# Bubble-free interest-rate rules*

Olivier Loisel†

September 2007

**Abstract**: we design, for a broad class of rational-expectations dynamic stochastic general-equilibrium models, "bubble-free" interest-rate rules that not only ensure the local determinacy of the targeted equilibrium in the neighbourhood of the targeted steady state, but also prevent the economy from gradually leaving this neighbourhood. We show that these rules can still be effective when the perfect-information, rational-expectations and forever-commitment assumptions are slightly relaxed. The design and study of these rules also lead us to both generalize and qualify existing results on whether and to what extent an interest-rate rule should be forward-looking to ensure equilibrium determinacy.

**Keywords**: DSGE models, interest-rate rules, local determinacy, global determinacy.

**JEL codes**: E52, E61.

## Introduction

Today's most common practice to design monetary policy in a rational-expectations dynamic stochastic general-equilibrium (DSGE) model proceeds in two steps. First, the exogenous stochastic disturbances are assumed to be small enough for the targeted equilibrium to be found in the neighbourhood of the targeted steady state. Second, an interest-rate rule is chosen such that the system of equations linearised in this neighbourhood admits that equilibrium as its unique stationary solution. Such an interest-rate rule, whose locally linearised form is often a Taylor rule satisfying the Taylor principle, enables the central bank to preclude the kind of macroeconomic fluctuations that, according to Clarida, Galí and Gertler (2000) and Lubik and Schorfheide (2004), occurred in the U.S. before 1979. However, as first shown by Benhabib, Schmitt-Grohé and Uribe (2001a), these interest-rate rules can be consistent with equilibrium trajectories that originate from the neighbourhood of the targeted steady state and gradually leave this neighbourhood − for instance to fall eventually into the neighbourhood of another steady state interpreted as the liquidity trap, as arguably did the Japanese economy in the 1990s-2000s.

*This paper is a revised version of Loisel (2006). I am grateful in particular to Klaus Adam, Lawrence Christiano, Daniel Cohen, Harris Dellas, Jordi Galí, Marc Giannoni, Michel Juillard, Hubert Kempf, Guy Laroque, Philippe Martin and Pedro Teles for their comments at various stages of this long project. The usual disclaimer applies.

†Banque de France and CEPREMAP, Paris, France. Address: Banque de France, DGEI-DIR-POMONE-41-1422, 39 rue Croix-des-Petits-Champs, 75049 Paris Cédex 01, France. Phone number: +33 1 42 92 49 42. Fax number: +33 1 42 92 62 92. Email address: olivier.loisel@banque-france.fr. The views expressed in this paper are those of the author and should not be interpreted as reflecting those of the Banque de France.

This paper designs locally linearised interest-rate rules that preclude all the undesirable developments mentioned above by not only ensuring the local determinacy of the targeted equilibrium in the neighbourhood of the targeted steady state, but also preventing the economy from gradually leaving this neighbourhood. To that aim, we consider a broad class of rational-expectations dynamic stochastic linear systems of equations, meant to represent the locally linearised reduced form of rational-expectations DSGE models. Provided that the exogenous stochastic disturbances are small enough, a necessary condition for the economy to gradually leave the neighbourhood of a steady state is that the locally linearised system admit at least one unstable eigenvalue, *i.e.* one eigenvalue of modulus higher than or equal to one. By removing all unstable eigenvalues from the locally linearised system, the interest-rate rules put forward in this paper therefore prevent the economy from gradually leaving the neighbourhood of the steady state considered. They moreover manage to ensure the existence and uniqueness of a local equilibrium by removing all non-predetermined variables from the locally linearised system, thus making this system satisfy Blanchard and Kahn's (1980) conditions. We call them "bubble-free interest-rate rules" because, in the fictitious linear model corresponding to this locally linearised system, they would eliminate all mean-divergent rational bubbles of the type first identified by Blanchard (1979), unlike the interest-rate rules commonly considered in the literature.

Loosely speaking, bubble-free interest-rate rules manage to remove all non-predetermined variables from the locally linearised system by mimicking the locally linearised structural equations, *i.e.* the locally linearised system without the interest-rate rule, so as to disconnect current variables − other than the current interest rate − from the private agents' expectations of future variables. We point out that, as a consequence, under a certain condition (likely to be met by most DSGE models of the broad class considered), these interest-rate rules are forward-looking, *i.e.* make the current interest rate conditional on the private agents' current expectations of future variables. We moreover show that, for any given stationary solution of the locally linearised structural equations, there also exists a backward-looking interest-rate rule consistent with this solution, ensuring its local determinacy and preventing the economy from gradually leaving the neighbourhood of the targeted steady state. These two findings enable us to both generalize and qualify existing results on whether an interest-rate rule should be backward- or forward-looking to ensure equilibrium determinacy. The consideration of bubble-free interest-rate rules also enables us to contribute to the literature on how much forward-looking a forward-looking interest-rate rule should be in order to ensure equilibrium determinacy.

Since, loosely speaking, bubble-free interest-rate rules mimic the locally linearised structural equations, some of their coefficients are tied to the structural parameters by equality constraints, rather than by inequality constraints as is typically the case for the coefficients of interest-rate rules ensuring only local equilibrium determinacy. This naturally raises the issue of what happens when the central bank has imperfect knowledge of the structural parameters and accordingly follows an interest-rate rule close to, but not exactly coinciding with a bubble-free interest-rate rule. We show that such a rule still ensures local equilibrium determinacy and, by using the structural equations as a lever on private agents' expectations, still prevents the economy from gradually leaving the neighbourhood of the targeted steady state. We also show that bubble-free interest-rate rules can still be effective when the private agents form myopic rational expectations and when the central bank cannot credibly commit to forever following an interest-rate rule, while by contrast conventional interest-rate rules then prove even more problematic.

The remaining of the paper is organized as follows. Section 1 presents our general framework. Section 2 designs bubble-free interest-rate rules. Section 3 uses both the results and the methods of section 2 to examine whether and to what extent an interest-rate rule should be forward-looking to ensure equilibrium determinacy. Section 4 discusses the robustness of the results of section 2 to departures from various assumptions. We then conclude and provide a technical appendix.

# 1 A general locally linearisable model

This section presents our general framework.

## 1.1 Locally linearised system

We consider a rational-expectations DSGE model with one policy-maker and many private agents, whether infinitely-lived or in overlapping generations. This model has $N+1$ endogenous scalar variables, where $N \in \mathbb{N}^*$[1]. Only one of them, called the control variable or policy instrument, is directly controlled by the policy-maker. We make this restriction because we will consider only monetary policy applications in the paper and most central banks use the short-term nominal interest rate as their single monetary policy instrument. But this restriction is without any loss in generality since, in the case of several control variables, the policy-maker could always exogenize all but one.

The model admits at least one steady state. If there are several steady states, then we call "targeted steady state" the one that is preferred by the policy-maker, *e.g.* the social-welfare-maximizing steady state. If there is only one, then for simplicity we call it "targeted steady state" too. The model is linearisable in the neighbourhood of the targeted steady state. The reduced form of the model, linearised in this neighbourhood, is made of $N+1$ time-invariant linear equations that can be further decomposed into $N$ structural equations, which describe the private agents'

[1] In the following, we sometimes use for convenience notations that implicitly assume $N \geq 2$. In such cases, the reader should easily infer the notation rigorously adapted to the case $N = 1$.

behaviour, and one policy feedback rule. Time being discrete, indexed by $t \in \mathbb{Z}$, let $z_t$ denote the deviation of the control variable at date $t$ from its value at the targeted steady state, $\mathbf{Y}_t$ the $N$-dimension vector made of the deviations of the non-control variables at date $t$ from their values at the targeted steady state, $L$ the lag operator, $\boldsymbol{\xi}_t$ a $N$-dimension vector of exogenous shocks and $E_t\{.\}$ the rational-expectations operator conditionally on $\{\mathbf{Y}_{t-k}, z_{t-k}\}_{k\geq 1}$ and $\left\{\boldsymbol{\xi}_{t-k}\right\}_{k\geq 0}$.

The $N$ structural equations are written as follows[2]:

$$E_t\left\{\mathbf{A}(L)\mathbf{Y}_t + \mathbf{B}(L)z_t\right\} + \mathbf{C}(L)\boldsymbol{\xi}_t = \mathbf{0} \tag{1}$$

$$\text{with } \underset{(N\times N)}{\mathbf{A}(L)} \equiv \sum\nolimits_{k=-m^a}^{n^a} \mathbf{A}_k L^k,\ \underset{(N\times 1)}{\mathbf{B}(L)} \equiv \sum\nolimits_{k=-m^b}^{n^b} \mathbf{B}_k L^k \text{ and } \underset{(N\times N)}{\mathbf{C}(L)} \equiv \sum\nolimits_{k=0}^{n^c} \mathbf{C}_k L^k,$$

where $\left(m^a, m^b, n^a, n^b, n^c\right) \in \mathbb{N}^5$, all $\mathbf{A}_k$, $\mathbf{B}_k$ and $\mathbf{C}_k$ have real numbers as elements and all the eigenvalues of $\mathbf{C}(L)$ are of modulus strictly lower than one. Each exogenous shock is assumed to follow a centered stationary autoregressive process of finite order[3]:

$$\mathbf{D}(L)\boldsymbol{\xi}_t = \boldsymbol{\varepsilon}_t \text{ with } \underset{(N\times N)}{\mathbf{D}(L)} \equiv \begin{bmatrix} D_1(L) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & D_N(L) \end{bmatrix}$$

and $D_i(L) \equiv \sum_{k=0}^{n^d} d_{i,k}L^k$ for $1 \leq i \leq N$, where $n^d \in \mathbb{N}$, all $d_{i,k}$ are real numbers, $\mathbf{D}(0)$ is invertible, all the eigenvalues of $\mathbf{D}(L)$ are of modulus strictly lower than one and $\boldsymbol{\varepsilon}_t$ is a $N$-dimension white noise vector that has a bounded probability distribution.

Let us call "fundamental shocks" the exogenous stochastic disturbances that feature in the model's equilibrium conditions and "sunspot shock" any exogenous stochastic process that is independent from all fundamental shocks. We define an equilibrium of the model as a sequence for the endogenous variables, conditional on current and past fundamental and/or sunspot shocks, that satisfies all the model's equilibrium conditions. Importantly, we assume that (1) is a valid first-order approximation of the structural equations at date $t$ provided that, at the equilibrium considered, all $\mathbf{Y}_{t-k}$ and $z_{t-k}$ for $k \in \left\{1, ..., \max\left(n^a, n^b\right)\right\}$ and all possible realizations of $\mathbf{Y}_{t+k}$ and $z_{t+k}$ for $k \in \left\{0, ..., \max\left(m^a, m^b\right)\right\}$ are close enough to zero, even when there exist some realizations of $\mathbf{Y}_{t+k}$ and $z_{t+k}$ for $k > \max\left(m^a, m^b\right)$ that substantially differ from zero.

Finally, we consider the set of policy feedback rules whose locally linearised form can be written as follows:

$$E_t\left\{\mathbf{F}(L)\mathbf{Y}_t\right\} + G(L)z_t + \mathbf{H}(L)\boldsymbol{\xi}_t = \mathbf{0} \tag{2}$$

[2] The focus of the paper, namely the design and the study of policy feedback rules in a general framework, forbids us to start from the commonly used expression with only one lag and one lead for the locally linearised system.

[3] This assumption is not restrictive in the sense that if each element of $\boldsymbol{\xi}_t$ followed instead a centered stationary finite-order ARMA process, then $\mathbf{C}(L)\boldsymbol{\xi}_t$ could easily be rewritten in the form $\mathbf{C}^*(L)\boldsymbol{\xi}_t^*$ with $\mathbf{C}^*(L) \equiv \sum_{k=0}^{n^{c*}} \mathbf{C}_k^* L^k$, where $n^{c*} \in \mathbb{N}$, all $\mathbf{C}_k^*$ are $N\times N$ matrices with real numbers as elements, all the eigenvalues of $\mathbf{C}^*(L)$ are of modulus strictly lower than one and each element of $\boldsymbol{\xi}_t^*$ follows a centered stationary autoregressive process of finite order.

$$\text{with } \underset{(1\times N)}{\mathbf{F}(L)} \equiv \sum\nolimits_{k=-m^f}^{n^f} \mathbf{F}_k L^k,\ G(L) \equiv \sum\nolimits_{k=0}^{n^g} g_k L^k \text{ and } \underset{(1\times N)}{\mathbf{H}(L)} \equiv \sum\nolimits_{k=0}^{n^h} \mathbf{H}_k L^k,$$

where $\left(m^f, n^f, n^g, n^h\right) \in \mathbb{N}^4$, all $g_k$ are real numbers, $g_0 \neq 0$ and all $\mathbf{F}_k$, $\mathbf{H}_k$ have real numbers as elements. Such rules qualify as "instrument rules" since their $z_t$-coefficient $g_0$ is non-zero. We assume throughout the paper, except in subsection 4.3, that the policy-maker can credibly commit to forever following a policy feedback rule whose locally linearised form is of type (2).

## 1.2 Three additional assumptions

This subsection presents three additional assumptions that we make about the $N$ structural equations (1). To that aim, let $\mathbf{e}_i$ denote, for all $i \in \{1, ..., N\}$, the $N$-element vector whose $i^{th}$ element is equal to one and whose other elements are equal to zero. Let $I_A$ denote the set of $i \in \{1, ..., N\}$ such that $\mathbf{e}_i'\mathbf{A}(L) \neq \mathbf{0}$ and $I_B$ the set of $i \in \{1, ..., N\}$ such that $\mathbf{e}_i'\mathbf{B}(L) \neq 0$. Let $m_i^a \equiv -\min\left[k \in \{-m^a, ..., n^a\}, \mathbf{e}_i'\mathbf{A}_k \neq \mathbf{0}\right]$ for $i \in I_A$ and $m_i^b \equiv -\min\left[k \in \left\{-m^b, ..., n^b\right\}, \mathbf{e}_i'\mathbf{B}_k \neq 0\right]$ for $i \in I_B$. Lastly, let us note, when $I_A = \{1, ..., N\}$,

$$\widehat{\mathbf{A}}(L) \equiv \begin{bmatrix} \mathbf{e}_1' L^{m_1^a} \\ \vdots \\ \mathbf{e}_N' L^{m_N^a} \end{bmatrix} \mathbf{A}(L).$$

We make the following two assumptions[4]:

**Assumption 1**: *(i)* $I_A = \{1, ..., N\}$*; (ii)* $\forall i \in \{1, ..., N\}$, $m_i^a \geq 0$*; (iii)* $\widehat{\mathbf{A}}(0)$ *is invertible.*

**Assumption 2**: *(i)* $1 \in I_B$*; (ii)* $m_1^b \geq 0$*; (iii)* $\forall i \in I_B \setminus \{1\}$, $m_i^a - m_i^b > \max\left[0, m_1^a - m_1^b\right]$.

Moreover, let $\Delta_i(X) \in \mathbb{R}[X]$ for $i \in \{1, ..., N+1\}$ denote the determinant of the $N \times N$ matrix obtained by removing its $i^{th}$ column from $N \times (N+1)$ matrix $X^{\max\left[n^a, n^b\right]}\left[\ \mathbf{A}\left(X^{-1}\right) \mid \mathbf{B}\left(X^{-1}\right)\ \right]$, and let $\mathcal{D}(X) \in \mathbb{R}[X]$ denote the greatest common divisor, defined up to a non-zero multiplicative scalar, of all non-zero $\Delta_i(X)$ for $i \in \{1, ..., N+1\}$. We make the following assumption:

**Assumption 3**: *all roots of* $\mathcal{D}(X)$ *have their modulus strictly lower than one.*

To our knowledge, rational-expectations DSGE models whose locally linearised structural equations are of type (1) typically satisfy assumptions 1 and 3. We view assumption 2 as slightly more restrictive. However, the specification made of (1) together with assumptions 1, 2 and 3 is general enough to encompass the suitably rewritten locally linearised structural equations of many rational-expectations DSGE models, and in particular of all those models whose locally linearised

[4] All the propositions of the paper would still hold if assumption 2.iii were replaced by $\forall i \in I_B \setminus \{1\}$, $m_i^a - m_i^b > m_1^a - m_1^b$: we make the assumption $\forall i \in I_B \setminus \{1\}$, $m_i^a - m_i^b > 0$ for convenience only, to keep the proofs of these propositions simple.

structural equations are written in a form of type (1) satisfying assumption 1 and 3 and featuring the short-term nominal interest rate only as a current variable (*i.e.* such that $m^b = n^b = 0$), for instance in the Euler equation, the Tobin's Q equation or the uncovered interest-rate parity. Indeed, the locally linearised structural equations of these models can easily be rewritten in an equivalent form of type (1) satisfying assumptions 1, 2 and 3, by re-ordering these equations so that $1 \in I_B$ and $\forall i \in I_B \setminus \{1\}$, $m_i^a \geq m_1^a$ and by replacing $\mathbf{e}_i'(1)$ by $\mathbf{e}_i'\mathbf{B}_0\mathbf{e}_1'(1) - \mathbf{e}_1'\mathbf{B}_0\mathbf{e}_i'(1)$ for all those $i \in I_B \setminus \{1\}$ such that $m_i^a = m_1^a$, where, for any $N$-equation system $(S)$, $\mathbf{e}_i'(S)$ denotes the $i^{th}$ equation of $(S)$.

