Increasing Returns to Scale and Welfare: Ranking the Multiple Deterministic Equilibria

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Abstract

We consider a real business cycle model with productive externalities and an aggregate non-convex technology *à la* Benhabib and Farmer, which exhibits indeterminacy of the steady state and multiplicity of deterministic equilibria. The aim of the paper is to rank these different equilibria according to the initial values of consumption using both a non-naive quadratic approximation, extensively explained by Benigno and Woodford [2006]. We study the implications of such a ranking in terms of smoothness of the welfare-maximizing solution and show that maximizing welfare consumption and labor paths are all the smoother than the level of increasing returns is low. At last, this solution provides a good benchmark for judging the desirability of the stabilization Guo and Lansing policy.

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JEL classification: E32, E4, H61, O42, O47.

1 Introduction

Despite the concavity of the utility function, Christiano and Harrison [2002] have established that increasing volatility of labor may raise welfare in economies with non-convex technology sets *à la* Benhabib and Farmer [1994]. In absence of any productive externality, fluctuations in consumption and labor are welfare-diminishing compared to a smooth consumption/investment plan when the utility function is concave. However, in the presence of productive externalities, the welfare loss implied by fluctuations may be more than compensated by the gain inherited from the increasing returns to scale: for a given capital stock, by bunching hard work, agents are able to increase the average level of consumption without raising the average level of labor. When disutility of labor does not raise disproportionately compared to the additional utility procured by consumption, this "bunching" effect dominates the former negative "concavity" effect and makes the agents better-off. Thus, when the steady state equilibrium is locally indeterminate, that is when there is multiplicity of deterministic equilibria around the steady state, stochastic sunspot equilibria may be welfare-improving.

In the literature the possibility of stabilizing an economy characterized by local indeterminacy has been analyzed in such a framework by Guo and Lansing [1998].[1] However no much attention has been dedicated to the choice of the best equilibrium path on which the economy has to be stabilized. It is clear, from Christiano and Harrison's estimates, that a stabilizing policy can make the agents worse-off when expectations are pinned down on a suboptimal path. From Pareto's criterion viewpoint, any (decentralized) deterministic equilibrium path of the Benhabib and Farmer economy is not efficient as long as agents do not grasp the externality of production. Nevertheless, from a welfare viewpoint, these deterministic equilibria do not display the same level of utility: the optimization programme fails to determine which of them provides the maximum amount of welfare since they all satisfy the first order conditions and the transversality condition. Thus, when agents jump from one path to another, the

[1] Economic policy constructed to stabilize the economy by minimizing the variance of output have also been analyzed in models in which the level of externality required to get indeterminacy is less stringent than in the current framework. See for instance Guo and Harrison [2002] or Sim [2005].

stochastic equilibrium so obtained may increase their welfare provided they leave a welfare-dominated deterministic path for a welfare-improving deterministic path.

In this paper we provide a welfare ranking of the different deterministic equilibria in an exogenous growth model with a non-convex technology and a locally indeterminate equilibrium. We determine the conditions under which a deterministic path switching makes the agents better-off.[2] We prove that the starting value of consumption determines simultaneously the speed of the capital accumulation and the desirability of a deterministic path switching. Actually, we translate and investigate the concavity and the bunching effects developed by Christiano and Harrison in terms of monotonicity of the consumption/investment plan. It will be shown that the (decentralized) welfare-maximizing equilibrium displays a path all the less monotonic and a ratio of the initial level of consumption over its steady state value all the lower since increasing returns to scale are high.

Bunching hard work in the very first periods makes capital accumulation faster. In the next periods, agents can benefit from the high level of capital stock by maintaining a high level of consumption but decreasing labor significantly. When increasing returns are high enough, reaching the welfare-maximizing capital stock requires relatively few time and effort, which explains the non-monotonicity of the equilibria during the first periods. However, when increasing returns are close to the Benhabib-Farmer condition for indeterminacy, bunching hard work in the first periods is not sufficient to accumulate quickly a large capital stock, which would require huge levels of labor and a loss of welfare that next periods consumption cannot offset. Thus, when increasing returns to scale are not high enough, the second best equilibrium consists in minimizing the variance of labor, smoothing consumption and accumulating the capital stock progressively.

Finally, we compare the welfare-maximizing path so obtained with the Guo-Lansing solution. We prove that the stabilizing tax policy they propose to prevent from endogenous fluctuations always deteriorates welfare compared to the second best equilibrium of the Benhabib-Farmer economy. We provide an alternative policy able to pin down expectations on the deterministic equilibrium we found. This can be done by fixing the rental rate on capital or the real wage at one or several periods of time.

After presenting briefly the main characteristics of the Benhabib-Farmer model in the second section – including uniqueness of the steady state equilibrium and the condition for indeterminacy, we will assume this condition satisfied and will specify in section 3 the set of monotonic consumption paths for any values of the parameters. These results will be helpful in establishing the welfare ranking of section 4 when we use a quadratic approximation of the utility function. We will confirm and extend these results with simulation methods. Before concluding, we will compare the welfare-maximizing equilibrium with the Guo-Lansing solution in section 5, and we will draw the conclusions for economic policy.

2 Model Setup

2.1 Agents' behavior

In this paper we analyze the welfare properties of different equilibrium paths in the Benhabib-Farmer model [1994]. This deterministic continuous-time model with infinitely lived agents is characterized by social increasing returns to scale due to factor-specific externalities in the aggregate production function. However, the representative firm is assumed not to take into account the externality of production and then faces a Cobb Douglas production function Y with constant returns to scale at the micro-level. Formally:

$$Y(t) = A(t)K(t)^a L(t)^b \quad \text{with} \quad 0 < a < 1, \quad \text{and} \quad a + b = 1, \tag{1}$$

$$A(t) = \bar{K}(t)^{a\gamma} \bar{L}(t)^{b\gamma} \quad \text{with} \quad \gamma > 0, \tag{2}$$

where $\bar{K}$ and $\bar{L}$ represent the average economy-wide levels of capital and labor. In equilibrium, $K = \bar{K}$ and $L = \bar{L}$ and by making the parameters substitutions $\alpha = a(1 + \gamma)$ and $\beta = b(1 + \gamma)$, we get the aggregate production function:

$$Y(t) = K(t)^\alpha L(t)^\beta,$$

[2] It is worth noting however that we only compare the differences in utility at the different deterministic equilibria after a definite change of path. We do not investigate stochastic equilibria for which agents permanently switch the equilibrium paths.

which obviously exhibits increasing returns to scale since $\alpha + \beta > 1$. In the same time, the economy is populated by a large number of identical consumers. As usual, firms maximize profit, which breaks even because of the constant returns, while the representative consumer, owner of the firms, faces the following optimal control problem depending on the two controls C, consumption, and L, labor:

$$\max_{C,L} \int_0^\infty \left(\log C(t) - \frac{L(t)^{1-\chi}}{1-\chi} \right) e^{-\rho t} dt,$$

subject to:

$$\dot{K}(t) = (r(t) - \delta) K(t) + w(t)L(t) - C(t),$$

where $\chi \leq 0$ is the inverse of the Frisch elasticity of labor supply, $\rho > 0$ is the discount rate and $\delta > 0$ the depreciation rate. We call $r(t)$ and $w(t)$ respectively the rate of return on capital and the real wage at time t.

