (I), second-order stationarity (II), and fourth moment existence (III) for a $\log GARCH\text{-}SV$ model where e_t is normally distributed.

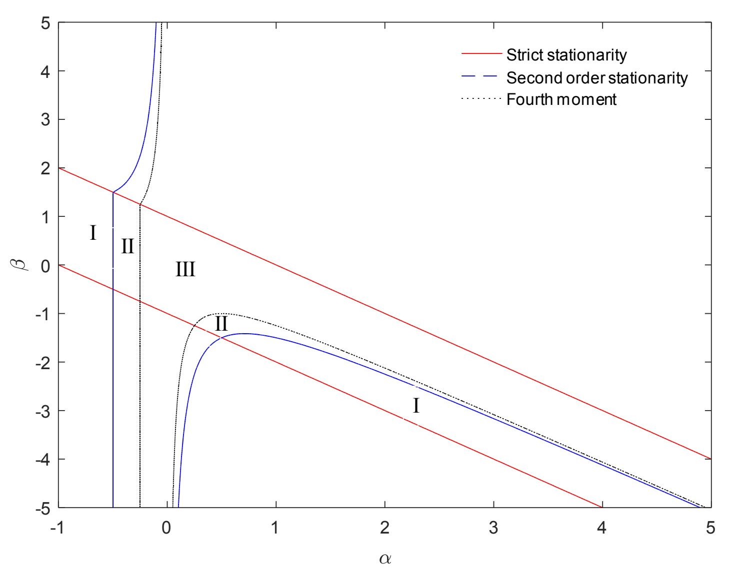

Figure 2. Regions of strict stationarity (I), second-order stationarity (II), and fourth moment existence (III) for a $\log GARCH\text{-}SV$ model where $e_t \sim \mathcal{N}(0,1)$.

Now we aim to analyze the dependency pattern in a stationary solution $\{\varepsilon_t; t \in \mathbb{Z}\}$ of (1). To achieve this, we first focus on the autocovariance function $\gamma_{\varepsilon,h} := cov(\varepsilon_t, \varepsilon_{t-h})$, which is a commonly used tool by statisticians, theorists, and practitioners.

Proposition 1. Let $\{\varepsilon_t; t \in \mathbb{Z}\}$ be a stationary solution of (1), with $\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right) < \infty$. Then $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is a weak white noise process with variance

$$\sigma_\varepsilon^2 := var(\varepsilon_t) = \exp\left\{\frac{\gamma}{1-\alpha-\beta}\right\}\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right).$$

Proof. We have for each $t \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{E}(\varepsilon_t) &= \mathbb{E}(\eta_t)\exp\left\{\frac{\gamma}{2(1-\alpha-\beta)}\right\}\prod_{l\geq 0} M_w\left(\frac{1}{2}(\alpha+\beta)^l\right),\\ &= 0,\ \text{provided that} \prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right) < \infty.\end{aligned}$$

From Theorem 2, we also have

$$\sigma_\varepsilon^2 = \mathbb{E}\left(\varepsilon_t^2\right) = \exp\left\{\frac{\gamma}{1-\alpha-\beta}\right\}\prod_{l\geq 0} M_w\left((\alpha+\beta)^l\right) < \infty.$$

Hence, for each $(t,h) \in \mathbb{Z}\times\mathbb{Z}^*$, we have $\gamma_{\varepsilon,h} = 0$, as an immediate consequence of the Cauchy-Schwarz inequality.

Although (ε_t) is a white noise process, it is possible that there may exist non-linear forms of dependence between its successive terms. To understand the nature of this dependence in a $\log GARCH\text{-}SV$ process defined by (1), we study the covariance structure of the observations raised to the power $r \in \mathbb{N}^*$.

We have from (2)

$$x_t = \sum_{i=0}^{h-1}(\alpha+\beta)^i(\gamma + w_{t-i}) + (\alpha+\beta)^h x_{t-h},$$

which gives for all $(t,h) \in \mathbb{Z}\times\mathbb{Z}^*$,

$$\begin{aligned}\mathbb{E}\left(\varepsilon_t^r\varepsilon_{t-h}^r\right) &= \mathbb{E}\left(\eta_t^r\right)\mathbb{E}\left(\eta_{t-h}^r\right)\mathbb{E}\left(\exp\left\{\frac{r}{2}(x_t + x_{t-h})\right\}\right)\\ &= \mathbb{E}\left(\eta_t^r\right)\mathbb{E}\left(\eta_{t-h}^r\right)\mathbb{E}\left(\exp\left\{\frac{r}{2}\left(\sum_{l=0}^{h-1}(\alpha+\beta)^l(\gamma + w_{t-l}) + \left((\alpha+\beta)^h + 1\right)x_{t-h}\right)\right\}\right)\\ &= \left[\mathbb{E}\left(\eta_t^r\right)\right]^2\exp\left\{\frac{r\gamma\left(1-(\alpha+\beta)^h\right)}{2(1-\alpha-\beta)}\right\}\prod_{l=0}^{h-1} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)M_x\left(\frac{r}{2}\left((\alpha+\beta)^h + 1\right)\right)\end{aligned}$$

Moreover

$$\mathbb{E}\left(\varepsilon_t^r\right) = \mathbb{E}\left(\eta_t^r\right)\exp\left\{\frac{r\gamma}{2(1-\alpha-\beta)}\right\}\prod_{l\geq 0} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right).$$

Therefore, the autocovariance function at lag $h \in \mathbb{Z}^*$ of the process $\{\varepsilon_t^r; t \in \mathbb{Z}, r \in \mathbb{N}^*\}$ can be expressed as follows:

$$
\begin{aligned}
\gamma_{\varepsilon^r,h} &= \mathbb{E}\left(\varepsilon_t^r \varepsilon_{t-h}^r\right) - \mathbb{E}\left(\varepsilon_t^r\right)\mathbb{E}\left(\varepsilon_{t-h}^r\right) \\
&= \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\gamma\left(1-(\alpha+\beta)^h\right)}{2(1-\alpha-\beta)}\right\}\prod_{l=0}^{h-1} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right) M_x\left(\frac{r}{2}\left((\alpha+\beta)^h+1\right)\right) \\
&- \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\gamma}{1-\alpha-\beta}\right\}\prod_{l\geq 0}\left(M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)\right)^2 . \\
&= \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\gamma\left(1-(\alpha+\beta)^h\right)}{2(1-\alpha-\beta)}\right\}\prod_{l=0}^{h-1} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right) M_x\left(\frac{r}{2}\left((\alpha+\beta)^h+1\right)\right) \\
&- \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\gamma}{1-\alpha-\beta}\right\}\prod_{l\geq 0}\left(M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)\right)^2 \\
&= \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\gamma}{1-\alpha-\beta}\right\}\left[\prod_{l=0}^{h-1} M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)\prod_{l\geq 0} M_w\left(\frac{r}{2}\left((\alpha+\beta)^h+1\right)(\alpha+\beta)^l\right)\right. \\
&\left. - \prod_{l\geq 0}\left(M_w\left(\frac{r}{2}(\alpha+\beta)^l\right)\right)^2\right]
\end{aligned}
$$