### 1.3 Targeted and non-targeted equilibria

In most rational-expectations DSGE models, the targeted equilibrium (*e.g.* the globally-social-welfare-maximizing equilibrium) does not depend on sunspot shocks, and the fundamental shocks are assumed to be small enough for this equilibrium to be found in the neighbourhood of the targeted steady state. As a consequence, the targeted equilibrium can typically be locally linearised in a stationary VARMA form that does not involve white noises other than those featuring in the structural equations. We accordingly assume that the policy-maker seeks to implement a given sequence $\{\mathbf{Y}_t, z_t\}_{t\in\mathbb{Z}}$ that satisfies (1) for $t \in \mathbb{Z}$ and can be written in the form

$$\begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} = \mathbf{S}(L) \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} + \mathbf{T}(L)\,\boldsymbol{\varepsilon}_t \tag{3}$$

$$\text{with} \underset{((N+1)\times(N+1))}{\mathbf{S}(L)} \equiv \sum\nolimits_{k=1}^{n^s} \mathbf{S}_k L^k \text{ and } \underset{((N+1)\times N)}{\mathbf{T}(L)} \equiv \sum\nolimits_{k=0}^{n^t} \mathbf{T}_k L^k$$

where $n^s \in N^*$, $n^t \in N$, all $\mathbf{S}_k$ and $\mathbf{T}_k$ have real numbers as elements and all eigenvalues of the systems $\mathbf{I} - \mathbf{S}(L)$ and $\mathbf{T}(L)$ are of modulus strictly lower than one, where $\mathbf{I}$ denotes the $(N+1)\times(N+1)$ identity matrix.

Let $\tau \equiv \max\left[m_1^a, m_1^b\right] + \sum_{i=2}^N m_i^a$. Let $\mathfrak{N}$ denote the neighbourhood of the targeted steady state within which (1) is deemed an acceptable approximation of the model's structural equations, and $\mathfrak{n}$ a given open neighbourhood of the targeted steady state such that: i) $\mathfrak{n} \subset \mathfrak{N}$; ii) whatever the realization of the exogenous shocks, the endogenous variables constantly remain in $\mathfrak{n}$ at the targeted equilibrium. The global interest-rate rule chosen should ideally not only be consistent with the targeted local equilibrium presented above, but also eliminate all other equilibria. In this paper, we focus on the following two kinds of non-targeted equilibrium:

**Definition 1 (type-A equilibria)**: *an equilibrium of the model is said to be of type A if: i) whatever the realization of the exogenous shocks, the endogenous variables constantly remain in* $\mathfrak{N}$ *at this equilibrium; and ii) this equilibrium differs from the targeted equilibrium.*

Type-A equilibria correspond to the non-targeted local equilibria. As is well-known, they may exist only if the locally linearised system admits more stable eigenvalues (*i.e.* eigenvalues of modulus strictly lower than one) than required by Blanchard and Kahn's (1980) conditions.

**Definition 2 (type-B equilibria)**: *an equilibrium of the model is said to be of type B if there exists $t \in \mathbb{Z}$ such that, at this equilibrium: i) whatever the realization of the exogenous shocks, the endogenous variables remain in $\mathfrak{n}$ up to date $t-1$; ii) the endogenous variables are in $\mathfrak{N} \setminus \mathfrak{n}$ at date t; iii) whatever the realization of the exogenous shocks, the endogenous variables are in $\mathfrak{N}$ from date t to date $t+\tau$; and iv) the endogenous variables are outside $\mathfrak{N}$ at some date later than $t+\tau$.*

Type-B equilibria correspond to the equilibria that originate from the neighbourhood of the targeted steady state and gradually leave this neighbourhood. The possibility of their existence has been shown, among others, by Benhabib and Eusepi (2005), Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b, 2002a, 2002b, 2003) and Christiano and Rostagno (2001). In these frameworks, type-B equilibria take the form of equilibrium trajectories originating arbitrarily close to the targeted steady state and gradually leaving its neighbourhood to eventually converge towards a deterministic cycle, a chaotic cycle or a non-targeted steady state interpreted as the liquidity trap. In Benhabib, Schmitt-Grohé and Uribe's (2001a) framework, notably, they exist for empirically plausible parameterizations and are robust to wide parameter perturbations.

Importantly, provided that the exogenous shocks are small enough, a necessary condition for type-B equilibria to exist is that the locally linearised system admit at least one unstable eigenvalue, *i.e.* one eigenvalue of modulus higher than or equal to one (otherwise all equilibria originating locally would remain local). Note, interestingly, that Benhabib, Schmitt-Grohé and Uribe (2001a, 2002a, 2002b, 2003) provide one reason to suspect that type-B equilibria exist in many frameworks satisfying this condition. Indeed, they point out that when the interest-rate rule respects the zero nominal interest-rate lower bound and makes the interest rate react positively and, at the targeted steady state, more than one-to-one to the inflation rate, there typically exist equilibrium trajectories originating arbitrarily close to the targeted steady state and gradually leaving its neighbourhood to eventually converge towards a second steady state at which the inflation rate is lower than its targeted value and the interest rate reacts less than one-to-one to the inflation rate.

Naturally, type-B equilibria do not constitute the only possible kind of non-local equilibria. In particular, equibria may exist that abruptly leave the neighbourhood of the targeted steady state, as opposed to gradually, or that may not even originate from this neighbourhood. However, these other equilibria seem to us less relevant than type-B equilibria, for essentially the same reason as the one put forward by Benhabib, Schmitt-Grohé and Uribe (2001a, p. 43): "in major industrialized countries [...] observed inflation dynamics are in general quite smooth, giving little credence to a model in which movements in inflation at business-cycle frequency are due to jumps from one steady state to another".

# 2 Design of bubble-free interest-rate rules

This section designs locally linearised interest-rate rules of type (2) that not only are consistent with the targeted local equilibrium and eliminate type-A equilibria, but also eliminate type-B equilibria as well. We call them "bubble-free interest-rate rules" because, in the fictitious linear model corresponding to our locally linearised system, they would eliminate all mean-divergent rational bubbles of the type first identified by Blanchard (1979), unlike the interest-rate rules commonly considered in the literature.

## 2.1 Bubble-free policy feedback rules

Let us adopt the convention $\sum_{i=u}^{v}\{.\}=0$ for $u>v$. Consider the policy feedback rules of the following form:

$$\begin{aligned} & E_t\left\{L^{m_1^b}\mathbf{e}_1'\mathbf{A}(L)\mathbf{Y}_t\right\}+L^{m_1^b}\mathbf{e}_1'\mathbf{B}(L)z_t+L^{m_1^b}\mathbf{e}_1'\mathbf{C}(L)\boldsymbol{\xi}_t+\mathbf{e}_1'\mathbf{O}(L)\mathbf{D}(L)\boldsymbol{\xi}_t \\ & +\sum_{i=2}^{N}L^{m_1^b+\sum_{j=2}^{i}m_j^a}\left[\mathbf{e}_i'\mathbf{A}(L)\mathbf{Y}_t+\mathbf{e}_i'\mathbf{B}(L)z_t+\mathbf{e}_i'\mathbf{C}(L)\boldsymbol{\xi}_t+L^{-m_i^a}\mathbf{e}_i'\mathbf{O}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right] \\ & +L^{m_1^b+\sum_{i=2}^{N}m_i^a}\left[\mathbf{P}(L)\mathbf{Y}_t+Q(L)z_t+\mathbf{R}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right]=\mathbf{0}, \end{aligned} \tag{4}$$

where $\mathbf{O}(L)$, $\mathbf{P}(L)$, $Q(L)$ and $\mathbf{R}(L)$ satisfy the following conditions:

**Condition 1**: $\underset{(N\times N)}{\mathbf{O}(L)}\equiv\begin{bmatrix}\sum_{k=0}^{m_1^b-1}\mathbf{O}_{1,k}L^k \\ \sum_{k=0}^{m_2^a-1}\mathbf{O}_{2,k}L^k \\ \vdots \\ \sum_{k=0}^{m_N^a-1}\mathbf{O}_{N,k}L^k\end{bmatrix}$, *where all* $\mathbf{O}_{i,k}$ *have real numbers as elements.*

**Condition 2**: $\underset{(1\times N)}{\mathbf{P}(L)}\equiv\sum_{k=0}^{n^p}\mathbf{P}_kL^k$, *where* $n^p\in\mathbb{N}$ *and all* $\mathbf{P}_k$ *have real numbers as elements, and*

$$\underset{(N\times N)}{\boldsymbol{\Omega}}\equiv\begin{bmatrix}\mathbf{P}_0 \\ \mathbf{e}_2'\widehat{\mathbf{A}}(0) \\ \vdots \\ \mathbf{e}_N'\widehat{\mathbf{A}}(0)\end{bmatrix}$$ *is invertible.*

**Condition 3**: $Q(L)\equiv\sum_{k=\max\left[0,m_1^a-m_1^b+1\right]}^{n^q}q_kL^k$, *where* $n^q\in\mathbb{N}$ *and all* $q_k$ *are real numbers.*

**Condition 4**: *the system*

$$\begin{bmatrix}\mathbf{A}(L) & \mathbf{B}(L) \\ \mathbf{P}(L) & Q(L)\end{bmatrix} \tag{5}$$

*has all its eigenvalues of modulus strictly lower than one.*

**Condition 5**: $\underset{(1\times N)}{\mathbf{R}}(L) \equiv \sum_{k=0}^{n^r} \mathbf{R}_k L^k$, *where* $n^r \in \mathbb{N}$ *and all* $\mathbf{R}_k$ *have real numbers as elements.*

Rules of type (4) satisfying conditions 1 to 5 belong to the class of rules (2), in particular because their $z_t$-coefficient $\mathbf{e}_1'\mathbf{B}_{-m_1^b}$ is non-zero. Two features of these rules, which may seem unpleasant at first sight, are worth acknowledging at this stage: first, their use requires perfect knowledge of the structural parameters, since some of their coefficients are linked to these parameters by equality constraints; second, they rapidly become more and more complex as the number of forward-looking structural equations increases. These two points are addressed in subsections 4.1 and 4.2 of the paper.

For any system of equations $(S)$, let $L(S)$ denote the system obtained by applying operator $L$ on both the left- and the right-hand sides of each equation of $(S)$. We first show that the fictitious linear model corresponding to our locally linearised system made of (1) and any rule of type (4) satisfying conditions 1 to 5 admits a unique solution and that this solution is non-explosive:

**Proposition 1 (determinacy)**: *whatever* $t \in \mathbb{Z}$ *and* $\mathbf{O}(L)$, $\mathbf{P}(L)$, $Q(L)$, $\mathbf{R}(L)$ *satisfying conditions 1 to 5, the system made of* $L^k(1)$ *and* $L^k(4)$ *for all* $k \geq -\tau$ *admits a unique solution* $\{\mathbf{Y}_{t-j}, z_{t-j}\}_{j\in\mathbb{N}}$ *and this solution is stationary.*

**Proof**: *cf* appendix A. ■

As made clear in appendix A, rules of type (4) satisfying conditions 1 to 5 achieve the existence and uniqueness of the solution $\{\mathbf{Y}_t, z_t\}$ by insulating $\mathbf{Y}_t$ from $E_t\{\mathbf{Y}_{t+k}\}$ and $E_t\{z_{t+k}\}$ for $k \geq 1$, thus pinning down $\mathbf{Y}_t$ uniquely, and by making $z_t$ uniquely recoverable from $\mathbf{Y}_t$. Since the structural equations may express $\mathbf{Y}_t$ as a function of $E_t\{\mathbf{Y}_{t+k}\}$ and $E_t\{z_{t+k}\}$ for $k \geq 1$, these rules are designed to "mimic" the structural equations in such a way that, when combined with these structural equations, they disconnect $\mathbf{Y}_t$ from $E_t\{\mathbf{Y}_{t+k}\}$ and $E_t\{z_{t+k}\}$ for $k \geq 1$. More precisely, the expectation at date $t$ of one of these rules taken at date $t + m_1^b$ has the same forward-looking part as the first structural equation, so that subtracting one from the other leads to a backward-looking equation; similarly, the expectation at date $t$ of this backward-looking equation taken at date $t + m_2^a$ has the same forward-looking part as the second structural equation, and so on. This explains why the time needed by these rules to be effective, equal to $\tau$ periods, is a function of the length of the forward-looking part of the structural equations. Since the system made of $L^k(1)$ and $L^k(4)$ for all $k \geq -\tau$ would be a valid first-order approximation of the model's equilibrium conditions along a candidate type-B equilibrium trajectory, proposition 1 implies that rules of type (4) satisfying conditions 1 to 5 not only ensure local equilibrium determinacy, but also eliminate all type-B equilibria.

We then show that whatever targeted local equilibrium of type (3) satisfying (1) can be implemented by a suitably chosen policy feedback rule of type (4) satisfying conditions 1 to 5:

**Proposition 2 (controllability)**: *for any sequence* $\{\mathbf{Y}_t, z_t\}_{t\in\mathbb{Z}}$ *of type (3) that satisfies (1) for all* $t \in \mathbb{Z}$, *there exist* $\mathbf{O}(L)$, $\mathbf{P}(L)$, $Q(L)$ *and* $\mathbf{R}(L)$ *satisfying conditions 1 to 5 and such that* $\{\mathbf{Y}_t, z_t\}_{t\in\mathbb{Z}}$ *is the unique solution of the system made of (1) and (4) for all* $t \in \mathbb{Z}$.

**Proof**: *cf* appendix B. ■

Technically speaking, after choosing $\mathbf{O}(L)$ satisfying condition 1, appendix B uses the generalized identity of Bezout and the Euclidian division to choose some $\mathbf{P}(L)$ and $Q(L)$ that satisfy conditions 2, 3 and 4 and are such that the eigenvalues of $\mathbf{I} - \mathbf{S}(L)$ are also eigenvalues of the system made of (1) and (4), and Cramer's rule to residually choose an $\mathbf{R}(L)$ that satisfies condition 5 and is such that the unique solution of this system coincides with the targeted stationary VARMA process (3). Hence, propositions 1 and 2 together imply that rules of type (4) satisfying conditions 1 to 5 not only are consistent with the targeted local equilibrium, but also eliminate all type-A and -B equilibria.

## 2.2 A New Keynesian illustration

Our assumption that the linearization of the model's equilibrium conditions in the neighbourhood of the targeted steady state is valid even when endogenous variables are expected to leave this neighbourhood at some sufficiently distant point in the future forbids us to consider models with a Calvo-type price-setting mechanism, such as the New Keynesian model[5]. Nevertheless, we feel that the best way to illustrate propositions 1 and 2 is to consider this well-known model and proceed as if it satisfied this assumption.

Let us therefore consider the New Keynesian model, for simplicity in its deterministic version, and assume that the targeted steady state is the globally social-welfare-maximizing steady state. The structural equations linearised in the neighbourhood of this steady state are then of type (1), with $N = 2$, $m_1^a = m_2^a = 1$ and $m_1^b = 0$, and satisfy assumptions 1, 2 and 3. They consist of an IS equation and a Phillips curve whose reduced forms are respectively written:

$$x_t = E_t\{x_{t+1}\} - \sigma(i_t - E_t\{\pi_{t+1}\}), \tag{6}$$

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa x_t, \tag{7}$$

where $x_t$, $\pi_t$ and $i_t$ respectively denote the deviations at date $t$ of the output gap, the inflation rate and the short-term nominal interest rate from their values at the targeted steady state, while

[5] In such models, although they may well not be able to eliminate type-B equilibria, bubble-free interest-rate rules can still be useful, as argued in subsection 4.3, by eliminating another kind of non-targeted equilibria called "type-C equilibria".

$\beta$, $\kappa$ and $\sigma$ are three parameters such that $0 < \beta < 1$, $\kappa > 0$ and $\sigma > 0$. The targeted equilibrium, assumed to be the globally social-welfare-maximizing equilibrium, here coincides with the targeted steady state, *i.e.* corresponds to $\pi_t = x_t = i_t = 0$ for $t \in \mathbb{Z}$.

Suppose for a moment that the central bank sets the short-term nominal interest rate according to a contemporaneous Taylor rule

$$i_t = \phi_\pi \pi_t + \phi_x x_t \tag{8}$$

with $(\phi_\pi, \phi_x) \in \mathbb{R}^2$. The locally linearised system is then made of (6), (7) and (8). As can be easily seen by putting it into Blanchard and Kahn's (1980) form, this system has two non-predetermined variables and two eigenvalues whatever the value taken by $(\phi_\pi, \phi_x) \in \mathbb{R}^2$. As a consequence, if $(\phi_\pi, \phi_x)$ is chosen so that these two eigenvalues are unstable, then Blanchard and Kahn's (1980) conditions are satisfied, *i.e.* type-A equilibria are eliminated, but type-B equilibria may exist[6]. Alternatively, if $(\phi_\pi, \phi_x)$ is chosen so that these two eigenvalues are stable, then the economy jumps out of the frying pan into the fire as type-B equilibria can no longer exist but type-A equilibria do. In other words, contemporaneous Taylor rules do not enable the central bank to have the cake and eat it. This result naturally holds for any interest-rate rule that is not designed to control the number of non-predetermined variables of the locally linearised system and explains why Benhabib, Schmitt-Grohé and Uribe (2001a, 2002a, 2002b, 2003) find that type-B equilibria exist precisely when type-A equilibria are eliminated by a locally "active" interest-rate rule.

By contrast, bubble-free interest-rate rules manage to eliminate both type-A and -B equilibria by removing all non-predetermined variables and all unstable eigenvalues from the locally linearised system. Such is the case, for instance, of the following kind of interest-rate rules:

$$i_t = E_t \{\pi_{t+1}\} + \psi \pi_t + \frac{1}{\sigma} E_t \{\Delta x_{t+1}\} \tag{9}$$

where $\Delta$ denotes the first-difference operator and $\psi \in \mathbb{R}^*$, that is to say that, as an illustration of proposition 1, the system made of (6), (7) and (9) taken at dates $t$ to $t+2$ pins down $(\pi_t, x_t, i_t)$ uniquely. Indeed, the replacement of $i_t$ in (6) by the right-hand side of (9) leads to $\pi_t = 0$, as the terms in $E_t \{\pi_{t+1}\}$, $E_t \{x_{t+1}\}$ and $x_t$ cancel each other out; the same reasoning conducted one period ahead implies $E_t \{\pi_{t+1}\} = 0$; the replacement of $E_t \{\pi_{t+1}\}$ and $\pi_t$ in (7) by 0 then leads to $x_t = 0$; the same reasoning conducted one period ahead implies $E_t \{x_{t+1}\} = 0$; finally, (6) or (9) then leads to $i_t = 0$[7].