2.2 Dynamical system and steady state equilibrium

From the first order conditions and after some algebra, Benhabib and Farmer obtain the following two nonlinear ordinary differential equations system:

$$\dot{k} = e^{\mu_0 + \mu_1 k + \mu_2 c} - \delta - e^{c-k} \quad (3)$$

$$\dot{c} = a e^{\mu_0 + \mu_1 k + \mu_2 c} - \delta - \rho \quad (4)$$

where $x = \ln X$, $\mu_0 = \frac{-\beta \ln b}{\beta + \chi - 1}$, $\mu_1 = \frac{(\chi-1)(\alpha-1)-\beta}{\beta+\chi-1}$ and $\mu_2 = \frac{\beta}{\beta+\chi-1}$. It is worth noting that the system represents the global dynamics of the economy.

Taking into account such dynamics, we determine the steady state of the system:

$$L_s = \left(\frac{(\rho+\delta)(1-a)}{\rho + \delta(1-a)} \right)^{\frac{1}{1-\chi}}$$

$$K_s = \left(\frac{a L_s^\beta}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}$$

$$C_s = \frac{(\rho+\delta)(1-a)}{a} \left(\frac{a L_s^\beta}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}$$

Benhabib and Farmer show that under the condition $\beta - 1 + \chi > 0$, the aggregate labor demand curve is upward sloping and steeper than the labor supply curve, and the steady state equilibrium is indeterminate. In the neighborhood of such an equilibrium, there exists a continuum of paths converging to it, then satisfying both the first order conditions and the transversality condition. In this framework, the perfect foresight and the rational expectations hypotheses, which usually lead to a unique equilibrium path, cannot discriminate between the different paths: in absence of coordination, agents are allowed to jump from one path to another at any period. However, in terms of welfare, these paths are not equivalent.

3 Local analysis

The results of this section are closely related to the classical Grobman-Hartman theorem about the preservation of the topological properties of the system under linearization. It states that around a hyperbolic equilibrium the flow of a nonlinear differential equation is conjugate via a local homeomorphism to the flow of its linear approximation.[3] It is clear from Benhabib and Farmer [1994] that no eigenvalue crosses zero as the determinant changes sign and the steady state becomes stable.[4] Then, the stationary equilibrium remains hyperbolic as the level of increasing returns increases and the stability of the steady state changes, even for the minimum degree of externality necessary for local indeterminacy.

[3] In a continuous-time model, an equilibrium is said to be hyperbolic when there is no eigenvalue equal to zero.

[4] To be precise the change in the stability of the equilibrium is related to the presence of a discontinuity in the value of one of the two eigenvalues, say λ_1, as the externality γ_b increases: $\lim_{\beta-1+\chi\to 0^-} \lambda_1 = -\infty$ while $\lim_{\beta-1+\chi\to 0^+} \lambda_1 = +\infty$.

From now, it will be assumed that the model exhibits local indeterminacy and then that both eigenvalues have strictly negative real part.[5] In this section, the properties of the Grobman-Hartman theorem are used to describe qualitatively the different equilibrium paths in terms of monotonicity by solving the linear approximation system. The approximated values of capital and consumption so obtained help to redefine the optimization problem to be solved and to maximize the representative agent's utility respect to the initial value of consumption he/she chooses.

3.1 Linearization

We proceed to a first order approximation of equations (3) and (4) around the deterministic equilibrium and express the general solution in terms of deviation of the two variables $k(t)$ and $c(t)$ from their steady state values k_s and c_s, i.e. $\tilde{x}(t) = \ln X(t) - \ln X_s$. We get:

$$\begin{bmatrix} \tilde{k}(t) \\ \tilde{c}(t) \end{bmatrix} \simeq \begin{bmatrix} \eta_1 v_{11} & \eta_2 v_{12} \\ \eta_1 v_{21} & \eta_2 v_{22} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} \tag{5}$$

with

$$V = [\xi_1 : \xi_2] = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} (1-\mu_2)\Psi - \delta & (1-\mu_2)\Psi - \delta \\ (1+\mu_1)\Psi - \delta - \lambda_1 & (1+\mu_1)\Psi - \delta - \lambda_2 \end{bmatrix}, \tag{6}$$

where $\Psi \equiv (\rho+\delta)/a$ and ξ_1 and ξ_2 are eigenvectors associated to the eigenvalues λ_1 and λ_2, which can be obtained after computing the Jacobian of the system formed by equations (3) and (4). Moreover, given a starting point $[K(0), C(0)]$, we apply Cramer's rule and deduce:

$$\eta_1 = \frac{v_{22}\tilde{k}(0) - v_{12}\tilde{c}(0)}{v_{11}v_{22} - v_{12}v_{21}} \tag{7}$$

$$\eta_2 = \frac{v_{11}\tilde{c}(0) - v_{21}\tilde{k}(0)}{v_{11}v_{22} - v_{12}v_{21}}. \tag{8}$$

3.2 Monotonic equilibrium paths

In order to understand the economic implications of the equilibrium paths ranking in terms of intertemporal consumption smoothness, we study in this subsection the conditions on $c(0)$ under which the path is monotonic. It is reminded that monotonicity only occurs when eigenvalues are real. In the following, we will assume without loss of generality that $\lambda_1 < \lambda_2 < 0$.

Under the condition $\beta - 1 + \chi > 0$ the stable manifold has dimension 2. We call stable arms the two paths such that:

$$\tilde{c}(t) = \eta_i v_{2i} e^{\lambda_i t}, \quad i = \{1, 2\}.$$

Let c_{0,ξ_i}, $i = \{1, 2\}$, be the starting log-values of consumption of such a path for a given initial stock of capita $K(0)$. These values are such that respectively η_1 and η_2 equalize zero, that is:

$$c_{0,\xi_1} = c_s + \tilde{k}(0)\frac{v_{21}}{v_{11}} \tag{9}$$

$$c_{0,\xi_2} = c_s + \tilde{k}(0)\frac{v_{22}}{v_{12}}. \tag{10}$$

The following proposition holds:

Proposition 1 *For a given initial stock of capital $K(0) < K_s$ (resp. $K(0) > K_s$), there exists a strictly positive (resp. negative) ε^* such that for $c(0) \in [c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$ (resp. $[c_{0,\xi_1}, c_{0,\xi_2} - \varepsilon^*]$) equilibrium paths of consumption are monotonic.*

Proof. See Appendix A.2. ■

A specific case with $K(0) < K_s$ is reported in Figure 1 below:

[5] It must be noticed however that the linear approximation computed here remains valid whatever the sign of the eigenvalues.