Now suppose that e_t is also normally distributed. It follows that

$$
\begin{aligned}
\mathbb{E}\left(\varepsilon_t^r \varepsilon_{t-h}^r\right) &= \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\mu}{1-\alpha-\beta} + \frac{r^2\delta^2\left(1+(\alpha+\beta)^h\right)}{4\left(1-(\alpha+\beta)^2\right)}\right\} \\
&\times \prod_{l=0}^{h-1}\frac{\Gamma\left(\frac{ra_l+1}{2}\right)}{2^{-\frac{ra_l}{2}}\sqrt{\pi}}\prod_{l\geq 0}\frac{\Gamma\left(\frac{r}{2}\left((\alpha+\beta)^h+1\right)a_l+\frac{1}{2}\right)}{2^{-\frac{r}{2}\left((\alpha+\beta)^h+1\right)a_l}\sqrt{\pi}}
\end{aligned}
$$

and

$$
\mathbb{E}\left(\varepsilon_t^r\right) = \mathbb{E}\left(\eta_t^r\right)\exp\left\{\frac{r\mu}{2\left(1-\alpha-\beta\right)} + \frac{r^2\delta^2}{8\left(1-(\alpha+\beta)^2\right)}\right\}\prod_{l\geq 0}\frac{\Gamma\left(\frac{ra_l+1}{2}\right)}{2^{-\frac{ra_l}{2}}\sqrt{\pi}}.
$$

Hence,

$$\gamma_{\varepsilon^r,h} = \left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\mu}{1-\alpha-\beta} + \frac{r^2\delta^2\left(1+(\alpha+\beta)^h\right)}{4\left(1-(\alpha+\beta)^2\right)}\right\}\left(\prod_{l=0}^{h-1}\frac{\Gamma\left(\frac{ra_l+1}{2}\right)}{2^{-\frac{ra_l}{2}}\sqrt{\pi}}\right)$$
$$\times\left(\prod_{l\geq 0}\frac{\Gamma\left(\frac{r}{2}\left((\alpha+\beta)^h+1\right)a_l+\frac{1}{2}\right)}{2^{-\frac{r}{2}\left((\alpha+\beta)^h+1\right)a_l}\sqrt{\pi}}\right)$$
$$-\left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\mu}{1-\alpha-\beta} + \frac{r^2\delta^2}{4\left(1-(\alpha+\beta)^2\right)}\right\}\prod_{l\geq 0}\frac{\left(\Gamma\left(\frac{ra_l+1}{2}\right)\right)^2}{2^{-ra_l}\pi}$$
$$=\left[\mathbb{E}\left(\eta_t^r\right)\right]^2 \exp\left\{\frac{r\mu}{1-\alpha-\beta} + \frac{r^2\delta^2}{4\left(1-(\alpha+\beta)^2\right)}\right\}$$
$$\times\left[\exp\left\{\frac{r^2\delta^2(\alpha+\beta)^h}{4\left(1-(\alpha+\beta)^2\right)}\right\}\left(\prod_{l=0}^{h-1}\frac{\Gamma\left(\frac{ra_l+1}{2}\right)}{2^{-\frac{ra_l}{2}}\sqrt{\pi}}\right)\left(\prod_{l\geq 0}\frac{\Gamma\left(\frac{\left[r\left((\alpha+\beta)^h+1\right)a_l\right]+1}{2}\right)}{2^{-\frac{r\left((\alpha+\beta)^h+1\right)a_l}{2}}\sqrt{\pi}}\right)-\prod_{l\geq 0}\frac{\left(\Gamma\left(\frac{ra_l+1}{2}\right)\right)^2}{2^{-ra_l}\pi}\right],$$

and when $r=2$, we obtain

$$\gamma_{\varepsilon^2,h} = \exp\left\{\frac{2\mu}{1-\alpha-\beta} + \frac{\delta^2}{1-(\alpha+\beta)^2}\right\}$$
$$\times\left[\exp\left\{\frac{\delta^2(\alpha+\beta)^h}{1-(\alpha+\beta)^2}\right\}\left(\prod_{l=0}^{h-1}\frac{\Gamma\left(a_l+\frac{1}{2}\right)}{2^{-a_l}\sqrt{\pi}}\right)\left(\prod_{l\geq 0}\frac{\Gamma\left(\left((\alpha+\beta)^h+1\right)a_l+\frac{1}{2}\right)}{2^{-\left((\alpha+\beta)^h+1\right)a_l}\sqrt{\pi}}\right)-\prod_{l\geq 0}\frac{\left(\Gamma\left(a_l+\frac{1}{2}\right)\right)^2}{2^{-2a_l}\pi}\right].$$

Consequently, the autocorrelation function at lag $h\in\mathbb{Z}^*$ is given by

$$\rho_{\varepsilon^2,h} = \frac{\exp\left\{\frac{\delta^2(\alpha+\beta)^h}{1-(\alpha+\beta)^2}\right\}\left(\prod_{l=0}^{h-1}\frac{\Gamma\left(a_l+\frac{1}{2}\right)}{2^{-a_l}\sqrt{\pi}}\right)\left(\prod_{l\geq 0}\frac{\Gamma\left(\left((\alpha+\beta)^h+1\right)a_l+\frac{1}{2}\right)}{2^{-\left((\alpha+\beta)^h+1\right)a_l}\sqrt{\pi}}\right)-\prod_{l\geq 0}\frac{\left[\Gamma\left(a_l+\frac{1}{2}\right)\right]^2}{2^{-2a_l}\pi}}{3\exp\left\{\frac{\delta^2}{1-(\alpha+\beta)^2}\right\}\prod_{l\geq 0}\frac{\Gamma\left(2a_l+\frac{1}{2}\right)}{2^{-2a_l}\sqrt{\pi}}-\prod_{l\geq 0}\frac{\left[\Gamma\left(a_l+\frac{1}{2}\right)\right]^2}{2^{-2a_l}\pi}}. \quad \square$$

Based on this proposition, it is important to emphasize that, similar to the *AR-SV* case, the sequence of random variables following a $\log$ *GARCH-SV* model defined by equation (1) does not display significant correlation. However, when these values are raised to the power $r\in\mathbb{N}^*$, they exhibit correlation, and this correlation tends to zero as h goes to infinity.

It is important to point out that we can examine the disparity between a positive and negative value of α, we delve into additional characteristics beyond those discussed above, namely the variance and kurtosis coefficient of the process. Specifically, we scrutinize the autocorrelations of the squared process, while holding the persistence coefficient $(\alpha+\beta)$ and δ constant. Indeed, when $r=2$ and both e_t and η_t are normally distributed, we observe that the sign of $\rho_{\varepsilon^2,1}$ is determined by the sign of the persistence coefficient rather than α alone. Additionally, we notice that in instances where α is negative, the absolute values of $\rho_{\varepsilon^2,1}$ are higher compared to those associated with a positive α. This observation highlights that models with a negative α value exhibit stronger autocorrelation between ε_t^2 and ε_{t-1}^2 than models with a positive α value. Generally, squared returns (ε_t^2) show significant autocorrelation when α is negative compared to a positive α. As a result, substantial autocorrelations of squared returns could also lead to estimating a negative α. For illustration, Figure 3 displays the correlogram of the squared process in certain scenarios.