[6] Under Woodford's (2003) standard calibration ($\beta = 0.99$, $\kappa = 0.024$ and $\sigma = 6.25$), when both eigenvalues of the system are of modulus higher than one, the lowest modulus of these eigenvalues is typically strikingly close to one. Indeed, this modulus is equal to 1.03 when $(\phi_\pi, \phi_x) = (1.5, 0.5)$, as in Taylor's (1993) original formulation, and does not exceed 1.39 for $(\phi_\pi, \phi_x) \in [0; 2]^2$. This result suggests that equilibria that leave the neighbourhood of the targeted steady state, if they exist, can do so gradually and therefore correspond to type-B equilibria.

[7] The simplicity of this proof is partly due to the fact that the New Keynesian Phillips curve makes $x_t$ directly recoverable from $\pi_t$ and $E_t \{\pi_{t+1}\}$. However, as should be clear from the previous section, such a property is by no means necessary for the existence of bubble-free interest-rate rules. Indeed, bubble-free interest-rate rules would still

It is worth noting that, by making $i_t$ react to $E_t\{\pi_{t+1}\}$ with a coefficient unity and to $\pi_t$ with an arbitrary non-zero coefficient, rules (9) do not necessarily satisfy the so-called Taylor principle, which makes $i_t$ react strictly more than one-to-one to the current or the expected future inflation rate. This result is due to the fact that, in the standard New Keynesian model considered, the Taylor principle is a necessary condition to eliminate type-A equilibria for specific parametric families of interest-rate rules, for instance rules of type $i_t = \alpha\pi_t$ or $i_t = \alpha E_t\{\pi_{t+1}\}$ that make the locally linearised system have two non-predetermined variables whatever $\alpha \in \mathbb{R}$; but it ceases to be one for slightly more general parametric families of interest-rate rules, as shown *e.g.* by Woodford (2003, chap. 4), even though these rules do not affect the number of non-predetermined variables of the locally linearised system; and it is definitely not one for parametric families of interest-rate rules general enough to include rules (9) that remove all non-predetermined variables from the locally linearised system.

## 2.3 Related literature

This subsection briefly reviews the literature on how monetary policy can eliminate type-B equilibria and positions our paper within this literature.

The literature has mostly proposed two-tier monetary policies to eliminate type-B equilibria, in the spirit of Obstfeld and Rogoff's (1983) fractional-backing proposal to rule out speculative hyperinflations. These two-tier monetary policies, advocated most notably by Benhabib, Schmitt-Grohé and Uribe (2002a, 2002b, 2003), Christiano and Rostagno (2001) and Woodford (2003, chap. 2), consist in switching from an interest-rate rule eliminating type-A equilibria to another rule such as a money growth rate peg (possibly accompanied by a non-Ricardian fiscal policy) when the endogenous variables go outside a specified neighbourhood of the targeted steady state. However, as argued by Green (2005) and Cochrane (2006), one drawback of these two-tier policies is that their credibility, and consequently their effectiveness in eliminating type-B equilibria, cannot be taken for granted, in particular because they are typically aggressive out of equilibrium when the endogenous variables go off track. Given that they are immune from this drawback, since they act while the endogenous variables are still in the neighbourhood of the targeted steady state, bubble-free interest-rate rules represent a particularly interesting alternative or complement to these two-tier policies.

To our knowledge, only four papers make other monetary policy proposals enabling the central bank to eliminate type-B equilibria. First, Currie and Levine (1993, chap. 4) design "overstable feedback rules" that remove all unstable eigenvalues from linear systems without affecting the

exist if, for instance, a term $\alpha E_t\{x_{t+1}\}$ were artificially added to the right-hand side of (7), leaving (6) unchanged, provided that $\alpha \neq \frac{\beta}{\sigma}$ so as to satisfy assumption 1.iii.

number of non-predetermined variables. Applied to locally linearised systems, these rules would eliminate type-B equilibria but would fail to eliminate type-A equilibria. Second, Antinolfi, Azariadis and Bullard (2006) propose, in a particular framework, interest-rate rules that similarly eliminate type-B equilibria but fail to eliminate type-A equilibria. Third, Benhabib, Schmitt-Grohé and Uribe (2003) propose, in a specific model, an interest-rate rule which they show eliminates all type-A equilibria and some (but perhaps not all) type-B equilibria. Fourth, Adão, Correia and Teles (2005) design monetary policy rules ensuring global equilibrium determinacy in a simple non-linear cash-in-advance model. Although these rules are global and non-linear, and although their working mechanism is presented in a very different way (with emphasis laid on finite *vs.* infinite model horizon), this working mechanism could be considered similar to that of our bubble-free interest-rate rules in a particular, simple case[8].

# 3 Forward- and backward-looking interest-rate rules

This section uses both the results and the methods of the previous section to shed some new light on the answers to the following two questions: should the interest-rate rule be forward- or backward-looking in order to ensure equilibrium determinacy? And how much forward-looking should a forward-looking interest-rate rule be in order to ensure equilibrium determinacy?

## 3.1 Forward-looking *vs.* backward-looking rules

This subsection deals more specifically with the question whether the interest-rate rule should be forward- or backward-looking to ensure equilibrium determinacy. Let us first define the concepts of forward-looking and backward-looking rules:

**Definition 3 (forward- and backward-looking rules)**: *a policy feedback rule of type (2) is said to be forward-looking when* $m^f \geq 1$ *and* $\exists k \in \{1, ..., m^f\}$, $\mathbf{F}_k \neq \mathbf{0}$, *and backward-looking otherwise.*

On the one hand, if $m_1^a > m_1^b$, then rules of type (4) satisfying conditions 1 to 5 are forward-looking. Propositions 1 and 2 therefore imply that, when $m_1^a > m_1^b$, there exists a forward-looking interest-rate rule that is consistent with the targeted equilibrium and eliminates both type-A and -B equilibria. This result matters because the condition $m_1^a > m_1^b$ is typically met by rational-expectations DSGE models encompassed within our general specification, in particular because their structural equations typically include an Euler equation.

[8] The two works were conducted independently from each other. The first versions of the present paper go back to Loisel (2003, 2004).

On the other hand, whether $m_1^a > m_1^b$ or not, there exists a backward-looking interest-rate rule that is consistent with the targeted equilibrium and eliminates both type-A and -B equilibria. Indeed, let us consider the following proposition:

**Proposition 3**: *for any $\mu \in \mathbb{R}^+ \setminus [0;1]$ and any sequence $\{\mathbf{Y}_t, z_t\}_{t\in\mathbb{Z}}$ of type (3) that satisfies (1), there exists a rule of type (2) that is backward-looking and such that the system made of (1) and this rule: i) admits $\{\mathbf{Y}_t, z_t\}_{t\in\mathbb{Z}}$ as its unique stationary solution; ii) has at most $\tau$ non-predetermined variables; and iii) has no eigenvalue whose modulus is between 1 and $\mu$.*

**Proof**: *cf* appendix C. ■

Technically speaking, appendix C largely draws on appendix B as it uses the generalized identity of Bezout and the Euclidian division to choose $\mathbf{F}(L)$ with $m_f = 0$ and $G(L)$ such that: i) the system made of (1) and (2) admits one unique stationary solution and has no eigenvalue whose modulus is between 1 and $\mu$, and ii) the eigenvalues of $\mathbf{I} - \mathbf{S}(L)$ are also eigenvalues of this system; and Cramer's rule to residually choose $\mathbf{H}(L)$ such that the unique local solution of this system coincides with the targeted stationary VARMA process (3).

For any $\mu \in \mathbb{R}^+ \setminus [0;1]$, let us note $(R_\mu)$ the rule whose existence is stated in proposition 3. Let us consider a given $\mu \in \mathbb{R}^+ \setminus [0;1]$. Since the system made of (1) and $(R_\mu)$ admits the targeted equilibrium as its unique stationary solution and has at most $\tau$ non-predetermined variables, and since $\mathfrak{n}$ includes all the possible realizations of the endogenous variables at the targeted equilibrium, any candidate type-B equilibrium needs to involve, in its locally linearised analytical expression, at least one unstable eigenvalue of the system in order to leave $\mathfrak{n}$ while remaining in $\mathfrak{N}$ for at least $\tau$ more periods. However, for $\mu$ large enough, all the unstable eigenvalues of the system have too large a modulus for any candidate type-B equilibrium to remain in $\mathfrak{N}$ for at least $\tau$ periods after leaving $\mathfrak{n}$. Therefore, provided that $\mu$ is large enough, $(R_\mu)$ eliminates all type-B equilibria. As a consequence, proposition 3 implies that there exists a backward-looking interest-rate rule that is consistent with the targeted equilibrium and eliminates both type-A and -B equilibria.

When $m_1^a > m_1^b$, to sum up, there exist both a forward-looking (bubble-free) interest-rate rule and a backward-looking interest-rate rule that are consistent with the targeted equilibrium and eliminate all type-A and -B equilibria. The next proposition suggests one reason to prefer the forward-looking rule to the backward-looking one in this case:

**Proposition 4**: *if $m_1^a > m_1^b$ then, for any $n \in \mathbb{N}$ and $M \in \mathbb{R}^*$, there exists $\mu \in \mathbb{R}^+ \setminus [0;1]$ such that there exists no backward-looking rule of type (2) such that: i) $n^f \leq n$ and $n^g \leq n$; ii) $g_0$ is normalized to one and all the other coefficients of this rule are of modulus lower than $M$; and iii)* *the system made of (1) and this rule admits a unique stationary solution and has no eigenvalue whose modulus is between* 1 *and* $\mu$*.*

**Proof**: *cf* appendix D. ■

In this proposition, $n$ and $M$ represent how much "long-tailed" and aggressive respectively the policy-maker is prepared to choose her rule. She may want to restrict her choice in such a way because, if chosen beyond these limits, the rule would be too complex or, by magnifying her data-measurement errors, would generate too much macroeconomic volatility. Proposition 4 therefore suggests that, when $m_1^a > m_1^b$, if the $\mu$ whose existence is stated in this proposition is lower than the lowest $\mu$ required to eliminate all trajectories gradually leaving the neighbourhood of the targeted steady state, then there exists no backward-looking rule that: i) eliminates both type-A and -B equilibria; and ii) is sufficiently short-tailed and non-aggressive. By contrast, proposition 1 implies that, when $m_1^a > m_1^b$, provided that $n$ and $M$ are large enough, whatever the lowest $\mu$ required to eliminate all trajectories gradually leaving the neighbourhood of the targeted steady state, there exists a forward-looking rule that: i) eliminates both type-A and -B equilibria; and ii) is sufficiently short-tailed and non-aggressive.

## 3.2 Related literature

This subsection briefly reviews two closely related strands of the literature and positions our paper within each of these strands.

The first strand of the literature deals with the issue of whether the interest-rate rule should be forward- or backward-looking in order to eliminate type-A equilibria. This strand of the literature has usually come in favour of the use of a backward-looking rule. For instance, Bernanke and Woodford (1997) show that some commonly considered forward-looking rules lead to type-A equilibria in the New Keynesian model and therefore warn against following forward-looking rules without first developing a structural model of the economy. Carlstrom and Fuerst (2000, 2002, 2005) similarly show that some commonly considered forward-looking rules lead to type-A equilibria in various models, contrary to some commonly considered backward-looking rules, and therefore advocate the use of backward-looking rules. However, this strand of the literature usually restricts its analysis to specific models and, in particular, to specific low-dimensional parametric families of rules (typically Taylor-type rules). We thus generalize the results obtained by this strand of the literature by showing that not only type-A equilibria, but also type-B equilibria can be eliminated by a backward-looking rule in all models of the broad class considered and by a forward-looking rule in most of these models.

The second strand of the literature focuses on the issue of how much forward-looking a forward-looking interest-rate rule should be in order to eliminate type-A equilibria. Batini and Pearlman (2002), Batini, Levine and Pearlman (2004), Batini, Justiniano, Levine and Pearlman (2006) and Leitemo (2006) consider interest-rate rules of type $i_t = \alpha + \beta i_{t-1} + \gamma E_t \{\pi_{t+\theta}\}$ with $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ and $\theta \in \mathbb{N}$, where $i_t$ and $\pi_t$ respectively denote the short-term nominal interest rate and the inflation rate at date $t$, and find that the more forward-looking the interest-rate rule or equivalently the more distant the forecast horizon (*i.e.* the higher $\theta$), the higher the risks of type-A equilibria or macroeconomic instability − the latter arising when the locally linearised system admits more unstable eigenvalues than required by Blanchard and Kahn's (1980) conditions[9]. Worryingly enough, these risks often materialise for the one- to two-year forecast horizons typically adopted by inflation-targeting central banks. Similarly, Giannoni and Woodford (2003) show, in some simple models, that interest-rate rules that eliminate type-A equilibria, are consistent with the targeted equilibrium and are independent of the statistical properties of the exogenous shocks can be found that are only modestly (if at all) forward-looking, that is to say that the forward-looking part of these rules mainly features current expectations of only one- or two-quarter ahead endogenous variables. They point out that this result provides little support for monetary policies that make the current short-term nominal interest rate respond primarily to one- to two-year-ahead inflation forecasts, such as those of inflation-targeting central banks. Interestingly, our requirement that interest-rate rules should eliminate both type-A and -B equilibria and be consistent with the targeted equilibrium leads to a similar result (though for a different reason) in those of their models that are encompassed within our specification, namely the result that the forward-looking part of interest-rate rules can be limited to only one-quarter-ahead forecasts.

# 4 Robustness of bubble-free interest-rate rules

This section discusses the robustness of the effectiveness of bubble-free interest-rate rules to departures from three assumptions in turn: i) that the policy-maker has perfect knowledge of the structural parameters; ii) that the private agents form rational expectations; iii) that the policy-maker can credibly commit to locally following a given interest-rate rule.

[9] Levin, Wieland and Williams (2003) obtain a similar result while considering a slightly more general family of interest-rate rules. Technically speaking, this result can be interpreted as follows: the choice of a more forward-looking interest-rate rule (*i.e.* of a higher $\theta$) is most likely to increase the number of eigenvalues and non-predetermined variables of the locally linearised system and hence the risk that no $(\beta, \gamma)$ exists such that Blanchard and Kahn's (1980) condition is satisfied. By contrast, the Calvo-type interest-rate rules put forward by Levine, McAdam and Pearlman (2007), of the kind $i_t = \alpha + \beta i_{t-1} + \gamma E_t \left\{ \sum_{k=0}^{+\infty} \varphi^k \pi_{t+k} \right\}$ with $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ and $\varphi \in ]0;1[$, manage to be infinitely forward-looking while making the locally linearised system have a finite (and possibly even small) number of eigenvalues and non-predetermined variables, which might well explain why these rules are much more successful in eliminating type-A equilibria.

## 4.1 Policy-maker's imperfect knowledge

This subsection examines the sensitivity of propositions 1 and 2 to the assumption that the policy-maker has perfect knowledge of the structural parameters, *i.e.* the coefficients featuring in the structural equations (1). This robustness analysis is particularly welcome as some of the coefficients of rules (4) are tightly tied to the structural parameters by equality constraints. By contrast, all the coefficients of interest-rate rules ensuring only local equilibrium determinacy are more loosely tied to the structural parameters by inequality constraints, as exemplified by the well-known Taylor principle or by Rotemberg and Woodford's (1999) "superinertia principle" (generalized by Giannoni and Woodford, 2002, and Woodford, 2003, chap. 8).

Let us first define the metric $d$ by

$$\left(\underset{(N_1\times N_2)}{\mathbf{X}_1(L)},\underset{(N_1\times N_2)}{\mathbf{X}_2(L)}\right)\equiv\left(\sum_{k=-m^x}^{n^x}\mathbf{X}_{1,k}L^k,\sum_{k=-m^x}^{n^x}\mathbf{X}_{2,k}L^k\right)$$
$$\mapsto d\left(\mathbf{X}_1(L),\mathbf{X}_2(L)\right)\equiv\sup_{-m^x\leq k\leq n^x}\left[\max_{1\leq i\leq N_1}\left(\max_{1\leq j\leq N_2}\left|\mathbf{e}_{1,i}'\left(\mathbf{X}_{1,k}-\mathbf{X}_{2,k}\right)\mathbf{e}_{2,j}\right|\right)\right],$$

where $(m^x,n^x)\in\overline{\mathbb{N}}^2$, $(N_1,N_2)\in\mathbb{N}^{*2}$, all $\mathbf{X}_{1,k}$, $\mathbf{X}_{2,k}$ have real numbers as elements and, for $h\in\{1,2\}$ and $l\in\{1,...,N_h\}$, $\mathbf{e}_{h,l}$ is the $N_h$-element vector whose $l^{th}$ element is equal to one and whose other elements are equal to zero. Let us then consider a rule of type (4) satisfying conditions 1 to 5, noted $(R)$, which is consistent with the targeted local equilibrium of type (3) satisfying (1). Let us also rewrite this targeted equilibrium as

$$\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=\mathbf{X}(L)\,\boldsymbol{\varepsilon}_t\text{ with }\underset{((N+1)\times N)}{\mathbf{X}(L)}\equiv\sum_{k=0}^{+\infty}\mathbf{X}_kL^k,\tag{10}$$

where all $\mathbf{X}_k$ have real numbers as elements. Lastly, let $(\widetilde{R})$ denote the rule corresponding to the expression (4) of rule $(R)$ where some exogenous measurement errors, each of them randomly drawn from a continuous probability distribution supported on a bounded interval including zero[10], are added to the elements of $\mathbf{A}_k$ for $-m^a\leq k\leq n^a$, $\mathbf{B}_k$ for $-m^b\leq k\leq n^b$, $\mathbf{C}_k$ for $0\leq k\leq n^c$ and to $d_{i,k}$ for $1\leq i\leq N$ and $0\leq k\leq n^d$, and let $\varepsilon$ denote the maximal length of the distribution-supporting intervals. We get the following proposition:

**Proposition 5**: *(i)* $\exists\mu\in\mathbb{R}^+\setminus[0;1]$ *such that, first, for* $\varepsilon$ *close enough to* $0$*, with probability one, the system made of (1) and* $(\widetilde{R})$ *admits a unique stationary solution and has at most* $\tau$ *non-predetermined variables and no eigenvalue whose modulus is between* $1$ *and* $\mu$ *and, second,* $\mu\longrightarrow+\infty$ *as* $\varepsilon\longrightarrow0$*; (ii) noting*

$$\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=\widetilde{\mathbf{X}}(L)\,\boldsymbol{\varepsilon}_t\textit{ with }\underset{((N+1)\times N)}{\widetilde{\mathbf{X}}(L)}\equiv\sum_{k=0}^{+\infty}\widetilde{\mathbf{X}}_kL^k,\tag{11}$$

[10] This continuous-probability-distribution assumption enables us to disregard degenerate cases as they are of measure zero.