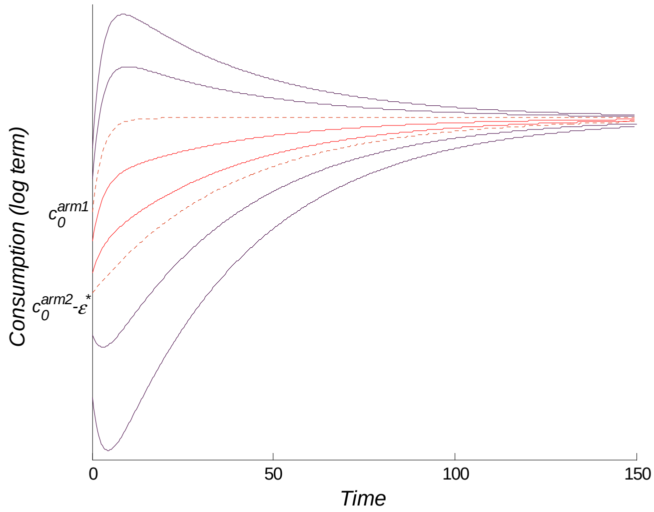


Figure 1: Consumption paths around the steady state.

3.3 Reformulation of the optimization programme

For a predetermined stock of capital $K(0)$ and a given initial level of consumption $C(0)$, the optimal paths of capital and consumption can be computed using equations (3) and (4). Furthermore, the associated optimal paths of labor can be derived using the following first order condition:

$$(\beta - 1 + \chi)l(t) = c(t) - \alpha k(t) - \ln b. \tag{11}$$

Searching the path making the agents better off consists of determining the initial level of consumption that maximizes welfare:

$$\max_{C(0)} \int_0^\infty \left(\log C(t) - \frac{L(t)^{1-\chi}}{1-\chi} \right) e^{-\rho t} dt. \tag{12}$$

Since $U(c_s, l_s)$ is a constant, it must be noticed that our optimization programme (12) can be rewritten as:

$$\max_{c(0)} \int_0^\infty \tilde{U}(c(t), l(t)) e^{-\rho t} dt, \tag{13}$$

where $\tilde{U}(c(t), l(t)) = U(c(t), l(t)) - U(c_s, l_s)$ and $U(c(t), l(t)) = c(t) - \frac{e^{l(t)(1-\chi)}}{1-\chi}$.

It is now well know that first-order approximation techniques are not always well suited to handle welfare comparisons. This is especially the case for stochastic models, as already shown by Kim and Kim [2004] or Schmitt-Grohe and Uribe [2004] among others. Although the current paper proposes a deterministic framework, we will follow the method developed by Fleming [1971], applied by Magill [1977] and recently revisited by Benigno and Woodford [2007a, b], and will make the different welfare comparisons using a second-order approximation of the utility function.

However, for any predetermined stock of capital and whatever the order of approximation, it is not obvious how to determine the range of values of initial consumption lying in the region where topological equivalence occurs and the approximation errors do not affect the welfare ranking.

4 Welfare Ranking of deterministic paths

When the dynamics is constrained to be linear and a quadratic approximation of utility function is provided, an analytical approach can be used to determine approximatively within the set of possible values the initial level of consumption maximizing welfare. However, the approximation error implied by such a technique requires us to work in a small neighborhood of the steady state.

The technique used by Benhabib and Farmer to prove indeterminate and which consists in linearizing the dynamic system and checking whether eigenvalues have negative real parts ensures that the stationary equilibrium is asymptotically stable, that is locally attractive.

Nevertheless, the technique does not provide any indication on how close to C_s the initial value of consumption must be chosen. Russell and Zecevic [1998, 2000] propose to identify the *region of attraction* for the Benhabib-Farmer model by the use of a Lyapunov function.[6] From this region and for different initial stocks of capital, they compute the largest interval of initial consumption levels such that the equilibrium paths are not constrained by the conditions of the Lyapunov stability but finally converge to the steady state. The concept of *region of stability*, as they call it, cannot be used for our problem since it does not guarantee that a path included in this set can be approximated far enough from the steady state by the linear approximation we derived.

To encompass the problem of the definition of the region of stability, ensure topological equivalence of the linearized system and limit the approximation error, we proceed in two steps.

First, we restrict the analysis to the minimum degree of externalities insuring indeterminacy. We thus determine analytically the welfare-maximizing starting value of consumption within the set of deterministic equilibria using a non-naive quadratic approximation of the utility function around the steady state.

Second, making use of simulation methods we enlarge the range of increasing returns to scale: we are then able to determine precisely a good approximation of the true value of the welfare-maximizing starting value of consumption and the behavior of the related consumption/investment plan whatever the values of the parameters.

4.1 Formal analysis

4.1.1 Approximation method

Following Fleming [1971], it can be shown that:

$$\begin{aligned}\tilde{U}(k,c) &= \underbrace{-(1-\chi)e^{(1-\chi)\bar{l}}\tilde{l}(k,c)^2}_{U_1(t)} + \underbrace{e^{-\bar{c}}[\Psi(\mu_1\tilde{k}+\mu_2\tilde{c})^2-(\Psi-\delta)(\tilde{c}-\tilde{k})^2]}_{U_2(t)} + \mathcal{O}(||(c,k)||^3)\\ &= \hat{U}(k,c)+\mathcal{O}(||(c,k)||^3)\end{aligned}$$

with $\tilde{l}(k,c)=\frac{\tilde{c}-\alpha\tilde{k}}{\beta-1+\chi}$ for $\beta-1+\chi\neq 0$.

The first term in the right hand side; $U_1(t)$ is always negative (or null) while the second term, $U_2(t)$, can be positive.

