Oligopoly Equilibria ’’*à la* Stackelberg’’

in Pure Exchange Economies

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Abstract. This paper introduces two equilibrium concepts which extend the notion of Stackelberg competition to cover a general equilibrium framework. In the framework of a pure exchange economy, asymptotic identification and welfare results are obtained.

1. Introduction

The Stackelberg concept of equilibrium has mainly been developed in production economies under partial equilibrium analysis (Tirole (1988)). The purpose of this paper is to insert the Stackelberg market structure in a pure exchange general equilibrium framework, which enables to capture the features of market power and the diversity of strategic interactions. We use the framework of oligopolistic competition developed by Codognato-Gabszewicz ((1991), (1993)), Gabszewicz-Michel (1997) and then pursued by Gabszewicz (2002).

According to the competition ‘*à la* Walras’, all the individuals behave the same non strategic way and all the sectors work the same perfect way. This double symmetry does not stand with the competition ‘*à la* Cournot’: an asymmetric treatment of the sectors is introduced, some being oligopolistic and others staying competitive. But the symmetry remains in the treatment of every individual on a given sector. When the competition ‘*à la* Stackelberg’ is introduced, a double asymmetry is possible: between the oligopolisitic and the competitive sectors, and moreover between the leader and the follower(s) in one same sector. It is then possible to associate a relative advantage for one sector upon another (uneven distribution of market power among the sectors) and a relative advantage for an agent upon another (uneven distribution of market power among the agents of a given sector).

Two concepts of Stackelberg general equilibrium are developed: the Stackelberg-Walras Equilibrium and the Stackelberg-Cournot Equilibrium. We compare these equilibria and obtain several results about asymptotic identifications and welfare.

2. A two-commodity economy

Consider a pure exchange economy with two consumption goods (1 and 2) and *n+2* consumers. It is assumed that good 2 is taken as the *numéraire*, so p is the price of good 1 as expressed in units of good 2.

The preferences of every consumer are represented by the same utility function:

$$U_h = x_{h1}x_{h2} \quad ,\forall h . \tag{1}$$

The structure of initial endowments in sector 1 and in sector 2 is assumed to be, respectively:

$$\omega_1 = (\alpha,0) \quad \text{and} \quad \omega_2 = (1-\alpha,0), \text{ with } \alpha \in (0,1) \tag{2}$$

$$\omega_h = \left(0,\frac{1}{n}\right) \quad , \; h = 3,..,n+2 . \tag{3}$$

In the first sector, each agent is an oligopolist: agent 1 is the leader and agent 2 is the follower. The pure strategies of agents 1 and 2 are denoted s_{11}, with $s_{11} \in [0,\alpha]$, and e_{21}, with $e_{21} \in [0,1-\alpha]$. In the second sector, agents are either price-takers or Cournotian oligopolists. The resulting equilibria are the Stackelberg Walras equilibrium (SWE) in the former case and the Stackelberg Cournot equilibrium (SCE) in the latter case. We study these two concepts of Stackelberg equilibria for pure exchange economies and compare them with the Cournot equilibrium (CE) and the Cournot-Walras equilibrium (CWE).

In the SWE framework, it is considered that agents having endowments in good 2 act competitively, whereas the other agents behave strategically. The story is solved by backward induction, considering first the behavior of the Walrasian agents, then the decision of the follower, and finally the choice of the leader.

The competitive plans of owners of good 2 come from a non-strategic maximization of the utility function subject to the budget constraint, i.e. $\underset{\{z_{h2}\}}{Arg\max}\left(\frac{1}{p}z_{h2}\right)\left(\frac{1}{n}-z_{h2}\right)$, $h = 3,...,n+2$, where z_{h2} represents the competitive supply of good 2 by agent *h*, $h = 3,...,n+2$. From (1) and (3), we deduce the competitive individual offer plan $z_{h2} = 1/(2n)$ and the demand functions $(x_{h1}, x_{h2}) = \left(\frac{1}{2np},\frac{1}{2n}\right)$, $h = 3,...,n+2$.

The aggregate demand in good 1 by owners of good 2 is $\sum_{h=3}^{h=n+2} x_{h1} = \sum_{h=3}^{h=n+2}[1/(2np)]$. The equilibrium price is then given by $\sum_{h=3}^{h=n+2}[1/(2np)] = s_{11} + e_{21}$, so $p = 1/[2(s_{11} + e_{21})]$.