*where all* $\widetilde{\mathbf{X}}_k$ *have real numbers as elements, the unique stationary solution (with probability one) of the system made of (1) and* $(\widetilde{R})$ *for* $\varepsilon$ *close enough to* $0$*, we have:* $d(\mathbf{X}(L), \widetilde{\mathbf{X}}(L)) \longrightarrow 0$ *as* $\varepsilon \longrightarrow 0$*.*

**Proof**: *cf* appendix E. ■

In other words, if the policy-maker's knowledge of the structural parameters is sufficiently accurate, then $(\widetilde{R})$ ensures local equilibrium determinacy with probability one. Moreover, as the policy-maker's knowledge of the structural parameters becomes perfect, the unique local equilibrium trajectory (with probability one) gets arbitrarily close to the targeted path for the endogenous variables. Finally, if the policy-maker's knowledge of the structural parameters is sufficiently accurate, then $(\widetilde{R})$ also eliminates type-B equilibria by using the structural equations as a lever on the private agents' expectations (as made clear in appendix E) to make all equilibrium trajectories leaving the neighbourhood of the targeted steady state do so too abruptly to qualify as type-B equilibria. In short and loosely speaking, both propositions 1 and 2 hold asymptotically with probability one. Therefore, the equality constraints tying some of the coefficients of bubble-free rules to the structural parameters, thus making the policy-maker manoeuvre on a Wicksellian-type razor's edge, prove not as restrictive as they may seem at first sight.

## 4.2 Private agents' myopic rational expectations

One way to relax the rational-expectations assumption is to suppose instead that the private agents form myopic rational expectations, *i.e.* rational expectations up to a given finite horizon $h \in \mathbb{N}^*$. Interestingly, this alternative assumption may make the economy more bubble-prone[11], that is to say in our context type-B equilibria more likely, under conventional interest-rate rules. By contrast, bubble-free interest-rate rules would remain effective in eliminating both type-A and -B equilibria provided that $h \geq \tau$, as clear from proposition 1.

## 4.3 Policy-maker's inability to commit

Instead of assuming that the policy-maker can credibly commit to forever following a policy feedback rule, suppose now more realistically that she can credibly commit only to following a policy feedback rule during at most $d$ periods, where $d \in \mathbb{N}^*$. As clear from proposition 1, bubble-free interest-rate rules then remain effective in eliminating both type-A and -B equilibria provided that $d \geq \tau$.

By contrast, under conventional interest-rate rules, not only type-B equilibria, but also interestingly a third kind of non-targeted equilibria may then exist, which we call type-C equilibria.

[11] This point was first made by Tirole (1982).

Indeed, suppose that the central banker credibly commits at a given date $t$ to follow, during the next $d$ periods, an interest-rate rule that makes the locally linearised system have at least one unstable eigenvalue. If the economy started to embark, during these $d$ periods, on a path leaving the neighbourhood of the targeted steady state, then the stability-concerned central banker would change its interest-rate rule at some date after $t+d$ in order to keep the variables within this neighbourhood or to bring them back into this neighbourhood, because she would find it both possible and desirable. Though initially diverging, the resulting boom-and-bust path remains bounded, and even local if the triggered interest-rate-rule adjustment occurs before the linear approximation of the structural equations becomes invalid. As a consequence, when the original non-linear model features infinitely-lived utility-maximizing private agents, this path does not violate the transversality condition typically required and hence typically qualifies as an equilibrium of this model. This "stabilization of last resort" raises a moral hazard problem since private agents, rightly expecting this reaction from the central banker, can settle on an initially diverging path even in the case where this path would not be an equilibrium if the central banker were compelled to stick forever to its interest-rate rule. In other words, these type-C equilibria, which the existing literature has so far ignored, can exist even when type-B equilibria do not[12].

# Conclusion

This paper aims to give a new insight into the design of interest-rate rules in a broad class of rational-expectations DSGE models. The literature has so far mostly focused on particular kinds of interest-rate rules that preclude unintended fluctuations around the targeted steady state (type-A equilibria), *e.g.* Taylor rules satisfying the Taylor principle. However, as first acknowledged by Benhabib, Schmitt-Grohé and Uribe (2001a), such rules do not prevent the economy from embarking on a path gradually leaving the neighbourhood of the targeted steady state and leading for instance to the liquidity trap (type-B equilibria). By contrast, the bubble-free interest-rate rules put forward in this paper manage to eliminate both types of equilibria, even when the perfect-information, rational-expectations and forever-commitment assumptions are slightly relaxed.

Obviously, bubble-free interest-rate rules make sense only insofar as the behaviour of private agents is at least partly forward-looking, since equilibrium indeterminacy would not be an issue otherwise. But most, if not all rational-expectations DSGE models based on explicit microeconomic foundations imply such a forward-looking behaviour for the private agents[13], which has led

[12] A parallel could be drawn between this escape-clause approach to interest-rate rules and the escape-clause approach to fixed exchange rate systems (*i.e.* the second-generation models of currency crises).

[13] Such a forward-looking behaviour is even less disputed for participants in asset markets than for private agents in macroeconomic models. Loisel (2006) therefore discusses the role that bubble-free interest-rate rules could play in a monetary policy reaction to perceived asset-price bubbles or exchange-rate misalignments.

Woodford (2003, chap. 1) to view the essence of central banking as the management of expectations. However, this now conventional view does not go as far as arguing that central banks should react to private agents' expectations through a forward-looking interest-rate rule: for instance, Bernanke and Woodford (1997) have famously warned against following forward-looking interest-rate rules without first developing a structural model of the economy. In this paper, we thus carry this view further still by arguing for the use of forward-looking (bubble-free) interest-rate rules on the basis of a broad class of structural models of the economy.

## References


Adão, B., I. H. Correia and P. Teles, 2005, Monetary policy with single instrument feedback rules, CEPR Working Paper No. 4948.

Antinolfi, G., C. Azariadis and J. Bullard, 2006, Monetary policy as equilibrium selection, mimeo.

Batini, N., Justiniano, A., Levine, P. and J. Pearlman, 2006, Robust inflation-forecast-based rules to shield against indeterminacy, Journal of Economic Dynamics and Control 30, 1491-1526.

Batini, N., P. Levine and J. Pearlman, 2004, Indeterminacy with inflation-forecast-based rules in a two-bloc model, ECB Working Paper No. 340.

Batini, N. and J. Pearlman, 2002, Too much too soon: instability and indeterminacy with forward-looking rules, External MPC Unit Discussion Paper No. 8.

Benhabib, J. and S. Eusepi, 2005, The design of monetary and fiscal policy: a global perspective, Journal of Economic Theory 123, 40-73.

Benhabib, J., S. Schmitt-Grohé and M. Uribe, 2001a, The perils of Taylor rules, Journal of Economic Theory 96, 40-69.

Benhabib, J., S. Schmitt-Grohé and M. Uribe, 2001b, Monetary policy and multiple equilibria, American Economic Review 91, No. 1, 167-186.

Benhabib, J., S. Schmitt-Grohé and M. Uribe, 2002a, Chaotic interest rate rules, American Economic Review 92, No. 2, 72-78.

Benhabib, J., S. Schmitt-Grohé and M. Uribe, 2002b, Avoiding liquidity traps, Journal of Political Economy 110, 535-563.

Benhabib, J., S. Schmitt-Grohé and M. Uribe, 2003, Backward-looking interest-rate rules, interest-rate smoothing and macroeconomic instability, Journal of Money, Credit and Banking 35, 1379-1412.

Bernanke, B. S. and M. Woodford, 1997, Inflation forecasts and monetary policy, Journal of Money, Credit and Banking 29, 653-684.

Blanchard, O. J., 1979, Speculative bubbles, crashes and rational expectations, Economics Letters 3, 387-389.

Blanchard, O. J. and C. M. Kahn, 1980, The solution of linear difference models under rational expectations, Econometrica 48, 1305-1311.

Carlstrom, C. T. and T. S. Fuerst, 2000, Forward-looking versus backward-looking Taylor rules, Federal Reserve Bank of Cleveland Working Paper No. 00-09.

Carlstrom, C. T. and T. S. Fuerst, 2002, Taylor rules in a model that satisfies the natural-rate hypothesis, American Economic Review 92, No. 2, 79-84.

Carlstrom, C. T. and T. S. Fuerst, 2005, Investment and interest rate policy: a discrete time analysis, Journal of Economic Theory 123, 4-20.

Christiano, L. J. and M. Rostagno, 2001, Money growth monitoring and the Taylor rule, NBER Working Paper No. 8539.

Clarida, R., J. Galí and M. Gertler, 2000, Monetary policy rules and macroeconomic stability: evidence and some theory, Quarterly Journal of Economics 115, 147-180.

Cochrane, J. H., 2006, Identification and price determination with Taylor rules: a critical review, mimeo.

Currie, D. and P. Levine, 1993, Rules, reputation and macroeconomic policy coordination (Cambridge University Press, Cambridge).

Giannoni, M. P. and M. Woodford, 2002, Optimal interest-rate rules: I. General theory, NBER Working Paper No. 9419.

Giannoni, M. P. and M. Woodford, 2003, How forward-looking is optimal monetary policy?, Journal of Money, Credit and Banking 35, 1425-1469.

Green, E. J., 2005, A review of *Interest and prices: foundations of a theory of monetary policy* by Michael Woodford, Journal of Economic Literature 43, 121-134.

Hamilton, J. D., 1994, Time series analysis (Princeton University Press, Princeton).

Leitemo, K., 2006, Targeting inflation by forecast feedback rules in small open economies, Journal of Economic Dynamics and Control 30, 393-413.

Levin, A., Wieland V. and J. C. Williams, 2003, The performance of forecast-based monetary policy rules under model uncertainty, American Economic Review 93, No. 3, 622-645.

Levine, P., P. McAdam and J. Pearlman, 2007, Inflation forecast-based-rules and indeterminacy: a puzzle and a resolution, International Journal of Central Banking, forthcoming.

Loisel, O., 2003, Forward-looking monetary policy rules to preclude multiple equilibria, in Four essays on macroeconomic volatility and instability under alternative exchange rate regimes, École Polytechnique, PhD dissertation, chap. 1 (http://www.imprimerie.polytechnique.fr/Theses/Files/loisel_web.pdf).

Loisel, O., 2004, Interest rate rules ensuring strong local equilibrium determinacy, CREST Working Paper No. 2004-03.

Loisel, O., 2006, Bubble-free interest-rate rules, Banque de France Working Paper No. 161.

Lubik, T. A. and F. Schorfheide, 2004, Testing for indeterminacy: an application to U.S. monetary policy, American Economic Review 94, No. 1, 190-217.

Obstfeld, M. and K. Rogoff, 1983, Speculative hyperinflations in maximizing models: can we rule them out?, Journal of Political Economy 91, 675-687.

Rotemberg, J. J. and M. Woodford, 1999, Interest rate rules in an estimated sticky price model, in J. B. Taylor editor, Monetary policy rules (NBER and University of Chicago Press, Chicago), 57-119.

Taylor, J., 1993, Discretion versus policy rules in practice, Carnegie-Rochester Conference Series on Public Policy 39, 195-214.

Tirole, J., 1982, On the possibility of speculation under rational expectations, Econometrica 50, 1163-1181.

Woodford, M., 2003, Interest and prices: foundations of a theory of monetary policy (Princeton University Press, Princeton).

# Appendix

For any system of equations $(S)$, let $E_t\{(S)\}$ denote the system obtained by applying operator $E_t$ on both the left- and the right-hand sides of each equation of $(S)$. For any polynomial $\mathcal{H}(X) \in \mathbb{R}[X]$, let $d_{\mathcal{H}}$ denote the degree of $\mathcal{H}(X)$. Let $|.|$ denote the determinant operator. Lastly, for any scalar $\lambda$, any non-zero integers $n$ and $p$ and any $n \times p$ matrix $\mathbf{K}$, let $\lambda\mathbf{K}$ denote the product of the $n \times n$ matrix whose diagonal elements are all equal to $\lambda$ and whose non-diagonal elements are all equal to 0 by matrix $\mathbf{K}$, and let $\mathbf{K}\lambda$ denote the product of matrix $\mathbf{K}$ by the $p \times p$ matrix whose diagonal elements are all equal to $\lambda$ and whose non-diagonal elements are all equal to 0 (note that $\lambda\mathbf{K} = \mathbf{K}\lambda$).

## A Proof of proposition 1

Consider a given $t \in \mathbb{Z}$ and suppose that $L^k(1)$ and $L^k(4)$ hold for all $k \geq -\max\left[m_1^a, m_1^b\right] - \sum_{i=2}^N m_i^a$. The substraction of $\mathbf{e}_1'(1)$ from $E_t\left\{L^{-m_1^b}(4)\right\}$ leads to equation $(\overrightarrow{1})$:

$$\sum_{i=2}^{N} \mathbf{e}_i' L^{\sum_{j=2}^{i} m_j^a}\left[\mathbf{A}(L)\mathbf{Y}_t + \mathbf{B}(L)z_t + \mathbf{C}(L)\boldsymbol{\xi}_t + L^{-m_i^a}\mathbf{O}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right] + L^{\sum_{i=2}^{N} m_i^a}\left[\mathbf{P}(L)\mathbf{Y}_t + Q(L)z_t + \mathbf{R}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right] = 0. \quad (\overrightarrow{1})$$

Similarly, $\forall k \in \{2, ..., N\}$, equation $(\overrightarrow{k})$ can be derived from equation $(\overrightarrow{k-1})$ by substracting

$\mathbf{e}'_k(1)$ from $E_t\left\{L^{-m^a_k}(\overrightarrow{k-1})\right\}$:

$$\sum_{i=k+1}^{N}\mathbf{e}'_i L^{\sum_{j=k+1}^{i} m^a_j}\left[\mathbf{A}(L)\mathbf{Y}_t+\mathbf{B}(L)z_t+\mathbf{C}(L)\boldsymbol{\xi}_t+L^{-m^a_i}\mathbf{O}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right]$$
$$+L^{\sum_{i=k+1}^{N} m^a_i}\left[\mathbf{P}(L)\mathbf{Y}_t+Q(L)z_t+\mathbf{R}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right]=0 \quad (\overrightarrow{k})$$

and in particular

$$\mathbf{P}(L)\mathbf{Y}_t+Q(L)z_t+\mathbf{R}(L)\mathbf{D}(L)\boldsymbol{\xi}_t=0. \qquad (\overrightarrow{N})$$

$\forall k\in\{2,...,N\}$, the substraction of $L^{m^a_k}(\overrightarrow{k})$ from $(\overrightarrow{k-1})$ leads to equation $(\overleftarrow{k})$:

$$\mathbf{e}'_k\left[L^{m^a_k}\left[\mathbf{A}(L)\mathbf{Y}_t+\mathbf{B}(L)z_t+\mathbf{C}(L)\boldsymbol{\xi}_t\right]+\mathbf{O}(L)\mathbf{D}(L)\boldsymbol{\xi}_t\right]=0. \qquad (\overleftarrow{k})$$

Equations $\overrightarrow{N}$ and $\overleftarrow{k}$ for $k\in\{2,...,N\}$ together can be re-written as follows:

$$\mathbf{U}(L)\mathbf{Y}_t+\mathbf{V}(L)z_t+\mathbf{W}(L)\boldsymbol{\xi}_t=\mathbf{0} \qquad (12)$$

with $\underset{(N\times N)}{\mathbf{U}(L)}=\begin{bmatrix}\mathbf{P}(L)\\ \mathbf{e}'_2\widehat{\mathbf{A}}(L)\\ \vdots\\ \mathbf{e}'_N\widehat{\mathbf{A}}(L)\end{bmatrix}\equiv\sum_{k=0}^{n^u}\mathbf{U}_kL^k$, $\underset{(N\times 1)}{\mathbf{V}(L)}=\begin{bmatrix}Q(L)\\ \mathbf{e}'_2L^{m^a_2}\mathbf{B}(L)\\ \vdots\\ \mathbf{e}'_NL^{m^a_N}\mathbf{B}(L)\end{bmatrix}\equiv\sum_{k=\max\left[0,m^a_1-m^b_1+1\right]}^{n^v}\mathbf{V}_kL^k$

due to assumption 2.iii and $\underset{(N\times N)}{\mathbf{W}(L)}=\begin{bmatrix}\mathbf{R}(L)\mathbf{D}(L)\\ \mathbf{e}'_2\left[L^{m^a_2}\mathbf{C}(L)+\mathbf{O}(L)\mathbf{D}(L)\right]\\ \vdots\\ \mathbf{e}'_N\left[L^{m^a_N}\mathbf{C}(L)+\mathbf{O}(L)\mathbf{D}(L)\right]\end{bmatrix}\equiv\sum_{k=0}^{n^w}\mathbf{W}_kL^k$,

where $(n^u,n^v,n^w)\in\mathbb{N}^3$ and all $\mathbf{U}_k$, $\mathbf{V}_k$, $\mathbf{W}_k$ have real numbers as elements. Since $\mathbf{U}(0)=\boldsymbol{\Omega}$ is invertible, (12) can be used to express $\mathbf{Y}_t$ as a function of $\mathbf{Y}_{t-1-k}$, $z_{t-\max\left[0,m^a_1-m^b_1+1\right]-k}$ and $\boldsymbol{\xi}_{t-k}$ for $k\geq 0$. If $m^a_1\geq m^b_1$, this expression can be used to sequentially replace $E_t\{\mathbf{Y}_{t+j}\}$ for $j\in\left\{0,...,m^a_1-m^b_1\right\}$ in (4) and thus get $z_t$ as a function of $\mathbf{Y}_{t-1-k}$, $z_{t-1-k}$ and $\boldsymbol{\xi}_{t-k}$ for $k\geq 0$. Alternatively, if $m^a_1<m^b_1$ then (4) directly expresses $z_t$ as a function of $\mathbf{Y}_{t-1-k}$, $z_{t-1-k}$ and $\boldsymbol{\xi}_{t-k}$ for $k\geq 0$. In both cases, the system thus obtained, made of this equation for $z_t$ and (12) for $\mathbf{Y}_t$, is backward-looking and non-degenerate and hence uniquely determines $\mathbf{Y}_t$ and $z_t$ as a function of $\mathbf{Y}_{t-1-k}$, $z_{t-1-k}$ and $\boldsymbol{\xi}_{t-k}$ for $k\geq 0$. This system, noted $(S)$, and the systems $L^k(S)$ for all $k\geq 1$ that can be similarly obtained, then uniquely determine $\mathbf{Y}_{t-j}$ and $z_{t-j}$ for $j\in\mathbb{N}$ as a function of $\boldsymbol{\xi}_{t-j-k}$ for $k\geq 0$. Given that the system made of $L^k(S)$ for all $k\geq 0$ is implied by the system made of $L^k(1)$ and $L^k(4)$ for all $k\geq -\tau$, the latter admits either zero or one unique solution for $\{\mathbf{Y}_{t-j},z_{t-j}\}_{j\in\mathbb{N}}$. Finally, given that the system made of $L^k(S)$ for all $k\in\mathbb{Z}$ admits one unique solution for $\{\mathbf{Y}_{t-k},z_{t-k}\}_{k\in\mathbb{Z}}$ and, as can be readily checked (using notably assumption 2.ii), is equivalent to the system made of $L^k(1)$ and $L^k(4)$ for all $k\in\mathbb{Z}$, we conclude that the system made of $L^k(1)$ and $L^k(4)$ for all $k\geq -\tau$ admits one unique solution for $\{\mathbf{Y}_{t-j},z_{t-j}\}_{j\in\mathbb{N}}$. This solution is stationary because the eigenvalues of the system made of $L^k(1)$ and $L^k(4)$ for all $k\in\mathbb{Z}$ are those of (5) which, given condition 4, are stable.