Proposition 2 *When β tends to $1-\chi$, the welfare-maximizing path is monotonic and starts with $c(0)=c_{0,\xi_2}$.*

Proof. According to equations (5) and (6), for $t >> 0$, $\tilde{c}(t)$ tends to $\eta_2 v_{22}e^{\bar{\lambda}_2 t}$ and $\tilde{k}(t)$ tends to $\eta_2 v_{12}e^{\bar{\lambda}_2 t}=(v_{12}/v_{22})\tilde{c}(t)\to\tilde{c}(t)/\alpha$ as β tends to $1-\chi$. The term $U_2(t)$ collapses to $e^{-\bar{c}}\delta\left(\frac{1-\alpha}{\alpha}\tilde{c}(0)e^{\bar{\lambda}_2 t}\right)^2$.

$\hat{U}(t)$ is maximum when the term of higher degree, $U_1(t)$, is maximum, that is for $\tilde{c}(t)=\alpha\tilde{k}(t)$ for all periods. At the first period, this means $\tilde{c}(0)=\alpha\tilde{k}(0)\sim(v_{22}/v_{12})\tilde{k}(0)$ as $\beta\to 1-\chi$, that is $c(0)=c_{0,\xi_2}$ as shown in equation 10. ■

Some conclusions about the welfare-maximizing path can be drawn as β increases from $1-\chi$: as shown in proposition 2, for $t\neq 0$, $U_1(t)$ is maximum provided $\tilde{c}(t)\sim\alpha\tilde{k}(t)$ whereas $U_2(t)\sim e^{-\bar{c}}\delta\left(\frac{1-\alpha}{\alpha}\tilde{c}(0)e^{\bar{\lambda}_2 t}\right)^2$ is maximum for $|c(0)|$ as high as possible, i.e. $c(0)=c_{0,\xi_2}-\varepsilon^*$.[7] When $\beta-1+\chi$ increases from zero,

[6] For an exhaustive definition of the concept of Lyapunov stability, see for instance.

[7] Since $\tilde{c}(t)\sim\alpha\tilde{k}(t)$, the optimal path must be monotonic and $c(0)$ must be lower or equal to $c_{0,\xi_2}-\varepsilon^*$.

increasing $|\tilde{l}(t)|$ yields a mid decrease of $U_1(t)$ while $U_2(t)$ can be increased provided $c(0)$ gets closer to $c_{0,\xi_2} - \varepsilon^*$.

Thus, as β increases from $1 - \chi$, we expect a relative decrease of $c(0)$.

4.1.2 Economic arguments

When the level of increasing returns to scale is just sufficient to insure indeterminacy, a government that wants to maximize welfare and that is able to pin down expectations on a given path has an incentive to coordinate consumers' expectations on a $c(0)$ as close as possible to c_{0,ξ_2} for any given stock of capital $k(0)$.

On this path:

$$\begin{aligned} \tilde{c}(t) &\sim \eta_2 v_{22} e^{\bar{\lambda}_2 t} \\ \tilde{k}(t) &\sim \eta_2 v_{12} e^{\bar{\lambda}_2 t}, \end{aligned}$$

log-deviations of capital and labor evolve monotonically and approximately at the same rate $\bar{\lambda}_2$. Actually, for mid levels of increasing returns to scale − when they are just sufficient to insure indeterminacy − the convexity of the utility function imposes to the welfare-maximizing path to minimize the variance of labor. Any degree of fluctuations is undesirable:

1. starting with a low level of consumption ($c_0 < c_{0,\xi_2} - \varepsilon^*$) that decreases on the first periods yields a higher instantaneous utility at the very beginning by reducing the desutility of labor but requires a sharp increase of labor afterwards to reach the steady state equilibrium, which in turn reduces the intertemporal utility. Increasing returns to scale are not sufficient to dampen the increase of the last periods average level of labor.

2. starting with a high level of consumption ($c_0 > c_{0,\xi_1}$) and accumulating rapidly a large stock of capital to benefit from this accumulation afterwards (since labor can decrease faster than consumption: the so-called *bunching effect*) yields a deep loss in the instantaneous utility by concentrating the desutility of labor in the first periods which is never compensated by the following decrease in the average level of labor. Increasing returns to scale are not sufficient to maintain a high level of consumption while decreasing labor once a large stock of capital has been accumulated.

The welfare-maximizing path is thus presents consumption, capital and labor paths as smooth and balanced as possible, that is exhibiting growth rates as flat as possible. In that case, agents do not expect the bunching effect to be optimal and would rather choose a monotonic path if they may pin down their expectations on it. But it is important not to conclude from this consideration alone that this bunching effect is not optimal from the social planner viewpoint for the welfare-maximizing path of the decentralized Benhabib and Farmer economy is not actually the optimal path. Whilst in Christiano and Harrison [1999] a stochastic sunspot equilibrium can make the agents better-off compared to a deterministic sunspot equilibrium by bunching hard work at some periods of time, a deterministic equilibrium is not able to welfare-dominate another one using the bunching effect when increasing returns to scale are just sufficient to insure indeterminacy. The intuition for such a finding is straightforward: an economy can be closer to the optimal solution when following a stochastic equilibrium that an economy whose agents' expectations are pined down on a deterministic path where the productive externalities are not internalized. Consequently, the agents expect the bunching effect not to be optimal while it is actually.

It would be interesting to enlarge the set of possible levels of increasing returns to scale in order to check whether or not the agents have interest to choose a non-monotonic equilibrium path, i.e. whose starting value of consumption would be lower than $c_{0,\xi_2} - \varepsilon^*$ or higher than c_{0,ξ_1}. This is the objective of the next subsection.

4.2 Computational analysis

Until now our analysis has focused on the minimal level of increasing returns (or close enough) required to get indeterminacy. Further from these values, an algebraical derivation of the welfare-maximizing value of $c(0)$ is not straightforward. In this section, we relax the size of increasing returns and switch to simulation methods to compute $c(0)$. We draw some qualitative predictions on the relation between the initial level of consumption and the level of increasing returns. Especially, it will be shown that the higher the increasing returns to scale, the higher the welfare-maximizing initial level of consumption and the less smooth the maximizing welfare paths of consumption, labor and investment.