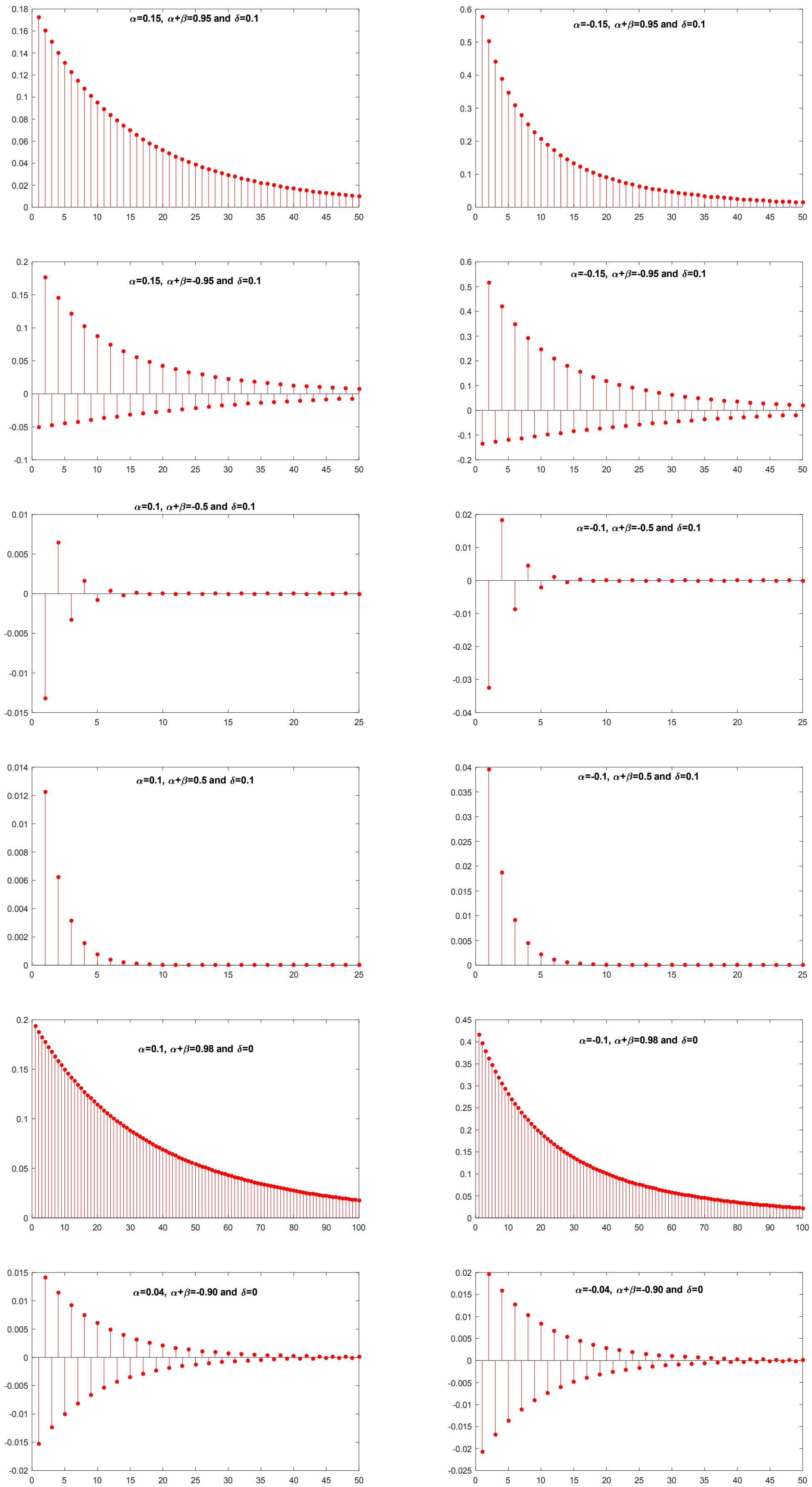

Figure 3. Illustration of the autocorelation function of the squared process in certain scenarios.

3 Estimating $\log GARCH\text{-}SV$ model

The estimation of parameters in the $\log GARCH\text{-}SV$ model (1) is challenging due to the unobservability of the volatility, which requires techniques for handling missing data. One such method is the EM algorithm (Dempster et al., 1977), a popular iterative method for computing maximum likelihood estimates in incomplete data problems. The EM algorithm consists of two steps in each iteration: the E-step and the M-step, where the expected likelihood is calculated and the parameter estimates are updated. However, for complex problems like $\log GARCH\text{-}SV$, the expected likelihood is often intractable, and Monte Carlo methods such as particle filters are more efficient. Particle filters are sequential Monte Carlo methods that can solve optimal estimation problems in nonlinear and non-Gaussian state-space models (see, e.g., Doucet and Johansen, 2011), making them a powerful tool in the $\log GARCH\text{-}SV$ framework.

Consider the linearized state-space representation obtained by applying the logarithm to the squared observed process ε_t^2. This yields a state-space representation of the model given by:

$$\begin{cases} y_t = x_t + d + u_t, \\ x_t = \mu + \alpha y_{t-1} + \beta x_{t-1} + \delta e_t, \end{cases} \quad t \in \mathbb{Z}, \tag{4}$$

This state-space representation involves three components: the log-transformed observation $y_t = \log(\varepsilon_t^2)$, the drift term d, and the centered random variable u_t. We can obtain the drift term d by taking the expected value of $\log(\eta_t^2)$. Finally, we define u_t as the difference between the logarithm of η_t^2 and its expected value. Note that the random variable u_t has a centered $\log \chi^2$ distribution with one degree of freedom when (η_t) is assumed to be Gaussian, which is a common assumption in the literature.

Let $\underline{Y} = (x_0,\ x_1, ..., x_n, y_1, y_2, ..., y_n)$ and $\underline{X} = (x_0,\ x_1, ..., x_n)$ denote the vector containing, respectively, the complete data and the log-volatility data. For a given realization $y = (y_1, y_2, ..., y_n)$ of stationary model (4), the twice-complete log-conditional likelihood function (given initial value y_0) of the parameter vector $\theta = (\mu, \alpha, \beta, \delta)'$, can be expressed as follows:

$$\begin{aligned} 2\mathbf{L}(\theta; \underline{Y}) = & -\log \sigma_x^2 - \frac{(x_0 - \mu_x)^2}{\sigma_x^2} - n \log \delta^2 - \sum_{t=1}^{n} \frac{(x_t - \mu - \alpha y_{t-1} - \beta x_{t-1})^2}{\delta^2} \\ & - \sum_{t=1}^{n} \exp(y_t - x_t) + \sum_{t=1}^{n} (y_t - x_t) + C, \end{aligned}$$

where $C = -(2n+1)\log(2\pi)$ is a constant independent of θ. To initiate the i-th iteration of the EM algorithm, we use the parameter estimates $\widehat{\theta}^{(i-1)}$ from the previous iteration and define the Q function in the E-step

$$\begin{aligned} Q\left(\theta, \widehat{\theta}^{(i-1)}\right) &= \mathbb{E}\left(2\mathbf{L}(\theta; Y)|Y, \widehat{\theta}^{(i-1)}\right) \\ &= -\log \sigma_x^2 - \frac{\left(x_0^{(n)} - \mu_x\right)^2 + P_0^{(n)}}{\sigma_x^2} - n \log \delta^2 \\ &\quad - \sum_{t=1}^{n} \frac{\left(x_t^{(n)} - \mu - \alpha y_{t-1} - \beta x_{t-1}^{(n)}\right)^2 + P_t^{(n)} + \beta^2 P_{t-1}^{(n)} - 2\beta P_{t,t-1}^{(n)}}{\delta^2} \\ &\quad - \sum_{t=1}^{n} \mathbb{E}\left[\exp(y_t - x_t)|\, Y, \widehat{\theta}^{(i-1)}\right] + \sum_{t=1}^{n} \left(y_t - x_t^{(n)}\right) + C, \end{aligned} \tag{5}$$

where

$$
\begin{aligned}
x_t^{(n)} &= \mathbb{E}\left(x_t | Y, \widehat{\theta}^{(i-1)}\right), \\
P_t^{(n)} &= \mathbb{E}\left(\left(x_t - x_t^{(n)}\right)^2 \middle| Y, \widehat{\theta}^{(i-1)}\right),
\end{aligned}
$$

and

$$
P_{t,t-1}^{(n)} = \mathbb{E}\left(\left(x_t - x_t^{(n)}\right)\left(x_{t-1} - x_{t-1}^{(n)}\right) \middle| Y, \widehat{\theta}^{(i-1)}\right).
$$