## B Proof of proposition 2

For the sake of expositional clarity, we omit the expression "for all $t \in \mathbb{Z}$" throughout this appendix. We prove proposition 2 by showing that, for any (3) satisfying (1), there exist $\mathbf{O}(L)$, $\mathbf{P}(L)$, $Q(L)$ and $\mathbf{R}(L)$ satisfying conditions 1 to 5 and such that (3) implies (1) and (4) and therefore, following proposition 1, such that (3) is the unique solution of the system made of (1) and (4). To that aim, suppose that a given (3) holds that satisfies (1).

**Step 1**: then, (1) holds as well. Moreover, there exists $\mathbf{O}(L)$ satisfying condition 1 and such that

$$\begin{bmatrix} E_t\left\{\mathbf{e}_1' L^{m_1^b}\mathbf{A}(L)\mathbf{Y}_t\right\} \\ \mathbf{e}_2' L^{m_2^a}\mathbf{A}(L)\mathbf{Y}_t \\ \vdots \\ \mathbf{e}_N' L^{m_N^a}\mathbf{A}(L)\mathbf{Y}_t \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1' L^{m_1^b}\mathbf{B}(L) \\ \mathbf{e}_2' L^{m_2^a}\mathbf{B}(L) \\ \vdots \\ \mathbf{e}_N' L^{m_N^a}\mathbf{B}(L) \end{bmatrix} z_t + \begin{bmatrix} \mathbf{e}_1'\left[L^{m_1^b}\mathbf{C}(L) + \mathbf{O}(L)\mathbf{D}(L)\right] \\ \mathbf{e}_2'\left[L^{m_2^a}\mathbf{C}(L) + \mathbf{O}(L)\mathbf{D}(L)\right] \\ \vdots \\ \mathbf{e}_N'\left[L^{m_N^a}\mathbf{C}(L) + \mathbf{O}(L)\mathbf{D}(L)\right] \end{bmatrix} \boldsymbol{\xi}_t = \mathbf{0}. \tag{13}$$

**Step 2**: the generalized identity of Bezout implies that there exists $(\mathcal{U}_1(X), ..., \mathcal{U}_{N+1}(X)) \in \mathbb{R}[X]^{N+1}$ such that

$$\sum\nolimits_{i=1}^{N+1} \mathcal{U}_i(X)\Delta_i(X) = \mathcal{D}(X). \tag{14}$$

Let $\Theta(X) \in \mathbb{R}[X]$ denote the polynomial, defined up to a non-zero multiplicative scalar, which has the same roots (whose modulus is strictly lower than one) with the same multiplicity as the eigenvalues of the system $\mathbf{I} - \mathbf{S}(L)$ corresponding to the autoregressive part of the targeted stationary VARMA process (3). Let $\mathcal{Z}(X) \in \mathbb{R}[X]$ be a given polynomial that: i) has all its roots of modulus strictly lower than one; and ii) is such that $\Theta(X)$ is a divisor of $\mathcal{Z}(X)\mathcal{D}(X)$. Given that assumption 1.iii implies the existence of $I \in \{1, ..., N\}$ such that

$$\begin{bmatrix} \mathbf{e}_I' \\ \mathbf{e}_2'\widehat{\mathbf{A}}(0) \\ \vdots \\ \mathbf{e}_N'\widehat{\mathbf{A}}(0) \end{bmatrix}$$

is invertible, let $n \in \mathbb{N}$ be such that

$$n \geq 2d_{\Delta_I} - d_{\mathcal{D}} + \max\left[\max_{i\in\{1,...,N\}}(d_{\mathcal{U}_i}),\ d_{\mathcal{U}_{N+1}} + \max\left(-1, m_1^a - m_1^b\right)\right] - d_{\mathcal{Z}}.$$

Let $\mathcal{Q}(X) \in \mathbb{R}[X]$ and $\mathcal{R}(X) \in \mathbb{R}[X]$ be respectively the quotient and the remainder of the Euclidian division of $X^n\mathcal{Z}(X)$ by $\Delta_I(X)$, *i.e.* the unique polynomials such that $X^n\mathcal{Z}(X) = \Delta_I(X)\mathcal{Q}(X) + \mathcal{R}(X)$ with $d_{\mathcal{R}} < d_{\Delta_I}$. Multiplying the left-hand side and the right-hand side of (14) by $\mathcal{R}(X)$, we obtain

$$\mathcal{R}(X)\sum\nolimits_{i=1}^{N+1} \mathcal{U}_i(X)\Delta_i(X) = \mathcal{R}(X)\mathcal{D}(X)$$

and therefore

$$\sum\nolimits_{\substack{i=1 \\ i\neq I}}^{N+1} \left[\mathcal{R}(X)\mathcal{U}_i(X)\right]\Delta_i(X) + \left[\mathcal{R}(X)\mathcal{U}_I(X) + \mathcal{Q}(X)\mathcal{D}(X)\right]\Delta_I(X) = X^n\mathcal{Z}(X)\mathcal{D}(X).$$

Let us note $\mathcal{P}_i(X) \equiv \mathcal{R}(X)\mathcal{U}_i(X)$ for $i \in \{1,...,N+1\} \smallsetminus \{I\}$ and $\mathcal{P}_I(X) \equiv \mathcal{R}(X)\mathcal{U}_I(X) + \mathcal{Q}(X)\mathcal{D}(X)$. Given that

$$\left\{\begin{array}{l} n \geq 2d_{\Delta_I} - d_{\mathcal{D}} + \max\left[\max\limits_{i\in\{1,...,N\}}(d_{\mathcal{U}_i}),\ d_{\mathcal{U}_{N+1}} + \max\left(-1, m_1^a - m_1^b\right)\right] - d_{\mathcal{Z}} \\ n = d_{\Delta_I} + d_{\mathcal{Q}} - d_{\mathcal{Z}} \\ d_{\Delta_I} > d_{\mathcal{R}} \end{array}\right.$$

$$\Longrightarrow d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{R}} + \max\left[\max\limits_{i\in\{1,...,N\}}(d_{\mathcal{U}_i}),\ d_{\mathcal{U}_{N+1}} + \max\left(-1, m_1^a - m_1^b\right)\right]$$

$$\Longrightarrow \left\{\begin{array}{l} d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{R}} + d_{\mathcal{U}_I} \\ d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{R}} + \max\limits_{i\in\{1,...,N\}\smallsetminus\{I\}}(d_{\mathcal{U}_i}) \\ d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{U}_{N+1}} + \max\left(-1, m_1^a - m_1^b\right) \end{array}\right. \Longrightarrow \left\{\begin{array}{l} d_{\mathcal{P}_I} = d_{\mathcal{Q}} + d_{\mathcal{D}} \\ d_{\mathcal{P}_I} > d_{\mathcal{P}_i} \text{ for } i \in \{1,...,N\} \smallsetminus \{I\} \\ d_{\mathcal{P}_I} \geq d_{\mathcal{P}_{N+1}} + \max\left(0, m_1^a - m_1^b + 1\right) \end{array}\right.,$$

the choice of

$$\mathbf{P}(L) = \sum\nolimits_{i=1}^{N} (-1)^{N+1-i} L^{d_{\mathcal{P}_I}} \mathcal{P}_i\left(L^{-1}\right) \mathbf{e}_i' \text{ and } Q(L) = L^{d_{\mathcal{P}_I}} \mathcal{P}_{N+1}\left(L^{-1}\right)$$

satisfies conditions 2 and 3.

**Step 3**: the non-zero eigenvalues of (5) are those of the system

$$\underset{((N+1)\times(N+1))}{\mathbf{\Psi}(L)} \equiv \sum\nolimits_{k=-m^\psi}^{n^\psi} \mathbf{\Psi}_k L^k = \left[\begin{array}{c|c} L^{m_1^b}\mathbf{e}_1'\mathbf{A}(L) & L^{m_1^b}\mathbf{e}_1'\mathbf{B}(L) \\ L^{m_2^a}\mathbf{e}_2'\mathbf{A}(L) & L^{m_2^a}\mathbf{e}_2'\mathbf{B}(L) \\ \vdots & \vdots \\ L^{m_N^a}\mathbf{e}_N'\mathbf{A}(L) & L^{m_N^a}\mathbf{e}_N'\mathbf{B}(L) \\ \hline \mathbf{P}(L) & Q(L) \end{array}\right],$$

where $\left(m^\psi, n^\psi\right) \in \mathbb{N}^2$ and all $\mathbf{\Psi}_k$ have real numbers as elements. If $m_1^b > m_1^a$ then $m^\psi = 0$ and

$$\mathbf{\Psi}_0 = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & \mathbf{e}_1'\mathbf{B}_{-m_1^b} \\ & \mathbf{e}_2'\widehat{\mathbf{A}}(0) & & 0 \\ & \vdots & & \vdots \\ & \mathbf{e}_N'\widehat{\mathbf{A}}(0) & & 0 \\ \hline & \mathbf{P}_0 & & Q(0) \end{array}\right]$$

is invertible (since condition 2 is satisfied). Therefore, according to a standard matricial result of time series analysis (*cf e.g.* Hamilton, 1994, chap. 10, prop. 10.1), the eigenvalues of $\mathbf{\Psi}(L)$ are the roots of polynomial $\left|X^{n^\psi}\mathbf{\Psi}\left(X^{-1}\right)\right| \in \mathbb{R}[X]$. Alternatively, if $m_1^b \leq m_1^a$ then the eigenvalues of $\mathbf{\Psi}(L)$ are those of the system that is obtained by using the last $N$ lines of $\mathbf{\Psi}(L)$ (given that condition 2 is satisfied) to sequentially remove the terms in $L^k$ for $k \in \{-m^\psi, ..., 0\}$ from the first line of $\mathbf{\Psi}(L)$. This system, noted $\mathbf{\Lambda}(L)$, is of the form $\sum_{k=0}^{n^\psi} \mathbf{\Lambda}_k L^k$, where all $\mathbf{\Lambda}_k$ have real numbers as elements and

$$\mathbf{\Lambda}_0 = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & \mathbf{e}_1'\mathbf{B}_{-m_1^b} \\ & \mathbf{e}_2'\widehat{\mathbf{A}}(0) & & 0 \\ & \vdots & & \vdots \\ & \mathbf{e}_N'\widehat{\mathbf{A}}(0) & & 0 \\ \hline & \mathbf{P}_0 & & Q(0) \end{array}\right]$$

is invertible. Therefore, according to a standard matricial result of time series analysis (*cf e.g.* Hamilton, 1994, chap. 10, prop. 10.1), the eigenvalues of $\mathbf{\Lambda}(L)$ − and hence those of $\mathbf{\Psi}(L)$ − are the roots of polynomial $\left|X^{n^\psi}\mathbf{\Lambda}\left(X^{-1}\right)\right|$, which is equal to polynomial $\left|X^{n^\psi}\mathbf{\Psi}\left(X^{-1}\right)\right|$. To sum up, whether $m_1^b > m_1^a$ or $m_1^b \leq m_1^a$, the non-zero eigenvalues of (5) are the non-zero roots of polynomial $\left|X^{n^\psi}\mathbf{\Psi}\left(X^{-1}\right)\right|$, that is to say those of

$$\sum_{i=1}^{N}\left[(-1)^{N+1-i}\mathbf{P}\left(X^{-1}\right)\mathbf{e}_i\right]\Delta_i(X)+\left[Q\left(X^{-1}\right)\right]\Delta_{N+1}(X)$$

and hence, by construction of $\mathbf{P}(L)$ and $Q(L)$, those of $\mathcal{Z}(X)\mathcal{D}(X)$. By definition of $\mathcal{Z}(X)$ and given assumption 3, all non-zero roots of $\mathcal{Z}(X)\mathcal{D}(X)$ are of modulus strictly lower than one, so that the $\mathbf{P}(L)$ and $Q(L)$ constructed at step 2 satisfy condition 4.

**Step 4**: (3) implies that there exists a unique $\underset{(1\times N)}{\mathbf{R}(L)}\equiv\sum_{k=0}^{+\infty}\mathbf{R}_k L^k$, where all $\mathbf{R}_k$ have real numbers as elements, such that

$$\mathbf{P}(L)\mathbf{Y}_t+Q(L)z_t+\mathbf{R}(L)\boldsymbol{\varepsilon}_t=\mathbf{0} \tag{15}$$

where $\mathbf{P}(L)$ and $Q(L)$ are the ones constructed at step 2. If $m_1^b > m_1^a$ then multiplying both (15) and (13) by $D(L)\equiv\prod_{i=1}^{N}D_i(L)$ leads to

$$D(L)\mathbf{P}(L)\mathbf{Y}_t+D(L)Q(L)z_t+D(L)\mathbf{R}(L)\boldsymbol{\varepsilon}_t=\mathbf{0}\text{ and} \tag{16}$$

$$D(L)\begin{bmatrix}\mathbf{e}_1'L^{m_1^b}\mathbf{A}(L) & \mathbf{e}_1'L^{m_1^b}\mathbf{B}(L)\\ \mathbf{e}_2'L^{m_2^a}\mathbf{A}(L) & \mathbf{e}_2'L^{m_2^a}\mathbf{B}(L)\\ \vdots & \vdots\\ \mathbf{e}_N'L^{m_N^a}\mathbf{A}(L) & \mathbf{e}_N'L^{m_N^a}\mathbf{B}(L)\end{bmatrix}\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}+\begin{bmatrix}\mathbf{e}_1'L^{m_1^b}\mathbf{C}(L)\\ \mathbf{e}_2'L^{m_2^a}\mathbf{C}(L)\\ \vdots\\ \mathbf{e}_N'L^{m_N^a}\mathbf{C}(L)\end{bmatrix}$$
$$\begin{bmatrix}\prod_{i=2}^{N}D_i(L) & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \prod_{i=1}^{N-1}D_i(L)\end{bmatrix}\boldsymbol{\varepsilon}_t+D(L)\mathbf{O}(L)\boldsymbol{\varepsilon}_t=\mathbf{0} \tag{17}$$

since, as a scalar, $D(L)$ is such that $D(L)\mathbf{K}=\mathbf{K}D(L)$ for any matrix $\mathbf{K}$. The system made of (16) and (17) is backward-looking (since $m_1^a > m_1^b$) and non-degenerate (since $D(0)=|\mathbf{D}(0)|\neq 0$ and $|\mathbf{\Psi}_0|\neq 0$). Cramer's rule then implies that there exist $(n_1,...,n_{N+1})\in\mathbb{N}^{N+1}$ with $n_i\geq d_{\Delta_i}$ for $i\in\{1,...,N+1\}$ and $\underset{((N+1)\times N)}{\mathbf{\Upsilon}_1(L)}\equiv\sum_{k=0}^{n^{v_1}}\mathbf{\Upsilon}_{1,k}L^k$, where $n^{v_1}\in\mathbb{N}$ and all $\mathbf{\Upsilon}_{1,k}$ have real numbers as elements, such that this system can be rewritten

$$D(L)L^{d_\mathcal{Z}+d_\mathcal{D}}\mathcal{Z}\left(L^{-1}\right)\mathcal{D}\left(L^{-1}\right)\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=$$
$$\mathbf{\Upsilon}_1(L)\boldsymbol{\varepsilon}_t+D(L)\begin{bmatrix}L^{n_1}\Delta_1\left(L^{-1}\right)\mathbf{R}(L)\boldsymbol{\varepsilon}_t\\ \vdots\\ L^{n_{N+1}}\Delta_{N+1}\left(L^{-1}\right)\mathbf{R}(L)\boldsymbol{\varepsilon}_t\end{bmatrix} \tag{18}$$

given step 3. But Cramer's rule also implies that there exists $\underset{((N+1)\times N)}{\boldsymbol{\Upsilon}_2(L)} \equiv \sum_{k=0}^{n^{\upsilon_2}} \boldsymbol{\Upsilon}_{2,k} L^k$, where $n^{\upsilon_2} \in \mathbb{N}$ and all $\boldsymbol{\Upsilon}_{2,k}$ have real numbers as elements, such that the targeted stationary VARMA process (3) can be rewritten

$$L^{d_\Theta} \Theta\left(L^{-1}\right) \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} = \boldsymbol{\Upsilon}_2(L) \boldsymbol{\varepsilon}_t,$$

which implies

$$D(L) L^{d_\mathcal{Z}+d_\mathcal{D}} \mathcal{Z}\left(L^{-1}\right) \mathcal{D}\left(L^{-1}\right) \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} = D(L) \frac{L^{d_\mathcal{Z}+d_\mathcal{D}} \mathcal{Z}\left(L^{-1}\right) \mathcal{D}\left(L^{-1}\right)}{L^{d_\Theta} \Theta\left(L^{-1}\right)} \boldsymbol{\Upsilon}_2(L) \boldsymbol{\varepsilon}_t \tag{19}$$

where $\frac{X^{d_\mathcal{Z}+d_\mathcal{D}} \mathcal{Z}\left(X^{-1}\right)\mathcal{D}\left(X^{-1}\right)}{X^{d_\Theta}\Theta\left(X^{-1}\right)} \in \mathbb{R}[X]$ by definition of $\mathcal{Z}(X)$. Given that $\Delta_{N+1}(X) \neq 0$ due to assumption 1.iii, the identification of (18) with (19) shows that $\exists n^r \in \mathbb{N}$, $\forall k > n^r$, $\mathbf{R}_k = \mathbf{0}$, so that $\mathbf{R}(L)$ satisfies condition 5. Alternatively, if $m_1^b \leq m_1^a$ then a similar reasoning (based on $\boldsymbol{\Lambda}(L)$ instead of $\boldsymbol{\Psi}(L)$, along the lines of step 3) shows that $\mathbf{R}(L)$ satisfies condition 5 again. To sum up, whether $m_1^b > m_1^a$ or $m_1^b \leq m_1^a$, there exists $\mathbf{R}(L)$ satisfying condition 5 and such that (15) holds. Finally, steps 1 to 4 together imply that (1) holds and that there exist $\mathbf{O}(L)$, $\mathbf{P}(L)$, $Q(L)$ and $\mathbf{R}(L)$ satisfying conditions 1 to 5 and such that (4) holds. Following proposition 1, we then conclude that (3) is the unique solution of the system made of (1) and (4).