4.2.1 Simulation methods

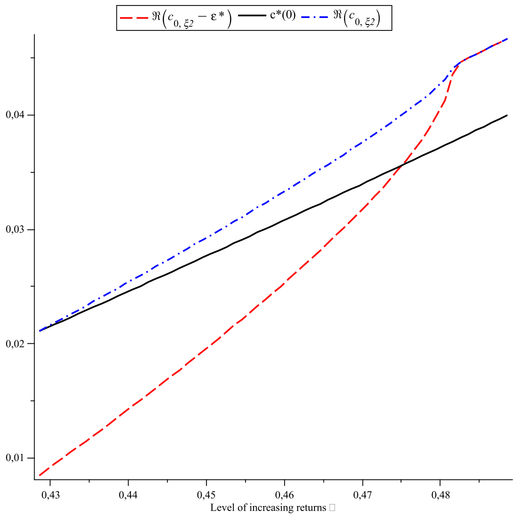
Computationally, we proceed as follows. First, we derive the eigenvalues of the Jacobian and the linear approximation of the log-values of capital and consumption, and we substitute them into the quadratic approximation of the utility function. We solve numerically the integral and obtain a quadratic approximation of the welfare function of the form:

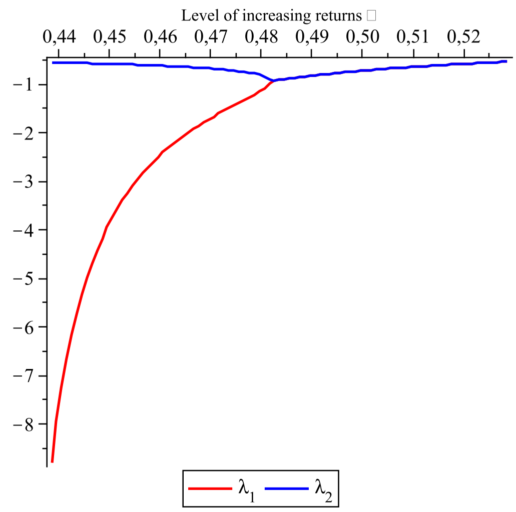
$$\hat{W}[c(0)] = n_1 c(0)^2 + n_2 c(0) + n_3,$$

where $n_1 < 0$ and whose maximum is reached for:

$$c^*(0) = -2n_1/n_2.$$

We first parameterize the economy as Benhabib and Farmer did: the capital share $a = 0.3$, the inverse of the Frisch elasticity of labor $\chi = 0$, the discount rate $\rho = 0.065$ and the depreciation rate $\delta = 0.1$. The initial value of capital is set close enough (but not to close) to the steady state at $k_0 = 0.9k_s$.[8] All the initial values of the parameters are checked to be in the attraction set.[9] Then, for different size of increasing returns to scale, γ, we get:





On figure 1, we have sketched the welfare-maximizing value of $c(0)$ in addition to $c_{0,\xi_2} - \varepsilon^*$ and c_{0,ξ_2} as the degree of increasing returns to scale γ increases from the minimal value satisfying the condition for indeterminacy. These results confirm proposition 2, which predicts that within the set of monotonic paths, the maximizing welfare equilibrium starts with an initial level of consumption $c(0) = c_{0,\xi_2}$. They also confirm our conjecture that this initial level tends to get closer to $c_{0,\xi_2} - \varepsilon^*$ as γ increases.

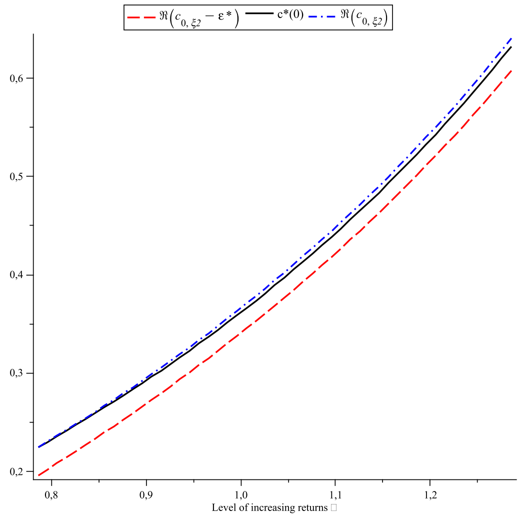
When the real part of both eigenvalues merge (as shown on figure 2) and become complex, it can be easily seen from equations (7) and (8) that $\Re(v_{21}/v_{11}) = \Re(v_{22}/v_{12})$ and then that $\Re(c_{0,\xi_1}) = \Re(c_{0,\xi_2}) = \Re(c_{0,\xi_2} - \varepsilon^*)$. For higher level of increasing returns, the welfare-maximizing path must be non-monotonic. However, it must be noticed that the welfare-maximizing path becomes non-monotonic before this threshold. Since $c_{0,\xi_2} - \varepsilon^*$ increases faster than $c^*(0)$ there is a level of increasing returns above which the maximum welfare is reached for an initial level of consumption $c^*(0)$ outside the range $[c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$,

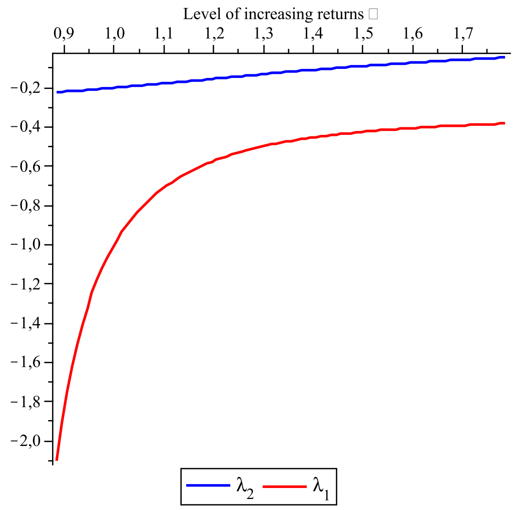
[8] Alternative computations have been derived with $k_0 = 0.99k_s$ and $k_0 = 0.999k_s$ but do not affect the conclusions. The difference in welfare from one path to another is however smaller.

[9] Taking into account table 1 in Russell and Zecevic (1998) it is, for example, possible to observe that $c(0)$ may be chosen in the interval $(c^{low}_{-34\%}, c^{\max}_{103\%})$ when $\beta = 1.26$.

meaning that the welfare-maximizing path is non-monotonic and the degree of consumption smoothness lower while the eigenvalues are real.