If we can evaluate these conditional expectations, we can proceed to the M-step, where we obtain the maximum likelihood (ML) estimate $\widehat{\theta}^{(i)}$ of θ as follows:

$$
\widehat{\theta}^{(i)} = \arg\max_{\theta} Q\left(\theta, \widehat{\theta}^{(i-1)}\right).
$$

To solve for the unknown parameters μ, α, β and δ, one can equate the first-order derivatives of (5) with respect to these parameters to zero. These derivatives are given by:

$$
\begin{aligned}
\frac{\partial Q\left(\theta, \widehat{\theta}^{(i-1)}\right)}{\partial \beta} &= \frac{2}{\delta^2} \sum_{t=1}^{n} \left(x_{t-1}^{(n)} z_t^{(n)} - \beta_s P_{t-1}^{(n)} + P_{t,t-1}^{(n)}\right), \\
\frac{\partial Q\left(\theta, \widehat{\theta}^{(i-1)}\right)}{\partial \alpha} &= \frac{2}{\delta^2} \sum_{t=1}^{n} y_{t-1} z_t^{(n)}, \\
\frac{\partial Q\left(\theta, \widehat{\theta}^{(i-1)}\right)}{\partial \mu} &= \frac{2}{\delta^2} \sum_{t=1}^{n} z_t^{(n)},
\end{aligned}
$$

and

$$
\frac{\partial Q\left(\theta, \widehat{\theta}^{(i-1)}\right)}{\partial \delta} = -\frac{2n}{\delta} + \frac{2}{\delta^3} \sum_{t=1}^{n} \left\{\left(z_t^{(n)}\right)^2 + P_t^{(n)} + \beta^2 P_{t-1}^{(n)} - 2\beta P_{t,t-1}^{(n)}\right\},
$$

where $z_t^{(n)} = x_t^{(n)} - \mu - \alpha y_{t-1} - \beta x_{t-1}^{(n)}$. Hence, at the i-th iteration, the parameter estimates of μ, α, β and δ are given by

$$
\widehat{\delta}^{(i)} = \left(\frac{1}{n} \sum_{t=1}^{n} \left\{\left(\widehat{z}_t^{(n)}\right)^2 + P_t^{(n)} + \left(\widehat{\beta}^{(i)}\right)^2 P_{t-1}^{(n)} - 2\widehat{\beta}^{(i)} P_{t,t-1}^{(n)}\right\}\right)^{1/2}, \tag{6}
$$

where $\widehat{z}_t^{(n)} = x_t^{(n)} - \widehat{\mu}^{(i)} - \widehat{\alpha}^{(i)} y_{t-1} - \widehat{\beta}^{(i)} x_{t-1}^{(n)}$, and the vector $\left(\widehat{\mu}^{(i)}, \widehat{\alpha}^{(i)}, \widehat{\beta}^{(i)}\right)'$, is given by

$$
\begin{pmatrix} \widehat{\mu}^{(i)} \\ \widehat{\alpha}^{(i)} \\ \widehat{\beta}^{(i)} \end{pmatrix} = A^{-1} b, \tag{7}
$$

with

$$
A = \begin{pmatrix} \sum_{t=1}^{n} x_{t-1}^{(n)} & \sum_{t=1}^{n} y_{t-1} x_{t-1}^{(n)} & \sum_{t=1}^{n} \left(\left(x_{t-1}^{(n)}\right)^2 + P_{t-1}^{(n)}\right) \\ \sum_{t=1}^{n} y_{t-1} & \sum_{t=1}^{n} y_{t-1}^2 & \sum_{t=1}^{n} x_{t-1}^{(n)} y_{t-1} \\ n & \sum_{t=1}^{n} y_{t-1} & \sum_{t=1}^{n} x_{t-1}^{(n)} \end{pmatrix} \text{ and } b = \begin{pmatrix} \sum_{t=1}^{n} \left(P_{t,t-1}^{(n)} + x_t^{(n)} x_{t-1}^{(n)}\right) \\ \sum_{t=1}^{n} x_t^{(n)} y_{t-1} \\ \sum_{t=1}^{n} x_t^{(n)} \end{pmatrix}.
$$

Note that the explicit expression of $\widehat{\theta}^{(i)}$ given by equations (6)-(7) depends on the values of $x_t^{(n)}$, $P_t^{(n)}$ and $P_{t,t-1}^{(n)}$, making it impossible to obtain exact values due to the complex dependence structure of the

$\log GARCH\text{-}SV$ model. To overcome this issue, we can use particle algorithms to sequentially approximate these quantities using the algorithm's output (e.g., Boussaha and Hamdi, 2018; Boussaha et al., 2023). In the following, we present a filtering algorithm that generates M samples (particles) from the probability density function $p\left(x_t \middle| \mathcal{F}_t, \widehat{\theta}^{(i-1)}\right)$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{y_s,\ s \leq t\}$.

Algorithm 1 (Particle filter algorithm)

1. *Initialization: for* $j = 1, ..., M$, *generate* $f_0^{(j)} \sim \mathcal{N}(\mu_x, \sigma_x^2)$ *and initial weights* $w_0^{(j)} = 1/M$, *where* M *is the number of the particles and* $\sim$ *means distributed according to.*

2. *For* $t = 1, ..., n$.

   (a) *For* $j = 1, ..., M$

      i. *Generate* $e_t^{(j)} \sim \mathcal{N}(0, 1)$.

      ii. *Compute* $p_t^{(j)} = \mu + \alpha y_{t-1} + \beta f_{t-1}^{(j)} + \delta e_t^{(j)}$.

      iii. *Update weights : compute*

$$w_t^{(j)} = w_{t-1}^{(j)} \exp\left\{-\frac{1}{2}\exp\left(y_t - p_t^{(j)}\right)\right\} \exp\left\{\frac{1}{2}\left(y_t - p_t^{(j)}\right)\right\}.$$

   (b) *For* $j = 1, ..., M$, *normalize weights: compute* $\widetilde{w}_t^{(j)} = w_t^{(j)} \Big/ \sum_{j=1}^{M} w_t^{(j)}$.

   (c) *Compute the measure of degeneracy* $n^{eff} = 1 \Big/ \sum_{j=1}^{M} \widetilde{w}_t^{(j)2}$. *If* $n^{eff} \leq n^T$ *(typically* $n^T = M/2$*), resample with replacement* M *equally weighted particles* $\left\{f_t^{(j)}, j = 1, ..., M\right\}$ *from the set* $\left\{p_t^{(j)}, j = 1, ..., M\right\}$ *according to the normalized weights* $\left\{\widetilde{w}_t^{(j)}, j = 1, ..., M\right\}$. *Else* $f_t^{(j)} = p_t^{(j)}, j = 1, ..., M$.