## C Proof of proposition 3

Let $\mu \in \mathbb{R}^+ \setminus [0;1]$. If $m_1^a \leq m_1^b$ then, given propositions 1 and 2, there exists a rule of type (4) that: i) satisfies conditions 1 to 5; ii) is backward-looking; iii) ensures the local determinacy of the given sequence of type (3) satisfying (1); and iv) is such that the system made of (1) and this rule has no non-predetermined variables and no eigenvalue whose modulus is between 1 and $\mu$. The remaining of the proof therefore deals with the case where $m_1^a > m_1^b$. We proceed in four steps: first, we show that (1) together with a backward-looking rule of type (2) can be written in Blanchard and Kahn's (1980) form with $\tau$ non-predetermined variables; second, we construct some particular $\mathbf{F}(L)$ and $G(L)$ such that $m_f = 0$, so that whatever $\mathbf{H}(L)$ the corresponding rule (2) is backward-looking; third, we show that whatever $\mathbf{H}(L)$ the system made of (1) and this rule admits at most one stationary solution and has no eigenvalue whose modulus is between 1 and $\mu$; fourth, we show that a suitable choice of $\mathbf{H}(L)$ makes this system admit exactly one stationary solution and makes this solution coincide with the targeted stationary VARMA process (3). Note that steps 2 to 4 of this appendix largely draw on steps 2 to 4 of appendix B, with $\mathbf{F}(L)$, $G(L)$, $\boldsymbol{\Xi}(L)$ and $\mathbf{H}(L)$ playing respectively the roles of $\mathbf{P}(L)$, $Q(L)$, $\mathbf{R}(L)$ and $\mathbf{R}(L)\mathbf{D}(L)$. In particular, we use in this appendix the polynomials $\mathcal{U}_i(X)$ for $i \in \{1, ..., N+1\}$ and $\Theta(X)$ introduced in appendix B.

**Step 1**: consider a system $(S)$ of type (1) and a backward-looking rule $(\widehat{R})$ of type (2). Let us rewrite $(S)$ step by step and keep for simplicity the same notation $(S)$ at each step. Re-order the lines of $(S)$ so that $m_1^a \geq ... \geq m_N^a$. Let $K \in \{1, ..., N\}$ and $\{i_1, ..., i_K\} \in \{1, ..., N\}^K$ be such that $m_1^a = ... = m_{i_1}^a > m_{i_1+1}^a = ... = m_{i_2}^a > ... > m_{i_{K-1}+1}^a = ... = m_{i_K}^a = m_N^a$. Re-order the elements of $\mathbf{Y}_t$ and accordingly the columns of $\mathbf{A}(L)$ so that $\forall i \in \{1, ..., N-1\}$, the $(N-i) \times (N-i)$ matrix noted $\mathbf{M}_i$ obtained by removing the first $i$ lines and the first $i$ columns from $\widehat{\mathbf{A}}(0)$ is invertible, this re-ordering being made possible by assumption 1.iii. Replace $\mathbf{e}_i'(S)$ by

$$\mathbf{e}_i'\widehat{\mathbf{A}}(0)^{-1} E_t \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & L^{m_2^a - m_1^a} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L^{m_N^a - m_1^a} \end{bmatrix} (S)$$

for $i \in \{1, ..., i_1\}$. If $K = 1$ then replace sequentially $E_t\left\{z_{t+m_{i_1}^a - k}\right\}$ for $k \in \left\{1, ..., m_{i_1}^a\right\}$ (if they appear) in $(S)$ by their expressions in $E_t\left\{L^{k-m_{i_1}^a}(\widehat{R})\right\}$. The resulting system $(S)$ is equivalent to the original one and together with $(\widehat{R})$ can easily be written in Blanchard and Kahn's (1980) form with $i_1 m_{i_1}^a = \sum_{i=1}^N m_i^a \equiv m$ non-predetermined variables. Otherwise (*i.e.* if $K \geq 2$), let us set $k = 1$. Replace $E_t\left\{z_{t+m_{i_1}^a - k}\right\}$ (if it appears) in $\mathbf{e}_i'(S)$ for $i \in \{1, ..., i_1\}$ by its expression in $E_t\left\{L^{k-m_{i_1}^a}(\widehat{R})\right\}$. Then, replace $E_t\left\{\mathbf{e}_i'\mathbf{Y}_{t+m_{i_1}^a - k}\right\}$ for $i \in \{i_1+1, ..., N\}$ (if they appear) in $\mathbf{e}_i'(S)$ for $i \in \{1, ..., i_1\}$ by their expression in

$$\mathbf{M}_i^{-1} \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & L^{m_{i_1+1}^a - m_{i_1}^a + k} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & L^{m_N^a - m_{i_1}^a + k} \end{array}\right] (S).$$

If $m_{i_1}^a > m_{i_2}^a + 1$ then repeat these last two steps sequentially for $k \in \left\{2, ..., m_{i_1}^a - m_{i_2}^a\right\}$. Proceed in a similar way as previously to transform $\mathbf{e}_i'(S)$ for $i \in \{i_1+1, ..., i_2\}$, then (if $K \geq 3$) $\mathbf{e}_i'(S)$ for $i \in \{i_2+1, ..., i_3\}$ and so on up to $\mathbf{e}_i'(S)$ for $i \in \{i_{K-1}+1, ..., i_K\}$. The final system $(S)$ is equivalent to the initial one and together with $(\widehat{R})$ can easily be written in Blanchard and Kahn's (1980) form with $\sum_{j=1}^K i_j m_{i_j}^a = \sum_{i=1}^N m_i^a = m$ non-predetermined variables. Note finally that this number $m$ of non-predetermined variables does not depend on the particular backward-looking rule $(\widehat{R})$ of type (2) considered and, since $m_1^a > m_1^b$, is equal to $\tau$.

**Step 2**: let $\mathcal{Z}(X) \in \mathbb{R}[X]$ be a given polynomial that: i) has exactly $m$ roots (taking into account their multiplicity) whose modulus is strictly higher than $\mu$; ii) has no root whose modulus is between 1 and $\mu$; and iii) is such that $\Theta(X)$ is a divisor of $\mathcal{Z}(X)\mathcal{D}(X)$. Let $n \in \mathbb{N}$ be such that

$$n \geq 2d_{\Delta_{N+1}} - d_{\mathcal{D}} + \max_{i \in \{1,...,N+1\}} (d_{\mathcal{U}_i}) - d_{\mathcal{Z}}.$$

Let $\mathcal{Q}(X) \in \mathbb{R}[X]$ and $\mathcal{R}(X) \in \mathbb{R}[X]$ be respectively the quotient and the remainder of the Euclidian division of $X^n \mathcal{Z}(X)$ by $\Delta_{N+1}(X)$, *i.e.* the unique polynomials such that $X^n \mathcal{Z}(X) = \Delta_{N+1}(X)\mathcal{Q}(X) + \mathcal{R}(X)$ with $d_{\mathcal{R}} < d_{\Delta_{N+1}}$. Multiplying the left-hand side and the right-hand side of (14) by $\mathcal{R}(X)$, we obtain

$$\mathcal{R}(X)\sum_{i=1}^{N+1} \mathcal{U}_i(X)\Delta_i(X) = \mathcal{R}(X)\mathcal{D}(X)$$

and therefore

$$\sum_{i=1}^{N} [\mathcal{R}(X)\mathcal{U}_i(X)]\Delta_i(X) + [\mathcal{R}(X)\mathcal{U}_{N+1}(X) + \mathcal{Q}(X)\mathcal{D}(X)]\Delta_{N+1}(X) = X^n \mathcal{Z}(X)\mathcal{D}(X).$$

Let us note $\mathcal{F}_i(X) \equiv \mathcal{R}(X)\mathcal{U}_i(X)$ for $i \in \{1, ..., N\}$ and $\mathcal{G}(X) \equiv \mathcal{R}(X)\mathcal{U}_{N+1}(X) + \mathcal{Q}(X)\mathcal{D}(X)$. The choice of $\mathbf{F}(L)\mathbf{e}_i = (-1)^{N+1-i} L^{d_{\mathcal{G}}}\mathcal{F}_i\left(L^{-1}\right)$ for $i \in \{1, ..., N\}$ and $G(L) = L^{d_{\mathcal{G}}}\mathcal{G}\left(L^{-1}\right)$ is admissible as it satisfies the requirements $G(X) \in \mathbb{R}[X]$ and $g_0 \neq 0$. Moreover, we have

$$\begin{cases} n \geq 2d_{\Delta_{N+1}} - d_{\mathcal{D}} + \max\limits_{i\in\{1,...,N+1\}}(d_{\mathcal{U}_i}) - d_{\mathcal{Z}} \\ n = d_{\Delta_{N+1}} + d_{\mathcal{Q}} - d_{\mathcal{Z}} \\ d_{\Delta_{N+1}} > d_{\mathcal{R}} \end{cases} \Longrightarrow d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{R}} + \max_{i\in\{1,...,N+1\}}(d_{\mathcal{U}_i})$$

$$\Longrightarrow \begin{cases} d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{R}} + d_{\mathcal{U}_{N+1}} \\ d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{R}} + \max\limits_{i\in\{1,...,N\}}(d_{\mathcal{U}_i}) \end{cases} \Longrightarrow d_{\mathcal{G}} = d_{\mathcal{Q}} + d_{\mathcal{D}} > d_{\mathcal{F}_i} \text{ for } i \in \{1, ..., N\},$$

so that the $\mathbf{F}(L)$ constructed at step 2 is such that $\forall i \in \{1, ..., N\}$, $\mathbf{F}(X)\mathbf{e}_i \in \mathbb{R}[X]$, in other words $m_f = 0$, *i.e.* any rule (2) with the $\mathbf{F}(L)$ and $G(L)$ constructed at step 2 is backward-looking.

**Step 3**: the non-zero eigenvalues of the system made of (1) and any rule (2) with the $\mathbf{F}(L)$ and $G(L)$ constructed at step 2 are those of the corresponding perfect-foresight deterministic system

$$\mathbf{\Psi}(L)\begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} = 0 \text{ where } \underset{((N+1)\times(N+1))}{\mathbf{\Psi}(L)} \equiv \sum_{k=0}^{n^\psi} \mathbf{\Psi}_k L^k = \left[\begin{array}{c|c} \widehat{\mathbf{A}}(L) & \begin{matrix} L^{m_1^a}\mathbf{e}_1'\mathbf{B}(L) \\ \vdots \\ L^{m_N^a}\mathbf{e}_N'\mathbf{B}(L) \end{matrix} \\ \hline \mathbf{F}(L) & G(L) \end{array}\right],$$

$n^\psi \in \mathbb{N}$ and all $\mathbf{\Psi}_k$ have real numbers as elements. Given that $m_1^a > m_1^b$, assumptions 1.iii and 2.iii together with $g_0 \neq 0$ make

$$\mathbf{\Psi}_0 = \left[\begin{array}{c|c} \widehat{\mathbf{A}}(0) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathbf{F}(0) & g_0 \end{array}\right]$$

invertible, so that according to a standard matricial result of time series analysis (*cf e.g.* Hamilton, 1994, chap. 10, prop. 10.1) this system's eigenvalues are the roots of polynomial $\left|X^{n^\psi}\mathbf{\Psi}\left(X^{-1}\right)\right| \in \mathbb{R}[X]$. As a consequence, the system's non-zero eigenvalues are the non-zero roots of

$$\sum_{i=1}^{N}\left[(-1)^{N+1-i}\mathbf{F}\left(X^{-1}\right)\mathbf{e}_i\right]\Delta_i(X) + \left[G\left(X^{-1}\right)\right]\Delta_{N+1}(X)$$

and hence, by construction of $\mathbf{F}(L)$ and $G(L)$, the non-zero roots of $\mathcal{Z}(X)\mathcal{D}(X)$. By definition of $\mathcal{Z}(X)$ and given assumption 3, $\mathcal{Z}(X)\mathcal{D}(X)$ has no root whose modulus is between 1 and $\mu$, and exactly $m$ roots whose modulus is strictly higher than $\mu$. Given step 1, this implies that the system made of (1) and any rule (2) with the $\mathbf{F}(L)$ and $G(L)$ constructed at step 2 admits either one or zero stationary solution, depending on whether Blanchard and Kahn's (1980) rank condition is satisfied or not, and has no eigenvalue whose modulus is between 1 and $\mu$.

**Step 4**: if the targeted stationary VARMA process (3) holds for $t \in \mathbb{Z}$, then: i) there exists a unique $\underset{(1\times N)}{\boldsymbol{\Xi}(L)} \equiv \sum_{k=0}^{+\infty} \boldsymbol{\Xi}_k L^k$, where all $\boldsymbol{\Xi}_k$ have real numbers as elements, such that

$$\mathbf{F}(L)\mathbf{Y}_t + G(L) z_t + \boldsymbol{\Xi}(L)\boldsymbol{\varepsilon}_t = \mathbf{0} \tag{20}$$

where $\mathbf{F}(L)$ and $G(L)$ are the ones constructed at step 2; and ii) there exists a unique

$$\underset{(N\times N)}{\boldsymbol{\Pi}(L)} \equiv \begin{bmatrix} \sum_{k=0}^{m_1^a-1} \boldsymbol{\Pi}_{1,k} L^k \\ \sum_{k=0}^{m_2^a-1} \boldsymbol{\Pi}_{2,k} L^k \\ \vdots \\ \sum_{k=0}^{m_N^a-1} \boldsymbol{\Pi}_{N,k} L^k \end{bmatrix},$$

where all $\boldsymbol{\Pi}_{i,k}$ have real numbers as elements, such that

$$\left[\begin{array}{c|c} \widehat{\mathbf{A}}(L) & \begin{array}{c} L^{m_1^a}\mathbf{e}_1'\mathbf{B}(L) \\ \vdots \\ L^{m_N^a}\mathbf{e}_N'\mathbf{B}(L) \end{array} \end{array}\right] \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} + \begin{bmatrix} L^{m_1^a}\mathbf{e}_1'\mathbf{C}(L) \\ \vdots \\ L^{m_N^a}\mathbf{e}_N'\mathbf{C}(L) \end{bmatrix} \boldsymbol{\xi}_t + \boldsymbol{\Pi}(L)\boldsymbol{\varepsilon}_t = \mathbf{0}. \tag{21}$$