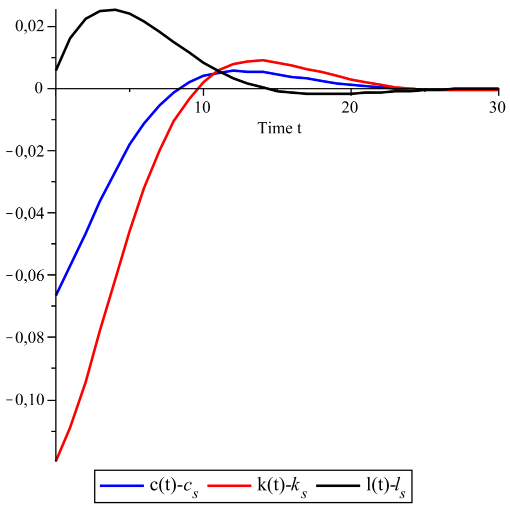
Actually, monotonicity occurs for almost all plausible values of the increasing returns. For lower elasticities of labor and higher capital shares, eigenvalues may not merge, the optimal values of initial consumption may remain included in $[c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$ and non-monotonic paths may never be welfare-maximizing. When $\chi = -.25$ (and the other parameters unchanged) for instance:





4.2.2 Economic arguments

In optimal growth models à la Benhabib and Farmer with social increasing returns to scale and productive externalities, Christiano and Harrison [1999] distinguish two effects affecting the consumption/investment plans. For a given technological coefficient (a given productive externality), the concavity of the utility function prevents from fluctuations which deteriorate welfare. This "concavity effect" leads to choose monotonic equilibria and smooth consumption and labor over time so as to maximize agent's welfare. However, when the externality varies with the average levels of capital and labor, increasing returns to scale appear at the aggregate level. It may be welfare improving to bunch hard work in the first periods to boost capital accumulation in order to benefit from higher productive externalities in the future for lower levels of labor.



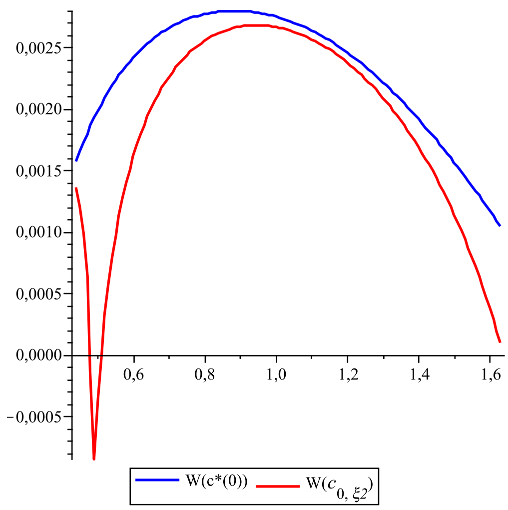
When this "bunching effect" dominates the "concavity effect", agents bring forward a part of their labor supply, raising the average level of consumption and decreasing labor after a while. On the figure above, we pictured the optimal paths of capital, consumption and labor for $\gamma = 0.9$.

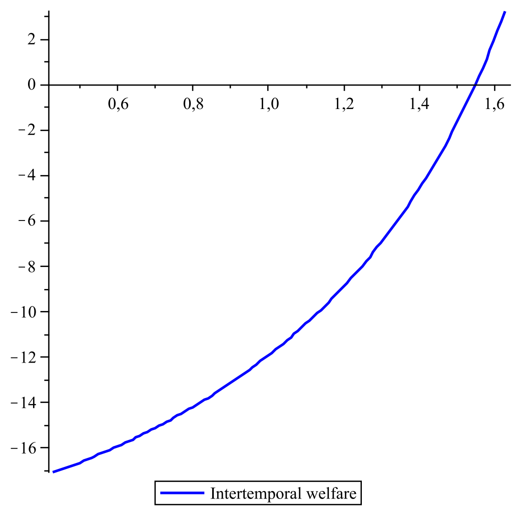
On the one hand, a higher level of increasing returns to scale makes capital accumulation larger for the same amount of worked hours or equivalently allows the representative agent to raise consumption without raising labor. On the other hand, the higher the level of increasing returns to scale, the higher the steady state value of capital: the average level of labor required to reach this steady state value appears to be higher when the consumption path is monotonic since agents do not benefit from the non-convexities of the production function. The conjunction of these two effects explain why it is important to bunch hard work when the degree of increasing returns to scale is huge.

When paths are monotonic, capital, consumption and labor lay below their steady state values forever. Here, this is no longer true: consumption and capital remain during several periods above their steady state values whereas labor remains below. Agents accumulate the capital stock during the first periods, which erodes gradually afterwards: they use this capital stock accumulated to conciliate a high level of consumption with a moderate level of labor. It is worth noting, however, that this happens only when eigenvalues are complex: non-monotonic paths when eigenvalues are real exhibit a decrease of consumption, capital and labor on the very first periods.

When increasing returns to scale are not sufficient, accumulating a large amount of capital requires to pay a stringent tribute in terms of disutility of labor that the increase in consumption cannot compensate. When the level of increasing returns is close to the minimum value to get indeterminacy, there is no level of capital stock such that the "bunching effect" dominates the "concavity effect": far from accelerating capital accumulation, agents are better-off when they smooth consumption and labor over time. As increasing returns become more and more important, the "bunching effect" is more likely to offset the "concavity effect": actually this configuration cannot happen if empirical values are given to the parameters. Although the welfare-maximizing path is generically monotonic, the welfare-maximizing value of initial consumption tends to diverge from c_{0,ξ_2}: this means that this path is all the less monotonic since the degree of externalities increases.

In any case, welfare deals more with the steady state value of welfare than to the monotonicity of the equilibrium paths. It is recalled that the instantaneous welfare $U(k(t), c(t)) = \hat{U}(k(t), c(t)) + U(c_s, k_s) + \mathcal{O}(||(c(t), k(t))||^3)$. As shown on the figures below, $\hat{U}$ is a negligible part of the intertemporal welfare.





This is an important observation for judging the desirability of the stabilization Guo and Lansing policy.

5 Stabilization policy

5.1 Guo and Lansing tax policy

We present here a slightly modified version of the Guo and Lansing stabilization policy, namely a countercyclical tax on production $\tau(t)$ (or equivalently a tax on both capital and labor incomes) which is redistributed in guise of a proportional lump sum transfer $T(t)$:

$$\begin{aligned}\tau(t) &= 1-\left(\frac{Y_s}{\bar{Y}(t)}\right)^{\phi} \qquad \text{with} \quad \phi>0 \\ T(t) &= \tau(t)\bar{Y}(t).\end{aligned}$$

In this case, each agent faces the same tax rate which increases with the national income. Finally, the law of motion of capital becomes:

$$\dot{K}(t) = [(1-\tau(t))r(t)-\delta]K(t) + (1-\tau(t))w(t)L(t) - C(t) + T(t).$$

It can be shown that the steady state does not vary from what we have previously computed and the new eigenvectors are such that:

$$V = [\xi_1 : \xi_2] = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} (1-\mu_2')\Psi-\delta & (1-\mu_2')\Psi-\delta \\ (1+\mu_1')\Psi-\delta-\lambda_1 & (1+\mu_1')\Psi-\delta-\lambda_2 \end{bmatrix},$$

where:

$$\begin{aligned}\mu_1' &= -\frac{\beta(1-\phi)+(1+\chi)(\alpha-1)}{\beta(1-\phi)-(1+\chi)} \\ \mu_2' &= -\frac{\beta}{\beta(1-\phi)-(1+\chi)}.\end{aligned}$$