The strategic plan of the follower is determined by two elements: she manipulates the market price and she takes the leader's strategy as given. Thus the follower's program is:

$$\underset{\{e_{21}\}}{Arg\max}\,\frac{1}{2}\big((1-\alpha)-e_{21}\big)\left(\frac{e_{21}}{s_{11}+e_{21}}\right), \tag{4}$$

which gives the following reaction function:

$$e_{21}(s_{11}) = \sqrt{s_{11}^2 + (1-\alpha)s_{11}} - s_{11} \tag{5}$$

We verify that this function is continuous and increasing, with $\partial e_{21}/\partial s_{11} > 0$ and $\partial^2 e_{21}/\partial s_{11}^2 < 0$. Moreover, $\partial e_{21}/\partial\alpha < 0$.

The strategic plan of the leader is determined by two elements: she manipulates the market price and the follower's strategy. The leader thus solves the following program:

$$\underset{\{s_{11}\}}{Arg\max}\,\frac{1}{2}(\alpha - s_{11})\left(\frac{s_{11}}{s_{11}+e_{21}(s_{11})}\right) , \tag{6}$$

which gives the following optimal strategy:

$$\tilde{s}_{11} = \frac{1}{4}\left[\varphi - 3(1-\alpha)\right] \quad , \tag{7}$$

where $\varphi \equiv \sqrt{(1-\alpha)(9-\alpha)}$, with $\varphi \in (0,3)$. We verify that this function is continuous and increasing, with $\partial\tilde{s}_{11} / \partial\alpha > 0$.

We can deduce the value of the follower's strategy $\tilde{e}_{21} = e_{21}(\tilde{s}_{11})$:

$$\tilde{e}_{21} = \frac{1}{4}\{\psi - \left[\varphi - 3(1-\alpha)\right]\} \ , \tag{8}$$

where $\psi \equiv \sqrt{(\varphi - 3(1-\alpha))(\varphi + (1-\alpha))}$.

Since $z_{h2} = 1/(2n)$, it is now possible to determine the equilibrium price:

$$\tilde{p} = \frac{2}{\psi} \ , \tag{9}$$

The individual allocations are thus:

$$(\tilde{x}_{11}, \tilde{x}_{12}) = \left(\frac{1}{4}(3+\alpha-\varphi), \frac{\varphi - 3(1-\alpha)}{2\psi}\right) \tag{10}$$

$$(\tilde{x}_{21}, \tilde{x}_{22}) = \left(\frac{1}{4}\left(\varphi - \psi + (1-\alpha)\right), \frac{\psi - \left[\varphi - 3(1-\alpha)\right]}{2\psi}\right) \tag{11}$$

$$(\tilde{x}_{h1}, \tilde{x}_{h2}) = \left(\frac{\psi}{4n}, \frac{1}{2n}\right) \quad , h = 3,..,n+2 \tag{12}$$

The associated utility levels are respectively:

$$\tilde{U}_1 = \frac{(3+\alpha-\varphi)(\varphi - 3(1-\alpha))}{8\psi} \tag{13}$$

$$\tilde{U}_2 = \left(\frac{1}{4}\left(\varphi - \psi + (1-\alpha)\right)\right)\left(\frac{\psi - \left[\varphi - 3(1-\alpha)\right]}{2\psi}\right) \tag{14}$$

$$\tilde{U}_h = \frac{\psi}{8n^2} \quad , h = 3,..,n+2 \tag{15}$$

In the SCE framework, it is considered that all agents behave strategically, with agent 1 as the only leader. The only difference with the previous case is that the agents endowed in good 2 behave oligopolistically. The story is solved by backward induction, considering first the decisions of the *(n+1)* Cournotian agents, and finally the choice of the leader.

We denote e_{h2} the pure strategy of agent *h*, $h = 3,...,n+2$, with $e_{h1} \in \left[0,1/n\right]$. The market price is given by $p = \frac{\sum_{h=3}^{h=n+2} e_{h2}}{s_{11} + e_{21}}$, which insures the market clearing.