3. *Finally, the sequence of* M *particles* $\left\{f_t^{(j)}, j = 1, ..., M\right\}$ *is a random sample from* $p\left(x_t | \mathcal{F}_t\right)$, *for* $t = 0, ..., n$.

Once the filtered particles are available, the particle smoothers required to approximate (6)-(7) can be performed using the following smoothing algorithm.

Algorithm 2 (Particle smoothing algorithm)

1. *For* $j = 1, ..., M$, *choose* $s_n^{(j)} = f_n^{(i)}$, *with probability* $\widetilde{w}_n^{(i)}$. *Pose* $W_n^{(j)} = 1/M$.

2. *For* $j = 1, ..., M$.

   (a) *For* $t = n, ..., 1$, *calculate for* $i = 1, ..., M$

$$W_{t-1|t}^{(i)} = \widetilde{w}_{t-1}^{(i)} \exp\left(\frac{-\left(s_t^{(j)} - \mu - \alpha y_{t-1} - \beta f_{t-1}^{(i)}\right)^2}{2\delta^2}\right).$$

   (b) *For* $i = 1, ..., M$, *normalize the modified weights: calculate*

$$\widetilde{W}_{t-1|t}^{(i)} = W_{t-1|t}^{(i)} \Bigg/ \sum_{j=1}^{M} W_{t-1|t}^{(i)}.$$

(c) *Choose* $s_{t-1}^{(j)} = f_{t-1}^{(i)}$, *with probability* $\widetilde{W}_{t-1|t}^{(i)}$.

3. *Finally, compute*

$$x_t^{(n)} \simeq x_t^{(n)} = \frac{\sum_{j=1}^{M} s_t^{(j)}}{M},$$

$$P_t^{(n)} \simeq \frac{\sum_{j=1}^{M} \left(s_t^{(j)} - \widehat{x}_t^{(n)}\right)^2}{M-1},$$

and

$$P_{t,t-1}^{(n)} \simeq \frac{\sum_{j=1}^{M} \left(s_t^{(j)} - \widehat{x}_t^{(n)}\right)\left(s_{t-1}^{(j)} - \widehat{x}_{t-1}^{(n)}\right)}{M}.$$

4 Simulation study

4.1 Simulation of time series with given coefficient of variation

Ding and Meade (2010) explored the accuracy of the empirical Square of Coefficient of Variation of volatility ($CV^2 := var\left(\sigma_t^2\right) \Big/ \left(\mathbb{E}\left(\sigma_t^2\right)\right)^2$) calculated from simulated series (generated from $GARCH$ and basic SV models) in reflecting the CV^2 used in the simulation. They conducted experiments across a range of parameter values commonly encountered in practical scenarios (with persistence values ranging between 0.9 and 0.98). Their findings revealed that, for a typical series length, such as 1000 observations, the sample CV^2 tends to underestimate the theoretical CV^2. This underestimation is more pronounced for higher values of CV^2 compared to lower values and is more conspicuous in $GARCH$ models than in SV models. They concluded that if a dataset exhibits a high value of CV^2, it is more likely that the underlying data-generating process follows SV rather than $GARCH$. Following the approach of Ding and Meade (2010), we conducted a simulation study on our novel model to contrast it with the two formulations of $GARCH$ and basic SV. A straightforward calculation yields:

$$\mathrm{CV}^2_{\log GARCH\text{-}SV} = \exp\left\{\frac{\delta^2}{1-(\alpha+\beta)^2}\right\} \prod_{l \geq 0} \left\{\frac{\sqrt{\pi}\,\Gamma\left(2a_l + \frac{1}{2}\right)}{\left(\Gamma\left(a_l + \frac{1}{2}\right)\right)^2}\right\} - 1.$$

Using this expression, we select a set of $(\mu, \alpha, \beta, \delta)$ parameter values for the $\log GARCH\text{-}SV$ model, adhering to the conditions set forth in Ding and Meade (2010) with respect to CV^2 value, persistence, and expected volatility. The Table 1 presents the chosen parameter sets, accompanied by the empirical mean ($\overline{CV}^2$) and standard deviation (Sd_{CV^2}) of $CV^2_{\log GARCH\text{-}SV}$ computed from 1000 runs.

Table 1. Sample versus theoretical CV^2 values for the $\log GARCH$-SV model.

						$n = 200$		$n = 1000$		$n = 4000$	
CV^2	μ	α	β	δ	$\alpha + \beta$	$\overline{CV}^2$	Sd_{CV^2}	$\overline{CV}^2$	Sd_{CV^2}	$\overline{CV}^2$	Sd_{CV^2}
10	−1.2177	−0.1500	1.0500	0.5255	0.90	3.90	3.64	5.89	5.07	7.77	7.86
	−0.7074	0.2800	0.6200	0.4682	0.90	4.65	3.32	6.91	4.74	8.65	5.61
	−0.6665	−0.1200	1.0700	0.3653	0.95	2.79	1.95	5.05	4.30	7.14	7.05
	−0.2430	0.2300	0.7200	0.2476	0.95	3.41	2.10	6.10	4.81	7.76	4.63
	−0.3259	−0.1100	1.0900	0.1228	0.98	1.59	1.16	3.33	2.48	5.84	5.11
	−0.0466	0.1500	0.8300	0.1163	0.98	2.16	1.39	4.47	2.89	6.88	4.68
1	−1.1129	−0.1300	1.0300	0.0887	0.90	0.75	0.56	0.92	0.68	0.97	0.38
	−0.8302	0.1000	0.8000	0.3028	0.90	0.82	0.36	0.95	0.26	0.98	0.16
	−0.5895	−0.0900	1.0400	0.1288	0.95	0.65	0.39	0.87	0.50	0.97	0.37
	−0.3287	0.1200	0.8300	0.1084	0.95	0.73	0.35	0.90	0.30	0.98	0.20
	−0.2569	−0.0600	1.0400	0.0842	0.98	0.48	0.35	0.78	0.48	0.93	0.49
	−1.1035	0.0800	0.9000	0.0348	0.98	0.55	0.34	0.83	0.38	0.96	0.30
0.1	−0.9890	−0.0500	0.9500	0.0638	0.90	0.09	0.03	0.10	0.02	0.10	0.01
	−0.8500	0.0600	0.8400	0.0505	0.90	0.09	0.03	0.10	0.01	0.10	0.01
	−0.5133	−0.0400	0.9900	0.0240	0.95	0.08	0.04	0.10	0.02	0.10	0.01
	−0.4125	0.0400	0.9100	0.0454	0.95	0.08	0.03	0.10	0.02	0.10	0.01
	−0.2072	−0.0200	1.0000	0.0417	0.98	0.07	0.04	0.09	0.03	0.10	0.02
	−0.1631	0.0200	0.9600	0.0438	0.98	0.07	0.04	0.09	0.03	0.10	0.02

To compare these results with those obtained from a basic SV model (our model with $\alpha = 0$), we employed the persistence of volatility and volatility of volatility parameters as provided by Ding and Meade (2010, Table 2, page 775). The μ parameter was calculated by also fixing the expected volatility at $(1\%)^2$. Table 2 presents the results of CV_{SV}^2 obtained from 1000 runs.

Table 2. Sample versus theoretical CV^2 values for the basic SV model.