Guo and Lansing [1998] have shown that the condition for determinacy is satisfied when $\beta(1-\phi)-1+\chi<0$. When β tends to $1-\chi$ a infinitely small ϕ can be chosen such that $\beta(1-\phi)$ tends to $1-\chi$. However, in the first case $\beta-1+\chi>0$ (the equilibrium is indeterminate) while in the second case $\beta(1-\phi)-1+\chi<0$ (the equilibrium is determined). Furthermore, when ϕ tends to zero, $\bar{\lambda}_2' = -\frac{(1-\alpha(1-\phi))[\rho+\delta(1-a)]}{\alpha}$ tends to $\bar{\lambda}_2$ and λ_1' tends to $-\infty$. Consequently, the unique equilibrium path tends to merge with the welfare-maximizing path starting with $c(0)=c_{0,\xi_2}$ when $\beta \sim 1-\chi$. In that case, Guo and Lansing policy is a good approximation of the second best policy in the Benhabib and Farmer model.

However, it is clear on the simulations above that the loss in welfare for an agent maintaining c_{0,ξ_2} as a starting level of consumption (which is the case of the Guo and Lansing policy) is increasing with the level of increasing returns. As this level goes up the "bunching effect" raises and theoretically may offset the "concavity effect": the welfare-maximizing path is less and less monotonic while the Guo and Lansing tax policy leads the agents to smooth their consumption/investment plans.

The difference of utility between the optimal path and the path starting with a level of consumption of c_{0,ξ_2} is maximum on the first period. Then, the gap tends to reduce but increases significantly when the welfare-maximizing paths become non-monotonic. Thus, the Guo and Lansing policy appears suboptimal (and all the more welfare-diminishing since the increasing returns are "low" or high): an alternative stabilization policy must be found.

5.2 Alternative economic policy

Actually, the second best equilibrium can be selected through a stabilization policy *à la* Saïdi [2005b] which is able to coordinate over time the agents on a given deterministic path.

Assume that the stationary equilibrium is indeterminate and that the government aims at coordinating the expectations on a deterministic indeterminate path characterized by the initial level of consumption and labor $(\bar{C}_0, \bar{L}_0)$. The expected rate of returns on capital is $\bar{r}_0 \equiv \alpha K_0^{a-1}\bar{L}_0^b$. The economic policy consists in subsidizing or taxing production such that the rate of returns on capital equals $\bar{r}_0$ by fixing a tax rate τ_0 (possibly negative) at the first period. Firms maximize their profit Π_0:

$$\Pi_0 = (1-\tau_0)Y_0 - r_0K_0 - w_0L_0,$$

with:

$$\tau_0 = 1-\bar{r}_0/r_0.$$

Since K_0 and $\bar{r}_0$ are predetermined, the equality of the after-tax rental rate of capital to the after-tax productivity of capital determines the quantity of labor at time 0:

$$L_0 = (\bar{r}_0/\alpha K_0^a)^{1/b} = \bar{L}_0.$$

Simultaneously, the couple (K_0, L_0) determines the equilibrium value of the first period after-tax real wage satisfying the second first order condition of profit maximization:

$$w_0 = (1-\tau_0)\frac{(1-\alpha)Y_0}{L_0}. \tag{14}$$

Finally, the first order condition (respect to labor) determines consumption at time 0, that is $\bar{C}_0$, which in turn determines the variation of capital and consumption at the next period via the law of motion of capital and the Euler equation. If the agents are rational, and we assume they are, they cannot switch to another equilibrium (which would be in contradiction with the FOCs) and are able to determine the triple (K_t, L_t, C_t) at any time t. Of course, expectations can be coordinated by fixing the real wage or the rental rate of any period, not especially the first one.

6 Conclusion

In this paper, we have shown that in a one-sector growth model with non-convex technology and productive externalities it is possible to rank the different equilibrium paths according to the initial value of consumption when the steady state is indeterminate. In the continuity of Christiano and Harrison's simulations, we have showed that welfare-improvement of stochastic sunspot equilibria is all the more powerful in the earlier periods of time since they condition the long run behavior of consumption and labor either by accelerating capital accumulation when the level of increasing returns is high (for a given elasticity of labor) or by decelerating the accumulation when it is low. Large fluctuations are then likely to be welfare-diminishing in the last case where the "concavity effect" dominates the "bunching effect". A direct implication of these findings is that progressive or countercyclical taxes able to pin down expectations as those developed by Guo and Lansing [1998] are more likely to be welfare-diminishing compared to any stochastic equilibrium when increasing returns are large since they smooth consumption and labor and decelerate capital accumulation.

Our analysis raises a question that deserve further investigations. Can we say something about the nature of the social planer's allocation? All the equilibria we considered are inefficient since the agents do not grasp the externality of production. In this case, the maximizing welfare deterministic equilibrium is more or less monotonic according to the aggregate level of increasing returns. Christiano and Harrison present an example of monotonic social planer's allocation while for different values of the externalities Dupor and Lenhert [2002] and Saïdi [2005b] show that this allocation is discontinuous and cycling. It can be conjectured that there is a close relationship between the monotonicity of the first best allocation and of the decentralized optimal solution.

A Appendix

A.1 Slopes of the stable arms

The Jacobian matrix of the system formed by equation (3) and (4) is:

$$J = \begin{pmatrix} (1+\mu_1)\,\Psi - \delta & (\mu_2 - 1)\,\Psi + \delta \\ a\mu_1\Psi & a\mu_2\Psi \end{pmatrix}$$

where $\Psi \equiv (\rho+\delta)/a$, $\mu_0 = \frac{-\beta \ln b}{\beta+\chi-1}$, $\mu_1 = \frac{(\chi-1)(\alpha-1)-\beta}{\beta+\chi-1}$ and $\mu_2 = \frac{\beta}{\beta+\chi-1}$. Let $\xi_i = (v_{1i}, v_{2i})^T$, $i = \{1,2\}$, the eigenvectors of the system defined such that:

$$\begin{pmatrix} (1+\mu_1)\Psi - \delta - \lambda_i & (\mu_2-1)\Psi + \delta \\ a\mu_1\Psi & a\mu_2\Psi - \lambda_i \end{pmatrix} \begin{pmatrix} v_{1i} \\ v_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

The slope of the stable arm associated to ξ_i at the stationary equilibrium is v_{2i}/v_{1i}. We want to show that:

$$\frac{v_{22}}{v_{12}} > \frac{v_{21}}{v_{11}} > 0,$$

or equivalently that the slope of the stable arm associated to ξ_2 is steeper than the slope of the stable arm associated to ξ_1 at the stationary equilibrium.