Multiplying both (20) and (21) by $D(L) \equiv \prod_{i=1}^{N} D_i(L)$ leads to

$$D(L)\mathbf{F}(L)\mathbf{Y}_t + D(L)G(L) z_t + D(L)\boldsymbol{\Xi}(L)\boldsymbol{\varepsilon}_t = \mathbf{0} \text{ and} \tag{22}$$

$$D(L)\left[\begin{array}{c|c} \widehat{\mathbf{A}}(L) & \begin{array}{c} L^{m_1^a}\mathbf{e}_1'\mathbf{B}(L) \\ \vdots \\ L^{m_N^a}\mathbf{e}_N'\mathbf{B}(L) \end{array} \end{array}\right] \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} + \begin{bmatrix} L^{m_1^a}\mathbf{e}_1'\mathbf{C}(L) \\ \vdots \\ L^{m_N^a}\mathbf{e}_N'\mathbf{C}(L) \end{bmatrix}$$
$$\begin{bmatrix} \prod_{i=2}^{N} D_i(L) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \prod_{i=1}^{N-1} D_i(L) \end{bmatrix} \boldsymbol{\varepsilon}_t + D(L)\boldsymbol{\Pi}(L)\boldsymbol{\varepsilon}_t = \mathbf{0} \tag{23}$$

since, as a scalar, $D(L)$ is such that $D(L)\mathbf{K} = \mathbf{K}D(L)$ for any matrix $\mathbf{K}$. The system made of (22) and (23) is backward-looking (since $m_1^a > m_1^b$) and non-degenerate (since $D(0) = |\mathbf{D}(0)| \neq 0$ and $|\boldsymbol{\Psi}_0| \neq 0$). Cramer's rule then implies that there exist $(n_1, ..., n_{N+1}) \in \mathbb{N}^{N+1}$ with $n_i \geq d_{\Delta_i}$ for $i \in \{1, ..., N+1\}$ and $\underset{((N+1)\times N)}{\boldsymbol{\Upsilon}_1(L)} \equiv \sum_{k=0}^{n^{v_1}} \boldsymbol{\Upsilon}_{1,k} L^k$, where $n^{v_1} \in \mathbb{N}$ and all $\boldsymbol{\Upsilon}_{1,k}$ have real numbers as elements, such that this system can be rewritten

$$D(L)\,L^{d_{\mathcal{Z}}+d_{\mathcal{D}}}\mathcal{Z}\left(L^{-1}\right)\mathcal{D}\left(L^{-1}\right)\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=$$

$$\boldsymbol{\Upsilon}_1(L)\,\boldsymbol{\varepsilon}_t+D(L)\begin{bmatrix}L^{n_1}\Delta_1\left(L^{-1}\right)\boldsymbol{\Xi}(L)\,\boldsymbol{\varepsilon}_t\\ \vdots\\ L^{n_{N+1}}\Delta_{N+1}\left(L^{-1}\right)\boldsymbol{\Xi}(L)\,\boldsymbol{\varepsilon}_t\end{bmatrix}\tag{24}$$

given step 3. But Cramer's rule also implies that there exists $\underset{((N+1)\times N)}{\boldsymbol{\Upsilon}_2(L)}\equiv\sum_{k=0}^{n^{\upsilon_2}}\boldsymbol{\Upsilon}_{2,k}L^k$, where $n^{\upsilon_2}\in\mathbb{N}$ and all $\boldsymbol{\Upsilon}_{2,k}$ have real numbers as elements, such that the targeted stationary VARMA process (3) can be rewritten

$$L^{d_\Theta}\Theta\left(L^{-1}\right)\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=\boldsymbol{\Upsilon}_2(L)\,\boldsymbol{\varepsilon}_t,$$

which implies

$$D(L)\,L^{d_{\mathcal{Z}}+d_{\mathcal{D}}}\mathcal{Z}\left(L^{-1}\right)\mathcal{D}\left(L^{-1}\right)\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=D(L)\,\frac{L^{d_{\mathcal{Z}}+d_{\mathcal{D}}}\mathcal{Z}\left(L^{-1}\right)\mathcal{D}\left(L^{-1}\right)}{L^{d_\Theta}\Theta\left(L^{-1}\right)}\boldsymbol{\Upsilon}_2(L)\,\boldsymbol{\varepsilon}_t\tag{25}$$

where $\frac{X^{d_{\mathcal{Z}}+d_{\mathcal{D}}}\mathcal{Z}\left(X^{-1}\right)\mathcal{D}\left(X^{-1}\right)}{X^{d_\Theta}\Theta\left(X^{-1}\right)}\in\mathbb{R}[X]$ by definition of $\mathcal{Z}(X)$. Given that $\Delta_{N+1}(X)\neq 0$ due to assumption 1.iii, the identification of (24) with (25) shows that $\exists n^{\xi}\in\mathbb{N}$, $\forall k>n^{\xi}$, $\boldsymbol{\Xi}_k=\mathbf{0}$. The choice of $\mathbf{H}(L)=\boldsymbol{\Xi}(L)\,\mathbf{D}(L)$ is therefore admissible. We have thus shown that, for any (3) satisfying (1), there exist $\mathbf{F}(L)$, $G(L)$ and $\mathbf{H}(L)$ with $m_f=0$ such that: i) (3) implies (1) and (2); and ii) the system made of (1) and (2) admits at most one stationary solution and has $\tau$ non-predetermined variables and no eigenvalue whose modulus is between 1 and $\mu$. Since there exists at least one (3) satisfying (1), due to assumption 3, proposition 3 follows.

## D Proof of proposition 4

Suppose $m_1^a>m_1^b$ and consider some given $n\in\mathbb{N}$ and $M\in\mathbb{R}^*$. Let us note $S_{n,M}$ the set of backward-looking rules of type (2) such that $n^f\leq n$, $n^g\leq n$, $g_0=1$, all $g_k$ for $k\in\{1,...,n^g\}$ (if $n^g\geq 1$) and all elements of $\mathbf{F}_k$ for $k\in\left\{0,...,n^f\right\}$ have an absolute value lower than $M$. Whatever the rule belonging to $S_{n,M}$ considered, the non-zero eigenvalues of the system made of (1) and this rule are those of the corresponding perfect-foresight deterministic system

$$\boldsymbol{\Psi}(L)\begin{bmatrix}\mathbf{Y}_t\\ z_t\end{bmatrix}=0\text{ where }\underset{((N+1)\times(N+1))}{\boldsymbol{\Psi}(L)}\equiv\sum_{k=0}^{n^\psi}\boldsymbol{\Psi}_kL^k\text{ with }\boldsymbol{\Psi}_0=\left[\begin{array}{c|c}\widehat{\mathbf{A}}(0) & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hline \mathbf{F}_0 & g_0\end{array}\right],$$

$n^\psi\in\mathbb{N}$, all $\boldsymbol{\Psi}_k$ have real numbers as elements and the zero elements in the last column of $\boldsymbol{\Psi}_0$ come from assumptions $m_1^a>m_1^b$ and 2.iii. Assumption 1.iii and the normalization $g_0=1$ make $\boldsymbol{\Psi}_0$ invertible, so that according to a standard matricial result of time series analysis (*cf* *e.g.* Hamilton, 1994, chap. 10, prop. 10.1) this system's eigenvalues are the roots of polynomial $\Phi(X) \equiv \left|X^{n^\psi}\boldsymbol{\Psi}\left(X^{-1}\right)\right| \in \mathbb{R}[X]$. Three results are then easily obtained: i) the coefficient $|\boldsymbol{\Psi}_0| = \left|\widehat{\mathbf{A}}(0)\right| g_0 = \left|\widehat{\mathbf{A}}(0)\right|$ of $X^{(N+1)n^\psi}$ in $\Phi(X)$ is non-zero and independent of the rule belonging to $S_{n,M}$ considered; ii) there exists $n' \in \mathbb{N}$ such that, whatever the rule belonging to $S_{n,M}$ considered, the degree $(N+1)\,n^\psi$ of $\Phi(X)$ is lower than $n'$; iii) partly as a consequence of the second result, there exists $M' \in \mathbb{R}^*$ such that, whatever the rule belonging to $S_{n,M}$ considered, all the coefficients of $\Phi(X)$ have an absolute value lower than $M'$. These three results together imply that there exists $\mu \in \mathbb{R}^+ \setminus [0;1]$ such that, whatever the rule belonging to $S_{n,M}$ considered, all the roots of $\Phi(X)$ have an absolute value lower than $\mu$ and, therefore, all the eigenvalues of the system made of (1) and this rule have a modulus lower than $\mu$. However, given assumptions 1 and 2 and since $m_1^a > m_1^b$, whatever the rule belonging to $S_{n,M}$ considered, the system made of (1) and this rule has at least one non-predetermined variable. As a consequence, there exists no rule belonging to $S_{n,M}$ and such that the system made of (1) and this rule admits a unique stationary solution and has no eigenvalue whose modulus is between 1 and $\mu$.

## E Proof of proposition 5

We proceed in three steps: first, we show that, with probability one, the system made of (1) and $(\widetilde{R})$ can be written in Blanchard and Kahn's (1980) form with at most $\tau$ non-predetermined variables; second, we show that this system admits a unique stationary solution and has no eigenvalue whose modulus is between 1 and $\mu$, with $\mu \longrightarrow +\infty$ as $\varepsilon \longrightarrow 0$; third, we show that $d(\mathbf{X}(L), \widetilde{\mathbf{X}}(L)) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

**Step 1**: consider a given system $(S)$ of type (1). Replace $E_t\left\{z_{t+m_1^b}\right\}$ in $\mathbf{e}_1'(S)$ by its expression in $E_t\left\{L^{-m_1^b}(\widetilde{R})\right\}$; if $m_1^a < m_1^b$, in which case $(\widetilde{R})$ is backward-looking, then replace sequentially $E_t\left\{z_{t+m_1^b-k}\right\}$ for $k \in \left\{1, ..., m_1^b - m_1^a\right\}$ (if they appear) in the resulting equation by their expressions in $E_t\left\{L^{k-m_1^b}(\widetilde{R})\right\}$; note $(\widetilde{E})$ the resulting equation. Consider

$$(\widetilde{S}) \equiv \left\{\begin{array}{c} (\widetilde{E}) \\ \mathbf{e}_2'(S) \\ \vdots \\ \mathbf{e}_N'(S) \end{array}\right. \quad \text{and } \widehat{\widetilde{\mathbf{A}}}(L) \equiv \left[\begin{array}{c} \mathbf{e}_1' L^{m_1^a} \\ \vdots \\ \mathbf{e}_N' L^{m_N^a} \end{array}\right] \widetilde{\mathbf{A}}(L)$$

where $\widetilde{\mathbf{A}}(L)$ is defined by writing $(\widetilde{S})$ in the form $E_t\left\{\widetilde{\mathbf{A}}(L)\mathbf{Y}_t + \widetilde{\mathbf{B}}(L)z_t\right\} + \widetilde{\mathbf{C}}(L)\boldsymbol{\xi}_t = \mathbf{0}$. Given that the probability distributions of the exogenous additive measurement errors are assumed to be continuous, $\widehat{\widetilde{\mathbf{A}}}(0)$ is invertible with probability one[14]. Rewrite then $(\widetilde{S})$ in a similar way as in step 1 of appendix C, with $(\widetilde{S})$, $\widetilde{\mathbf{A}}(L)$, $\widehat{\widetilde{\mathbf{A}}}(0)$ and $(\widetilde{R})$ playing the roles of $(S)$, $\mathbf{A}(L)$, $\widehat{\mathbf{A}}(0)$ and $(\widehat{R})$ respectively. If $m_1^a \leq m_1^b$ then this rewriting enables us to put the system made of $(\widetilde{S})$ and

[14] In the remaining of the proof, for simplicity, we may sometimes drop the expression "with probability one".

$(\widetilde{R})$ in Blanchard and Kahn's (1980) form since $(\widetilde{R})$ is backward-looking and since $m_i^a > m_i^b$ for $i \in I_B \setminus \{1\}$ due to assumption 2.iii. Alternatively, if $m_1^a > m_1^b$ then this rewriting also enables us to put the system made of $(\widetilde{S})$ and $(\widetilde{R})$ in Blanchard and Kahn's (1980) form, even though $(\widetilde{R})$ is forward-looking, because $m_i^a - m_i^b > m_1^a - m_1^b$ for $i \in I_B \setminus \{1\}$ due to assumption 2.iii and because the only variable of type $E_t\{z_{t+k}\}$ with $k \in \mathbb{N}$ appearing in the system made of the rewritten system $(\widetilde{S})$ and $(\widetilde{R})$ is $z_t$ in $(\widetilde{R})$. In both cases the number of non-predetermined variables is equal to $m \equiv \sum_{i=1}^{N} m_i^a$. Since the system made of $(S)$ and $(\widetilde{R})$ is equivalent to the system made of $(\widetilde{S})$ and $(\widetilde{R})$, we have thus shown that, with probability one, the system made of $(S)$ and $(\widetilde{R})$ can be written in Blanchard and Kahn's (1980) form with $m$ non-predetermined variables. Note finally that $m \leq \tau$.

**Step 2**: for any system or equation $(x)$, let $(\overline{x})$ denote the perfect-foresight deterministic form of $(x)$. The same reasoning as the one conducted at the beginning of appendix A, this time starting from $(\widetilde{E})$ instead of $(\overrightarrow{1})$ and using $(\widetilde{R})$ instead of $(R)$, leads to an equation $(\widetilde{\overrightarrow{N}})$, corresponding to equation $(\overrightarrow{N})$ in appendix A, such that $(\overline{\widetilde{\overrightarrow{N}}})$ is of the form

$$\widetilde{\mathbf{P}}(L)\,\mathbf{Y}_t + \widetilde{Q}(L)\,z_t = 0 \text{ with } \underset{(1\times N)}{\widetilde{\mathbf{P}}(L)} \equiv \sum_{k=-m}^{n^{\widetilde{p}}} \widetilde{\mathbf{P}}_k L^k \text{ and } \widetilde{Q}(L) \equiv \sum_{k=-m+1+\max\left[m_1^a - m_1^b,\ 0\right]}^{n^{\widetilde{q}}} \widetilde{q}_k L^k,$$

where $\left(n^{\widetilde{p}}, n^{\widetilde{q}}\right) \in \mathbb{N}^2$, all $\widetilde{\mathbf{P}}_k$ have real numbers as elements, all $\widetilde{q}_k$ are real numbers, $\widetilde{\mathbf{P}}_{-m} = \mathbf{e}_1' \widehat{\widetilde{\mathbf{A}}}(0)$, $d(\widetilde{\mathbf{P}}(L), \mathbf{P}(L)) \longrightarrow 0$ and $d(\widetilde{Q}(L), Q(L)) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. The non-zero eigenvalues of the system made of $(S)$ and $(\widetilde{R})$ are those of the system made of $(\overline{S})$ and $(\overline{\widetilde{R}})$ which in turn are those of the system made of $(\overline{S})$ and $(\overline{\widetilde{\overrightarrow{N}}})$. The latter system can be rewritten

$$\widetilde{\mathbf{\Gamma}}_1(L) \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} = \mathbf{0} \text{ with } \widetilde{\mathbf{\Gamma}}_1(L) \equiv \begin{bmatrix} L^m \widetilde{\mathbf{P}}(L) & L^m \widetilde{Q}(L) \\ \mathbf{e}_1' L^{\max\left[m_1^a, m_1^b\right]} \mathbf{A}(L) & \mathbf{e}_1' L^{\max\left[m_1^a, m_1^b\right]} \mathbf{B}(L) \\ \mathbf{e}_2' L^{m_2^a} \mathbf{A}(L) & \mathbf{e}_2' L^{m_2^a} \mathbf{B}(L) \\ \vdots & \vdots \\ \mathbf{e}_N' L^{m_N^a} \mathbf{A}(L) & \mathbf{e}_N' L^{m_N^a} \mathbf{B}(L) \end{bmatrix} \equiv \sum_{k=0}^{n^{\widetilde{\gamma}_1}} \widetilde{\mathbf{\Gamma}}_{1,k} L^k$$

where $n^{\widetilde{\gamma}_1} \in \mathbb{N}$ and all $\widetilde{\mathbf{\Gamma}}_{1,k}$ have real numbers as elements. Let us define

$$\underset{(N\times(N+1))}{\mathbf{J}_1} \equiv \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \quad \underset{((N+1)\times N)}{\mathbf{J}_2} \equiv \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$\underset{((N+1)\times 1)}{\mathbf{J}_3} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ and } \underset{(N\times(N+1))}{\mathbf{J}_4} \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

If $m_1^a \geq m_1^b$ then replace sequentially $\mathbf{Y}_{t-k}$ for $k \in \left\{0, ..., m_1^a - m_1^b\right\}$ in the second line of $\widetilde{\mathbf{\Gamma}}_1(L)\left[\begin{array}{cc}\mathbf{Y}_t & z_t\end{array}\right]' = \mathbf{0}$ by its expression in $(\mathbf{J}_1\widetilde{\mathbf{\Gamma}}_{1,0}\mathbf{J}_2)^{-1}\mathbf{J}_1\widetilde{\mathbf{\Gamma}}_1(L)L^k\left[\begin{array}{cc}\mathbf{Y}_t & z_t\end{array}\right]' = \mathbf{0}$, given that $|\mathbf{J}_1\widetilde{\mathbf{\Gamma}}_{1,0}\mathbf{J}_2| = |\widehat{\widehat{\mathbf{A}}}(0)| \neq 0$, and note $\widetilde{\mathbf{\Gamma}}_2(L)\left[\begin{array}{cc}\mathbf{Y}_t & z_t\end{array}\right]' = \mathbf{0}$ the resulting system, with $\widetilde{\mathbf{\Gamma}}_2(L) \equiv \sum_{k=0}^{n^{\widetilde{\gamma}_2}} \widetilde{\mathbf{\Gamma}}_{2,k}L^k$ where $n^{\widetilde{\gamma}_2} \in \mathbb{N}$ and all $\widetilde{\mathbf{\Gamma}}_{2,k}$ have real numbers as elements ($\widetilde{\mathbf{\Gamma}}_2(L) = \widetilde{\mathbf{\Gamma}}_1(L)$ if $m_1^a < m_1^b$). Given that $\mathbf{J}_1\widetilde{\mathbf{\Gamma}}_{1,k}\mathbf{J}_3 = \mathbf{0}$ for $k \in \left\{0, ..., \max\left[m_1^a - m_1^b, 0\right]\right\}$ due to assumption 2.iii, we have

$$\widetilde{\mathbf{\Gamma}}_{2,0} = \left[\begin{array}{cc}\mathbf{e}_1'\widehat{\widehat{\mathbf{A}}}(0) & 0 \\ \mathbf{0} & \mathbf{e}_1'\mathbf{B}_{-m_1^b} \\ \mathbf{e}_2'\widehat{\widehat{\mathbf{A}}}(0) & 0 \\ \vdots & \vdots \\ \mathbf{e}_N'\widehat{\widehat{\mathbf{A}}}(0) & 0\end{array}\right].$$