				$n = 200$		$n = 1000$		$n = 4000$	
CV^2	μ	δ	β	$\overline{CV}^2$	Sd_{CV^2}	$\overline{CV}^2$	Sd_{CV^2}	$\overline{CV}^2$	Sd_{CV^2}
10	−1.0409	0.6750	0.90	4.38	3.11	6.43	5.07	8.06	5.58
	−0.5206	0.4840	0.95	3.05	1.96	5.37	4.28	7.67	5.72
	−0.2082	0.3080	0.98	1.92	1.42	3.75	2.72	5.93	4.10
1	−0.9557	0.3630	0.90	0.81	0.36	0.93	0.27	0.98	0.17
	−0.4779	0.2600	0.95	0.69	0.37	0.91	0.40	0.96	0.22
	−0.1912	0.1660	0.98	0.50	0.31	0.76	0.36	0.92	0.32
0.1	−0.9257	0.1335	0.90	0.09	0.03	0.10	0.01	0.10	0.01
	−0.4631	0.0996	0.95	0.08	0.03	0.09	0.02	0.10	0.01
	−0.1851	0.0610	0.98	0.06	0.04	0.09	0.03	0.10	0.02

Contrasting these findings with those presented in Table 1, it becomes evident that in cases where a dataset exhibits an elevated CV^2 value, there is a higher probability that the underlying data-generating process corresponds to the $\log GARCH$-SV model rather than basic $GARCH$ and SV models. This highlights the flexibility of our proposed model to encompass a broad range in the modeling of real data.

4.2 Effectiveness of ML estimation procedure

To assess the effectiveness of our proposed ML estimation procedure, we conducted several Monte Carlo experiments and report four simulation studies in this section, each using different sample lengths. The sample sizes, n, considered in these simulation studies were 500, 1000, 1500, and 2000. In each case, we employed a sequence of $M = 200$ particles. It is important to note that in our experiments, the corresponding parameter values were chosen such that they satisfy the stationary condition $|\alpha + \beta| < 1$. Note also that in the third and fourth simulations, we adjusted the distribution of observation or log-volatility errors. Specifically, we replaced the Gaussian distribution with a Student-t distribution with

7 degrees of freedom for observation errors and log-volatility errors in the third and fourth simulations, respectively. This adjustment allows us to assess the estimator's performance in scenarios where the normality assumption may not hold.

For every set of observations, we computed the ML (quasi-ML in the Student case) estimate of the parameter vector θ. This process was repeated 1000 times, and we summarized the finite sample properties of the estimators in Table 3. The table provides the true values (TV) of the parameters for each $\log GARCH$-SV model under consideration, along with the empirical mean ($Mean$) and empirical standard deviation (Std) of the estimators.

Table 3. Results of the first and second simulation study for $\log GARCH$-SV model

		1^{st} simulation $e_t, \eta_t \sim \mathcal{N}(0,1)$				2^{nd} simulation $e_t, \eta_t \sim \mathcal{N}(0,1)$			
		μ	α	β	δ^2	μ	α	β	δ^2
	TV	1.2000	0.3000	0.6000	1.0000	0.9000	-0.1200	0.9000	1.6000
$n=500$	$Mean$	1.2871	0.3015	0.5878	1.0158	1.0931	-0.0810	0.8308	1.7301
	Std	0.2326	0.0436	0.0546	0.1814	0.2568	0.0590	0.0838	0.2958
$n=1000$	$Mean$	1.2576	0.3044	0.5892	1.0182	1.0556	-0.0823	0.8397	1.7257
	Std	0.1619	0.0321	0.0397	0.1266	0.1842	0.0413	0.0584	0.2150
$n=1500$	$Mean$	1.2536	0.3085	0.5864	1.0140	1.0550	-0.0815	0.8396	1.7224
	Std	0.1277	0.0264	0.0322	0.1033	0.1443	0.0351	0.0477	0.1663
$n=2000$	$Mean$	1.2199	0.3026	0.5962	1.0122	0.9655	-0.1013	0.8701	1.6527
	Std	0.0922	0.0155	0.0209	0.0789	0.1015	0.0264	0.0348	0.1362
		3^{rd} simulation: $\delta e_t \sim \mathcal{T}(7)$ and $\eta_t \sim \mathcal{N}(0,1)$				4^{th} simulation: $e_t \sim \mathcal{N}(0,1)$ and $\eta_t \sim \mathcal{T}(7)$			
		μ	α	β	$var(\delta e_t)$	μ	α	β	δ^2
	TV	1.2000	0.3000	0.6000	1.4000	0.9000	-0.1200	0.9000	1.6000
$n=500$	$Mean$	1.3227	0.3073	0.5789	1.3323	1.2507	-0.0887	0.8067	2.1081
	Std	0.2734	0.0492	0.0663	0.2515	0.3026	0.0686	0.0980	0.3526
$n=1000$	$Mean$	1.2574	0.3046	0.5892	1.3624	1.2264	-0.0941	0.8178	2.0535
	Std	0.1710	0.0375	0.0471	0.1898	0.2374	0.0491	0.0736	0.3038
$n=1500$	$Mean$	1.2711	0.3158	0.5777	1.3783	1.2085	-0.0941	0.8211	2.0618
	Std	0.1274	0.0279	0.0328	0.1523	0.1612	0.0367	0.0514	0.2073
$n=2000$	$Mean$	1.2807	0.3136	0.5801	1.3858	1.2001	-0.1014	0.8285	2.0582
	Std	0.1269	0.0278	0.0324	0.1397	0.1317	0.0277	0.0395	0.1762

Table 3 demonstrates that, in the $1st$ and $2nd$ simulations, the mean values of all parameters closely align with their true values. Furthermore, with an increasing sample size, the standard deviations decrease. These results confirm the desirable consistency of the ML estimators, indicating the effectiveness of the proposed estimation procedure. However, in the $3rd$ and $4th$ simulations, a small bias is observed when one of the errors is non-Gaussian. Nonetheless, this bias diminishes as the sample size increases. Additionally, we observe a reduction in variance with an increasing sample size, indicating the convergence of the QML estimators.

5 Empirical study

To showcase the practical contribution of the $\log GARCH$-SV model in modeling financial time series, we applied this newly proposed model to two financial time series: the exchange rates of the euro against the Algerian dinar (EUR/DZD) and the Standard and Poor Composite 500 ($S\&P$ 500) stock price index. The first series, EUR/DZD, comprises daily log-return prices obtained from the Bank of Algeria's opening spot quotation, spanning from January 4, 2000, to September 29, 2011, resulting in a total of 3055 observations. This series has been previously examined by Hamdi and Souam (2013, 2018) and Boussaha and Hamdi (2018). The second series consists of daily closing log-return in percentage prices of the $S\&P$ 500 stock. The observations encompass the period from January 3, 1991, to October 20, 2006, yielding a total of 3985 observations. This financial time series has been previously analyzed by Jungbacker and Koopman (2009). The log-return series for EUR/DZD and $S\&P$ 500 are illustrated in Figure 4. We applied a

$\log GARCH\text{-}SV$ model to these log-return series and subsequently compared the fitted model with the SV basic and $GARCH$ models. The estimation results for these three models can be found in Table 4.