According to system (15), notice first that:

$$\frac{v_{2i}}{v_{1i}} = \frac{\lambda_i + \delta - (1+\mu_1)\Psi}{(\mu_2 - 1)\Psi + \delta}. \tag{16}$$

Moreover, when Benhabib and Farmer's condition for indeterminacy is satisfied, that is when $\beta - 1 + \chi > 0$, $\mu_2 - 1 > 0$ and $1 + \mu_1 < 0$. Since the trace is equal to the sum of the two eigenvalues, the following relation holds for any $i, j = \{1, 2\}$ with $i \neq j$:

$$\begin{aligned} sign\left\{\frac{v_{2i}}{v_{1i}}\right\} &= sign\left\{-(1+\mu_1)\Psi + \delta + \lambda_i\right\} \\ &= sign\left\{a\mu_2\Psi - Trace(J) + \lambda_i\right\} \\ &= sign\left\{a\mu_2\Psi - \lambda_j\right\}. \end{aligned}$$

Under Benhabib and Farmer's condition for indeterminacy, both $a\mu_2\Psi$ and $-\lambda_j$ are positive.

Finally since $\lambda_1 < \lambda_2$ it follows immediately from equation (16) that the slope of the stable arm associated to ξ_2, v_{22}/v_{12}, is steeper than the slope of the stable arm associated to ξ_1, v_{21}/v_{11}. If we assume to start with an initial stock of capital lower (resp. greater) than its steady state value, $\tilde{k}(0) < 0$ (resp. $\tilde{k}(0) > 0$) and from equations (9) and (10) it is easily deduced that $c_{0,\xi_1} > c_{0,\xi_2}$ (resp. $c_{0,\xi_1} < c_{0,\xi_2}$).

A.2 Monotonic paths

Monotonicity of consumption paths occurs provided the equation $d\tilde{c}(t)/dt = 0$ has no solution. This means that there is no $t \in \mathbb{R}^+$ such that for any (η_1, η_2):

$$\eta_1 \lambda_1 v_{21} e^{\lambda_1 t} + \eta_2 \lambda_2 v_{22} e^{\lambda_2 t} = 0,$$

that is:

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{v_{22}\tilde{k}(0) - v_{12}\tilde{c}(0)}{v_{21}\tilde{k}(0) - v_{11}\tilde{c}(0)} \frac{\lambda_1}{\lambda_2} \frac{v_{21}}{v_{22}}\right). \tag{17}$$

A solution exists if and only if $E \equiv \frac{v_{22}\tilde{k}(0) - v_{12}\tilde{c}(0)}{v_{21}\tilde{k}(0) - v_{11}\tilde{c}(0)} > 0$. Define: $c(0) \equiv c_{0,\xi_2} + \varepsilon$. In this case relation E becomes

$$E = \frac{-v_{12}^2 \varepsilon}{[v_{12}v_{21} - v_{11}v_{22}]\tilde{k}(0) - v_{11}v_{12}\varepsilon},$$

where $v_{ij} < 0$ for any $i, j = \{1, 2\}$ (as shown in appendix A.1).

If $\varepsilon > 0$, equation (17) as a solution if and only if $[v_{12}v_{21} - v_{11}v_{22}]\tilde{k}(0) - v_{11}v_{12}\varepsilon < 0$, that is for:

$$\begin{aligned} \varepsilon &> \left[\frac{v_{21}}{v_{11}} - \frac{v_{22}}{v_{12}}\right]\tilde{k}(0) \\ &> c_{0,\xi_1} - c_{0,\xi_2}, \end{aligned}$$

or equivalently for:

$$c(0) > c_{0,\xi_1}.$$

If $\varepsilon < 0$, equation (17) as a solution if and only if $[v_{12}v_{21} - v_{11}v_{22}]\tilde{k}(0) - v_{11}v_{12}\varepsilon > 0$, that is for:

$$\begin{aligned} \varepsilon &< \left[\frac{v_{22}}{v_{12}} - \frac{v_{21}}{v_{11}}\right]\tilde{k}(0) \\ &< c_{0,\xi_2} - c_{0,\xi_1}, \end{aligned}$$

or equivalently for:

$$c(0) < c_{0,\xi_2} - \varepsilon^*,$$

with $\varepsilon^* \equiv c_{0,\xi_1} - c_{0,\xi_2}$, which is positive (resp. negative) according to Appendix A.1 provided $K(0) < K_s$ (resp. $K(0) > K_s$).

Thus consumption paths have a monotonic behavior if and only if $c_0 \in [c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$ for $K(0) < K_s$ and $c_0 \in [c_{0,\xi_1}, c_{0,\xi_2} - \varepsilon^*]$ for $K(0) > K_s$.

A.3 Solution of some limits

The trace and determinant of the Jacobian matrix J are the following:

$$\begin{aligned} Det(J) &= (\rho+\delta)\frac{\rho+\delta(1-a)}{a}\frac{(1-\alpha)(1-\chi)}{\beta-1+\chi} \\ Tr(J) &= \frac{(\rho+\delta)(\beta-(1+\gamma_a)(1-\chi))}{\beta-1+\chi}-\delta \end{aligned}$$

When the condition for indeterminacy holds, one can see immediately that $Tr(J)$ tends to $-\infty$ and $Det(J)$ tends to $+\infty$ as $\beta-1+\chi$ tends to zero. Moreover the two limits have the same "order" of convergence. Now consider the following limits:

$$\lim_{\beta\to 1-\chi}\lambda_1=\lim_{\beta\to 1-\chi}\frac{Tr(J)-|Tr(J)|\sqrt{1-4\frac{Det(J)}{Tr(J)^2}}}{2}=\lim_{\beta\to 1-\chi}\frac{Tr(J)-|Tr(J)|}{2}=-\infty$$

Multiplying and dividing by $Tr(J)-|Tr(J)|\sqrt{1-4\frac{Det(J)}{Tr(J)^2}}$, we get:

$$\begin{aligned} \lim_{\beta\to 1-\chi}\lambda_2 &= \lim_{\beta\to 1-\chi}\frac{Tr(J)+|Tr(J)|\sqrt{1-4\frac{Det(J)}{Tr(J)^2}}}{2} \\ &= \lim_{\beta\to 1-\chi}\frac{2\det J}{trJ-|trJ|\sqrt{1-4\frac{\det J}{(trJ)^2}}} \\ &= \lim_{\beta\to 1-\chi}\frac{Det(J)}{Tr(J)}=-\frac{(1-\alpha)[\rho+\delta(1-a)]}{\alpha}. \end{aligned}$$

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