Since $\widehat{\widehat{\mathbf{A}}}(0)$ is invertible, $\widetilde{\mathbf{\Gamma}}_{2,0}$ is invertible as well so that according to a standard matricial result of time series analysis (*cf e.g.* Hamilton, 1994, chap. 10, prop. 10.1) the non-zero eigenvalues of $\widetilde{\mathbf{\Gamma}}_2(L)$, which are those of $\widetilde{\mathbf{\Gamma}}_1(L)$, are the roots of polynomial $\widetilde{\mathcal{E}}(X) \equiv \left|X^{n^{\widetilde{\gamma}_2}}\widetilde{\mathbf{\Gamma}}_2\left(X^{-1}\right)\right| \in \mathbb{R}[X]$. Now $\widetilde{\mathcal{E}}(X) = \widetilde{\mathcal{E}}_1(X) + \widetilde{\mathcal{E}}_2(X)$ where

$$\widetilde{\mathcal{E}}_1(X) \equiv \left|\begin{array}{c}X^{n^{\widetilde{\gamma}_2}-m}\sum_{k=-m}^{-1}\widetilde{\mathbf{P}}_kX^{-k} \quad X^{n^{\widetilde{\gamma}_2}-m}\sum_{k=-m}^{-1}\widetilde{q}_kX^{-k} \\ \mathbf{J}_4X^{n^{\widetilde{\gamma}_2}}\widetilde{\mathbf{\Gamma}}_2\left(X^{-1}\right)\end{array}\right|$$

$$\text{and } \widetilde{\mathcal{E}}_2(X) \equiv \left|\begin{array}{c}X^{n^{\widetilde{\gamma}_2}-m}\sum_{k=0}^{n^{\widetilde{p}}}\widetilde{\mathbf{P}}_kX^{-k} \quad X^{n^{\widetilde{\gamma}_2}-m}\sum_{k=0}^{n^{\widetilde{q}}}\widetilde{q}_kX^{-k} \\ \mathbf{J}_4X^{n^{\widetilde{\gamma}_2}}\widetilde{\mathbf{\Gamma}}_2\left(X^{-1}\right)\end{array}\right|.$$

If $m = 0$ then $\widetilde{\mathcal{E}}_1(X) = 0$. Otherwise the degree of $\widetilde{\mathcal{E}}_1(X)$ is equal to $n^{\widetilde{\gamma}_2}(N+1)$ since the coefficient of $X^{n^{\widetilde{\gamma}_2}(N+1)}$ in $\widetilde{\mathcal{E}}_1(X)$ is $\left|\widetilde{\mathbf{\Gamma}}_{2,0}\right| \neq 0$. For $\varepsilon$ sufficiently close to 0, the degree of $\widetilde{\mathcal{E}}_2(X)$ is equal to $n^{\widetilde{\gamma}_2}(N+1) - m$ since the coefficient of $X^{n^{\widetilde{\gamma}_2}(N+1)-m}$ in $\widetilde{\mathcal{E}}_2(X)$ is

$$\left|\begin{array}{cc}\widetilde{\mathbf{P}}_0 & \widetilde{q}_0 \\ \mathbf{0} & \mathbf{e}_1'\mathbf{B}_{-m_1^b} \\ \mathbf{e}_2'\widehat{\mathbf{A}}(0) & 0 \\ \vdots & \vdots \\ \mathbf{e}_N'\widehat{\mathbf{A}}(0) & 0\end{array}\right| \longrightarrow (-1)^{N+1}\mathbf{e}_1'\mathbf{B}_{-m_1^b}|\mathbf{\Omega}| \neq 0 \text{ as } \varepsilon \longrightarrow 0.$$

Let us note $\widetilde{x}_1$, ..., $\widetilde{x}_{n^{\widetilde{\gamma}_2}(N+1)}$ the roots of $\widetilde{\mathcal{E}}(X)$, ranked first by increasing modulus (*i.e.* $|\widetilde{x}_1| \leq ... \leq \left|\widetilde{x}_{n^{\widetilde{\gamma}_2}(N+1)}\right|$) and second by increasing complex argument (*i.e.* if $\exists i \in \left\{1, ..., n^{\widetilde{\gamma}_2}(N+1) - 1\right\}$, $|\widetilde{x}_i| = |\widetilde{x}_{i+1}|$, then $\varphi(\widetilde{x}_i) \leq \varphi(\widetilde{x}_{i+1})$, where $\varphi : \mathbb{C} \longrightarrow [0; 2\pi[$ denotes the complex argument function). Similarly, let us note $x_1$, ..., $x_n$ the non-zero eigenvalues of system (5) ranked first by increasing modulus and second by increasing complex argument, which are all of modulus strictly lower than one since $(R)$ satisfies condition 4. Since $\widetilde{\mathcal{E}}_1(X) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, we have

$$\left(\widetilde{x}_1, ..., \widetilde{x}_{n^{\widetilde{\gamma}_2}(N+1)-m}\right) \longrightarrow (0, ..., 0, x_1, ..., x_n) \text{ as } \varepsilon \longrightarrow 0$$

$$\text{and } \forall k \in \{0,...,m-1\},\ \left|\widetilde{x}_{n^{\widetilde{\gamma}_2}(N+1)-k}\right| \longrightarrow +\infty \text{ as } \varepsilon \longrightarrow 0,$$

which implies: i) that the system made of (1) and $(\widetilde{R})$ has no eigenvalue whose modulus is between 1 and $\mu$, with $\mu \longrightarrow +\infty$ as $\varepsilon \longrightarrow 0$; ii) given step 1, that this system admits either one or zero stationary solution, depending on whether Blanchard and Kahn's (1980) rank condition is satisfied or not. Since assumption 3 and the continuous-probability-distribution assumption together ensure that this rank condition is satisfied with probability one, we further get that this system admits one unique stationary solution.

**Step 3**: let us write equations $(\overrightarrow{k})$ for $k \in \{1,...,N\}$, obtained in appendix A, in the form

$$\overrightarrow{\mathbf{U}}(L)\,\mathbf{Y}_t + \overrightarrow{\mathbf{V}}(L)\,z_t + \overrightarrow{\mathbf{W}}(L)\,\boldsymbol{\xi}_t = \mathbf{0}$$

$$\text{with } \underset{(N\times N)}{\overrightarrow{\mathbf{U}}(L)} \equiv \sum_{k=0}^{n^{\overrightarrow{u}}} \overrightarrow{\mathbf{U}}_k L^k,\ \underset{(N\times 1)}{\overrightarrow{\mathbf{V}}(L)} \equiv \sum_{k=\max\left[0,m_1^a-m_1^b+1\right]}^{n^{\overrightarrow{v}}} \overrightarrow{\mathbf{V}}_k L^k \text{ and } \underset{(N\times N)}{\overrightarrow{\mathbf{W}}(L)} \equiv \sum_{k=0}^{n^{\overrightarrow{w}}} \overrightarrow{\mathbf{W}}_k L^k,$$

where $\left(n^{\overrightarrow{u}}, n^{\overrightarrow{v}}, n^{\overrightarrow{w}}\right) \in \mathbb{N}^3$ and all $\overrightarrow{\mathbf{U}}_k$, $\overrightarrow{\mathbf{V}}_k$, $\overrightarrow{\mathbf{W}}_k$ have real numbers as elements. For all $(i,j) \in \{1,...,N\}^2$ such that $i \leq j$, let $\zeta_{i,j}$ be defined by $\zeta_{i,j} = 1$ if $\forall k \in \{i,...,j\}$, $m_k^a = 0$ and $\zeta_{i,j} = 0$ otherwise. We then have

$$\overrightarrow{\mathbf{U}}(0) = \begin{bmatrix} \zeta_{2,N} & 1 & \zeta_{2,2} & \cdots & \zeta_{2,N-1} \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \zeta_{N-1,N-1} \\ \zeta_{N,N} & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} \mathbf{U}(0),$$

so that since $\mathbf{U}(0) = \boldsymbol{\Omega}$ is invertible, $\overrightarrow{\mathbf{U}}(0)$ is invertible as well. In this case, the same reasoning as the one conducted at the end of appendix A, this time using $\overrightarrow{\mathbf{U}}(L)$, $\overrightarrow{\mathbf{V}}(L)$ and $\overrightarrow{\mathbf{W}}(L)$ instead of $\mathbf{U}(L)$, $\mathbf{V}(L)$ and $\mathbf{W}(L)$, leads to a system of the form

$$E_t\left\{\boldsymbol{\Lambda}_1(L)\begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} + \boldsymbol{\Lambda}_2(L)\,\boldsymbol{\xi}_t\right\} = \mathbf{0} \tag{26}$$

$$\text{with } \underset{(N+1)\times(N+1)}{\boldsymbol{\Lambda}_1(L)} \equiv \sum_{k=0}^{n^{\lambda_1}} \boldsymbol{\Lambda}_{1,k} L^k \text{ and } \underset{(N+1)\times N}{\boldsymbol{\Lambda}_2(L)} \equiv \sum_{k=0}^{n^{\lambda_2}} \boldsymbol{\Lambda}_{2,k} L^k,$$

where $\left(n^{\lambda_1}, n^{\lambda_2}\right) \in \mathbb{N}^2$, all $\boldsymbol{\Lambda}_{1,k}$, $\boldsymbol{\Lambda}_{2,k}$ have real numbers as elements, $\boldsymbol{\Lambda}_{1,0}$ is invertible and all eigenvalues of $\boldsymbol{\Lambda}_1(L)$ are of modulus strictly lower than one. Since (26) is equivalent to the system made of $(S)$ and $(R)$, (10) is the unique solution of (26).

Similarly, let us follow the same reasoning as the one conducted at the beginning of appendix A, this time starting from $(\widetilde{E})$ instead of $(\overrightarrow{1})$ and using $(\widetilde{R})$ instead of $(R)$, to get equations $(\widetilde{\overrightarrow{2}})$ to $(\widetilde{\overrightarrow{N}})$ corresponding to equations $(\overrightarrow{2})$ to $(\overrightarrow{N})$ in appendix A. Equations $(\widetilde{E})$ and $(\widetilde{\overrightarrow{k}})$ for $k \in \{2,...,N\}$ can then be re-written in the form

$$E_t\left\{\widetilde{\overrightarrow{\mathbf{U}}}(L)\,\mathbf{Y}_t + \widetilde{\overrightarrow{\mathbf{V}}}(L)\,z_t + \widetilde{\overrightarrow{\mathbf{W}}}(L)\,\boldsymbol{\xi}_t\right\} = \mathbf{0}$$

$$\text{with } \underset{(N\times N)}{\widetilde{\overrightarrow{\mathbf{U}}}(L)} \equiv \sum_{k=-m^{\widetilde{\overrightarrow{u}}}}^{n^{\widetilde{\overrightarrow{u}}}} \widetilde{\overrightarrow{\mathbf{U}}}_k L^k,\ \underset{(N\times 1)}{\widetilde{\overrightarrow{\mathbf{V}}}(L)} \equiv \sum_{k=-m^{\widetilde{\overrightarrow{v}}}}^{n^{\widetilde{\overrightarrow{v}}}} \widetilde{\overrightarrow{\mathbf{V}}}_k L^k \text{ and } \underset{(N\times N)}{\widetilde{\overrightarrow{\mathbf{W}}}(L)} \equiv \sum_{k=-m^{\widetilde{\overrightarrow{w}}}}^{n^{\widetilde{\overrightarrow{w}}}} \widetilde{\overrightarrow{\mathbf{W}}}_k L^k,$$

where $(m^{\widetilde{\overrightarrow{u}}}, m^{\widetilde{\overrightarrow{v}}}, m^{\widetilde{\overrightarrow{w}}}, n^{\widetilde{\overrightarrow{u}}}, n^{\widetilde{\overrightarrow{v}}}, n^{\widetilde{\overrightarrow{w}}}) \in \mathbb{N}^6$, all $\widetilde{\overrightarrow{\mathbf{U}}}_k$, $\widetilde{\overrightarrow{\mathbf{V}}}_k$, $\widetilde{\overrightarrow{\mathbf{W}}}_k$ have real numbers as elements, $d(\widetilde{\overrightarrow{\mathbf{U}}}(L), \overrightarrow{\mathbf{U}}(L)) \longrightarrow 0$, $d(\widetilde{\overrightarrow{\mathbf{V}}}(L), \overrightarrow{\mathbf{V}}(L)) \longrightarrow 0$ and $d(\widetilde{\overrightarrow{\mathbf{W}}}(L), \overrightarrow{\mathbf{W}}(L)) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Given that $\widetilde{\overrightarrow{\mathbf{U}}}(0) \longrightarrow \overrightarrow{\mathbf{U}}(0)$ as $\varepsilon \longrightarrow 0$, $\widetilde{\overrightarrow{\mathbf{U}}}(0)$ is invertible for $\varepsilon$ sufficiently small, so that the same reasoning as the one conducted at the end of appendix A leads to a system of the form

$$E_t \left\{ \widetilde{\mathbf{\Lambda}}_1(L) \begin{bmatrix} \mathbf{Y}_t \\ z_t \end{bmatrix} + \widetilde{\mathbf{\Lambda}}_2(L)\, \boldsymbol{\xi}_t \right\} = \mathbf{0} \tag{27}$$

$$\text{with } \underset{(N+1)\times(N+1)}{\widetilde{\mathbf{\Lambda}}_1(L)} \equiv \sum_{k=-m^{\widetilde{\lambda}_1}}^{n^{\widetilde{\lambda}_1}} \widetilde{\mathbf{\Lambda}}_{1,k} L^k \text{ and } \underset{(N+1)\times N}{\widetilde{\mathbf{\Lambda}}_2(L)} \equiv \sum_{k=-m^{\widetilde{\lambda}_2}}^{n^{\widetilde{\lambda}_2}} \widetilde{\mathbf{\Lambda}}_{2,k} L^k,$$

where $\left(m^{\widetilde{\lambda}_1}, m^{\widetilde{\lambda}_2}, n^{\widetilde{\lambda}_1}, n^{\widetilde{\lambda}_2}\right) \in \mathbb{N}^4$, all $\widetilde{\mathbf{\Lambda}}_{1,k}$, $\widetilde{\mathbf{\Lambda}}_{2,k}$ have real numbers as elements, $d(\widetilde{\mathbf{\Lambda}}_1(L), \mathbf{\Lambda}_1(L)) \longrightarrow 0$ and $d(\widetilde{\mathbf{\Lambda}}_2(L), \mathbf{\Lambda}_2(L)) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Since (27) is implied by the system made of $(S)$ and $(\widetilde{R})$, (11) is one solution of (27).

Finally, let us note $(\theta_1, ..., \theta_{n^\theta})$, where

$$n^\theta \equiv (m^a + n^a + 1) N^2 + \left(m^b + n^b + 1\right) N + (n^c + 1) N^2 + \left(n^d + 1\right) N,$$

the list (in a given order) of the true structural parameters, *i.e.* the elements of $\mathbf{A}_k$ for $-m^a \leq k \leq n^a$, $\mathbf{B}_k$ for $-m^b \leq k \leq n^b$ and $\mathbf{C}_k$ for $0 \leq k \leq n^c$ and $d_{i,k}$ for $1 \leq i \leq N$ and $0 \leq k \leq n^d$. Similarly, let us note $(\widetilde{\theta}_1, ..., \widetilde{\theta}_{n^\theta})$ the list (in the corresponding order) of the measured structural parameters, and let us consider a given infinite sequence of $(\widetilde{\theta}_1, ..., \widetilde{\theta}_{n^\theta})$ converging towards $(\theta_1, ..., \theta_{n^\theta})$. This sequence corresponds to a unique sequence of $\varepsilon$ converging towards zero and a unique sequence of $\widetilde{\mathbf{X}}(L)$. If $\widetilde{\mathbf{X}}_0$ did not converge towards $\mathbf{X}_0$ along this sequence of $\widetilde{\mathbf{X}}(L)$, then there would exist a strictly positive real number $\omega_0$ and an extracted sequence of $(\widetilde{\theta}_1, ..., \widetilde{\theta}_{n^\theta})$ such that $\left\|\widetilde{\mathbf{X}}_0 - \mathbf{X}_0\right\| \geq \omega_0$ for every element of the corresponding extracted sequence of $\widetilde{\mathbf{X}}(L)$, where $\|.\|$ denotes a given norm on matrices. From (26) and (27) it is easy to see, but tedious to show formally, that for any element of this extracted sequence of $(\widetilde{\theta}_1, ..., \widetilde{\theta}_{n^\theta})$ sufficiently close to $(\theta_1, ..., \theta_{n^\theta})$ there would then exist a strictly increasing sequence extracted from the sequence $(\left\|\widetilde{\mathbf{X}}_k\right\|)_{k\in\mathbb{N}}$ corresponding to this element, which is impossible given that $\widetilde{\mathbf{X}}_k \longrightarrow \mathbf{0}$ as $k \longrightarrow +\infty$, so that we conclude that $\widetilde{\mathbf{X}}_0 \longrightarrow \mathbf{X}_0$ along the sequence of $\widetilde{\mathbf{X}}(L)$ considered. By the same reasoning we obtain that $\forall k \in \mathbb{N}$, if $(\widetilde{\mathbf{X}}_0, ..., \widetilde{\mathbf{X}}_k) \longrightarrow (\mathbf{X}_0, ..., \mathbf{X}_k)$ along the sequence of $\widetilde{\mathbf{X}}(L)$ considered then $\widetilde{\mathbf{X}}_{k+1} \longrightarrow \mathbf{X}_{k+1}$ along this sequence. By recurrence on $k \in \mathbb{N}$ we therefore conclude that $\forall k \in \mathbb{N}$, $\widetilde{\mathbf{X}}_k \longrightarrow \mathbf{X}_k$ along this sequence. Given that there exists $(\overline{p}, \overline{q}) \in \mathbb{N}^2$ such that every element of the sequence of $\widetilde{\mathbf{X}}(L)$ considered is the Wold form of a VARMA$(p, q)$ process with $p \leq \overline{p}$ and $q \leq \overline{q}$, as implied by Blanchard and Kahn's (1980) results in our context, this "simply continuous" convergence ($\forall k \in \mathbb{N}$,

$\widetilde{\mathbf{X}}_k \longrightarrow \mathbf{X}_k$) implies in turn the "absolutely continuous" convergence $d(\widetilde{\mathbf{X}}\,(L)\,,\mathbf{X}\,(L)) \longrightarrow 0$ along the sequence of $\widetilde{\mathbf{X}}\,(L)$ considered.