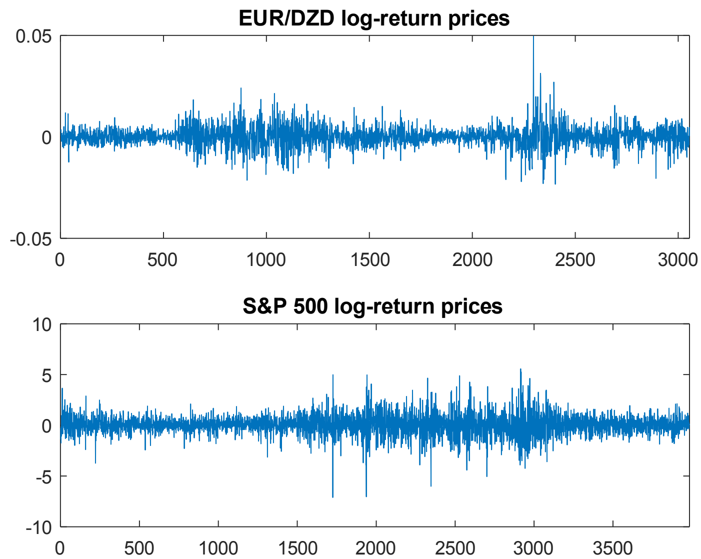

Figure 4. The EUR/DZD and $S\&P$ 500 log-return series.

Table 3. Parameter estimates for $\log GARCH\text{-}SV$, SV basic and $GARCH$ models for EUR/DZD and $S\&P$ 500 log-return series.

			EUR/DZD log-return series			
	μ	α	β	δ	$\mathbb{E}\left(\varepsilon_t^2\right)$	κ_ε
Empirical	–	–	–	–	0.000025	8.9678
Fitted $\log GARCH\text{-}SV$	-0.5731	-0.0163	0.9662	0.3016	0.000026	7.7263
Fitted SV basic	-0.0139	–	0.9987	0.0373	0.000030	5.1419
Fitted $GARCH$	0.0000002	0.0564	0.9359	–	0.000026	5.1221
			$S\&P$ 500 log-return series			
	μ	α	β	δ	$\mathbb{E}\left(\varepsilon_t^2\right)$	κ_ε
Empirical	–	–	–	–	0.9988	7.0780
Fitted $\log GARCH\text{-}SV$	-0.0326	-0.0073	0.9542	0.3010	1.0008	7.2172
Fitted SV basic	-0.0241	–	0.9443	0.3097	1.0102	7.2737
Fitted $GARCH$	0.0051	0.0548	0.9405	–	1.0730	8.2331

One way to assess the adequacy of $\log GARCH\text{-}SV$ models is by examining their ability to capture the features or stylized facts inherent in the series to be modeled. Expressions for moments, particularly variance and kurtosis of observations, are available for this purpose. These expressions enable us to determine, for example, whether a $\log GARCH\text{-}SV$ model is capable of reproducing the stylized facts present in the data. Indeed, as evident in Table 4, the variance and kurtosis of the estimated $\log GARCH\text{-}SV$ model closely match the empirical values of the two analyzed series, comparatively to those obtained from the two estimated SV basic and $GARCH$ models.

On the other hand, the predicted, filtered, and smoothed volatility for the various models can be obtained using the Kalman filter, particle filter, and particle smoother algorithms, respectively. These resulting volatilities are referred to as predicted, filtered volatility, and smoothed volatility, respectively. Figures 5-10 illustrate the estimated volatilities for the different models. It is evident that the $\log GARCH\text{-}SV$ models effectively capture the significant shocks in the EUR/DZD and $S\&P$ 500 log-return series compared to the basic SV and $GARCH$ models. Therefore, the estimated volatility from the $\log GARCH\text{-}SV$ model demonstrates greater accuracy, and our new model has proven to be globally superior to the SV basic and $GARCH$ models in fitting the EUR/DZD exchange rates and $S\&P$ 500 stock market index.

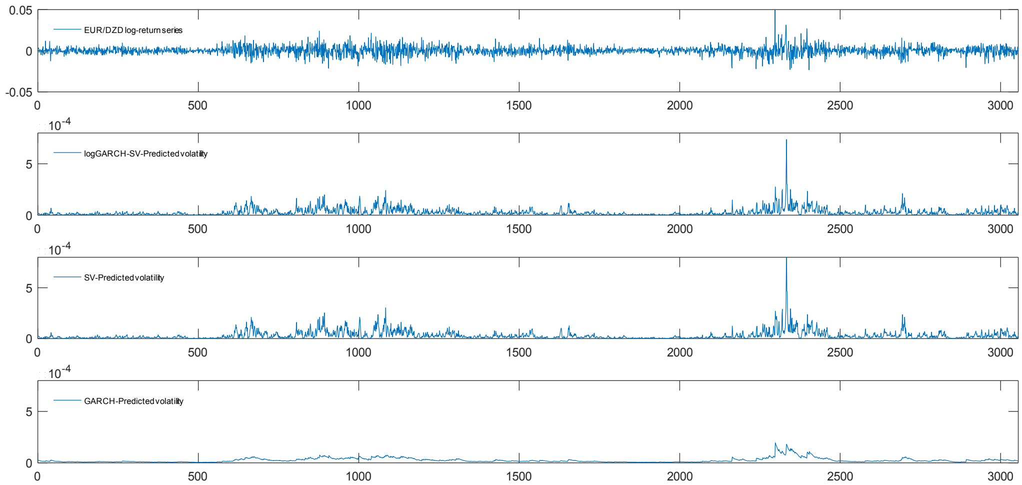

Figure 5. From top to bottom, graph of the EUR/DZD log-returns and predicted volatility obtained from $\log GARCH$-SV, SV and $GARCH$ fitted models.

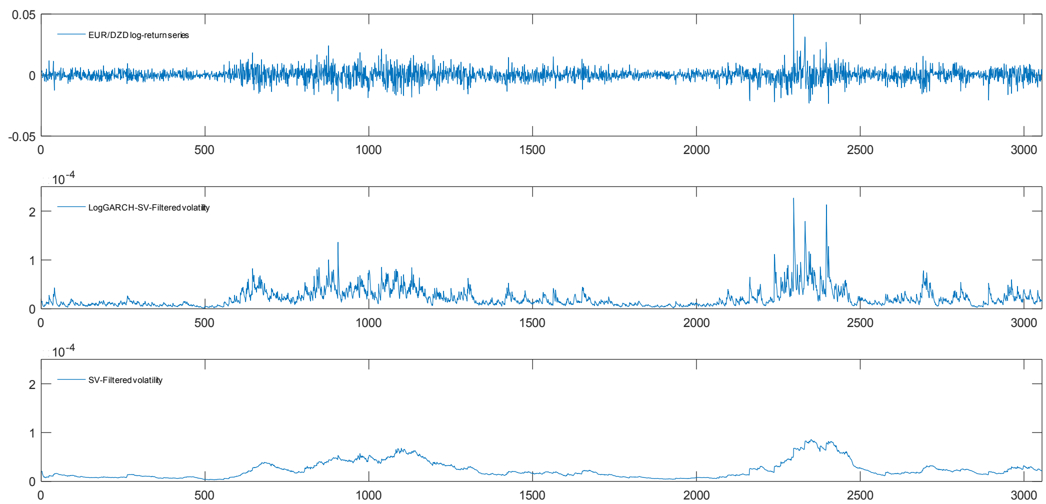

Figure 6. From top to bottom, graph of the EUR/DZD log-returns and filtered volatility obtained from $\log GARCH$-SV and SV fitted models.

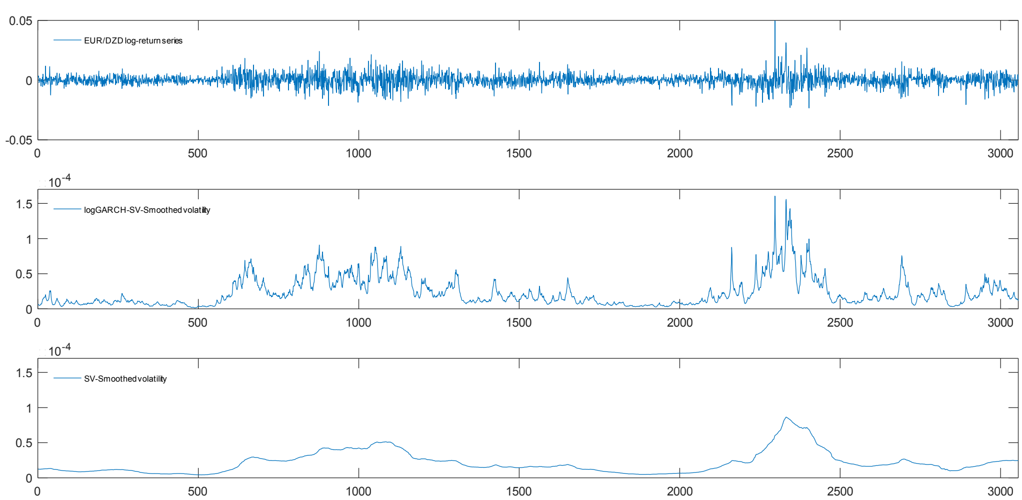

Figure 7. From top to bottom, graph of the EUR/DZD log-returns and smoothed volatility obtained from $\log GARCH$-SV and SV fitted models.

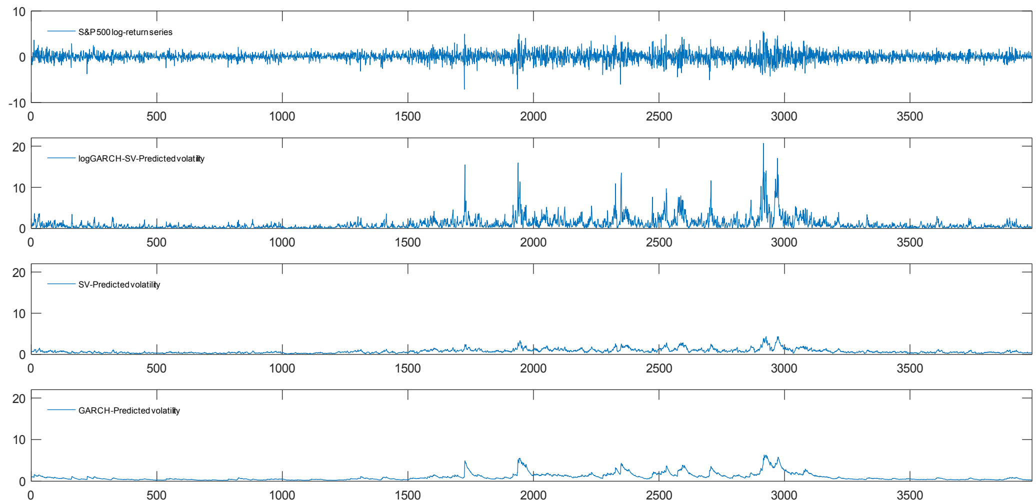

Figure 8. From top to bottom, graph of the $S\&P$ 500 log-returns and predicted volatility obtained from $\log GARCH$-SV, SV and $GARCH$ fitted models.

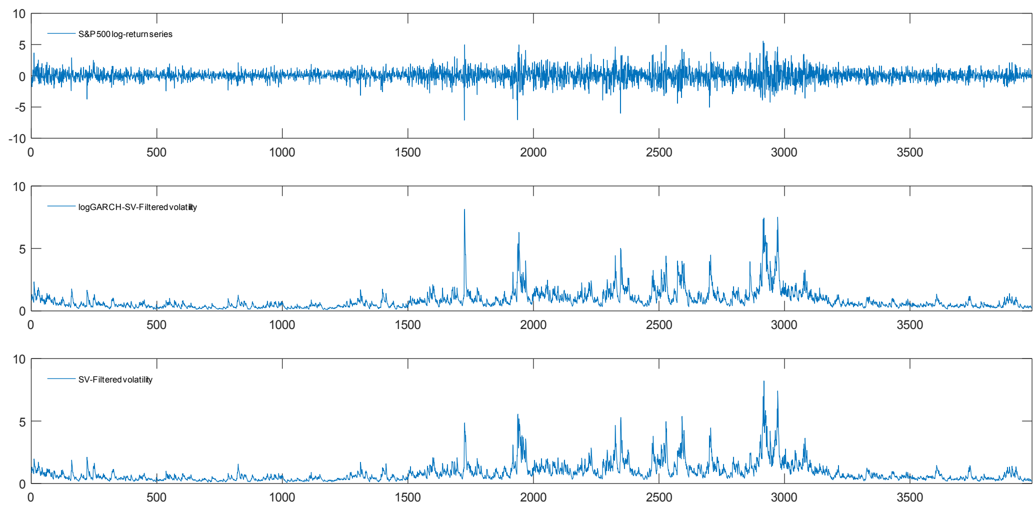

Figure 9. From top to bottom, graph of the $S\&P$ 500 log-returns and filtered volatility obtained from $\log GARCH$-SV and SV fitted models.

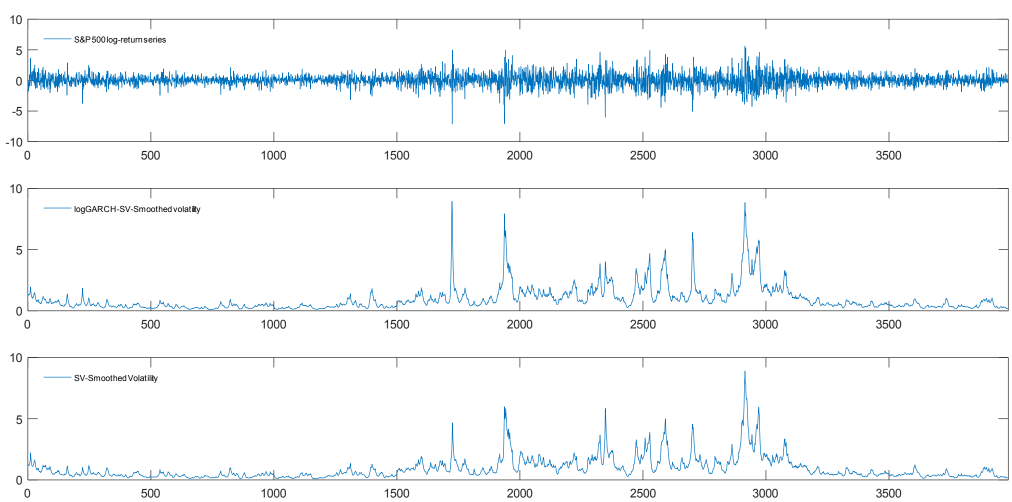

Figure 10. From top to bottom, graph of the $S\&P$ 500 log-returns and smoothed volatility obtained from $\log GARCH$-SV and SV fitted models.

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