Taking the $(n-1)e_{-h2}$, s_{11} , and e_{21} as given, each strategist *h* of sector 2 maximizes her utility:

$$\underset{\{e_{h2}\}}{Arg\max}\left(\frac{\left(s_{11} + e_{21}\right)}{e_{h2} + (n-1)e_{-h2}} e_{h2}\right)\left(\frac{1}{n} - e_{h2}\right) \quad , h = 3,...,n+2 \, , \tag{16}$$

which gives the following reaction function:

$$e_{h2} = \frac{n-1}{n(2n-1)} \quad , \ h = 3,...,n+2 \tag{17}$$

Taking the strategy s_{11} and the *n* strategies e_{h2} as given, the Cournotian follower of sector 1 maximizes her utility:

$$\underset{\{e_{21}\}}{Arg\max}\left((1-\alpha) - e_{21}\right)\frac{\sum_{h=3}^{h=n+2} e_{h2}}{s_{11} + e_{21}} e_{21} \ , \tag{18}$$

which gives the following reaction function:

$$e_{21}(s_{11}) = \sqrt{s_{11}^2 + (1-\alpha)s_{11}} - s_{11} \ . \tag{19}$$

Considering the best responses of all the followers, the leader maximizes her utility:

$$\underset{\{s_{11}\}}{Arg\max}\left(\alpha - s_{11}\right)\frac{\sum_{h=3}^{h=n+2} e_{h2}}{s_{11} + e_{21}(s_{11})} s_{11} \ , \tag{20}$$

which gives the optimal strategy:

$$\hat{s}_{11} = \frac{1}{4}\left[\varphi - 3(1-\alpha)\right] \tag{21}$$

The values of the Cournotian strategies follow:

$$\hat{e}_{21} = \frac{1}{4}\{\psi - \left[\varphi - 3(1-\alpha)\right]\} \tag{22}$$

$$\hat{e}_{h2} = \frac{n-1}{n(2n-1)} \quad , \ h = 3,...,n+2 \tag{23}$$

The equilibrium price $\hat{p} = \frac{\sum_{h=3}^{h=n+2} \hat{e}_{h2}}{\hat{s}_{11} + \hat{e}_{21}} = \frac{4(n-1)}{(2n-1)} \frac{1}{\psi}$ can be written:

$$\hat{p} = \frac{2(n-1)}{2n-1} \tilde{p} \tag{24}$$

The individual allocations are thus:

$$(\hat{x}_{11}, \hat{x}_{12}) = \left(\tilde{x}_{11}, \frac{2(n-1)}{2n-1}\tilde{x}_{12}\right) \tag{25}$$

$$(\hat{x}_{21}, \hat{x}_{22}) = \left(\tilde{x}_{21}, \frac{2(n-1)}{2n-1}\tilde{x}_{22}\right) \tag{26}$$

$$(\hat{x}_{h1}, \hat{x}_{h2}) = \left(\frac{n}{n-1}\tilde{x}_{h1}, \frac{2n}{2n-1}\tilde{x}_{h2}\right) \ , h = 3,...,n+2 \tag{27}$$

The utility levels reached are respectively:

$$\hat{U}_1 = \frac{2(n-1)}{2n-1}\tilde{U}_1 \tag{28}$$

$$\hat{U}_2 = \frac{2(n-1)}{2n-1}\tilde{U}_2 \tag{29}$$

$$\hat{U}_h = \frac{2n^2}{(n-1)(2n-1)}\tilde{U}_h \quad , \ h = 3,...,n+2 \tag{30}$$

Proposition 1. *When the number of agents tends to infinity, the Stackelberg-Cournot equilibrium identifies to the Stackelberg-Walras equilibrium.*

Proof. We have to show that the equilibrium price and optimal allocations in sector 2 converge toward the Stackelberg-Walras one when *n* becomes large. For the equilibrium price, we have $\lim_{n\to\infty}\left(\frac{2(n-1)}{2n-1}\right)\tilde{p} = \tilde{p}$. For the individual allocations, as $(\hat{x}_{11}, \hat{x}_{12}) = \left(\tilde{x}_{11}, \frac{2(n-1)}{2n-1}\tilde{x}_{12}\right)$ for the leader, $(\hat{x}_{21}, \hat{x}_{22}) = \left(\tilde{x}_{21}, \frac{2(n-1)}{2n-1}\tilde{x}_{22}\right)$ for the second agent and $(\hat{x}_{h1}, \hat{x}_{h2}) = \left(\frac{n}{n-1}\tilde{x}_{h1}, \frac{2n}{2n-1}\tilde{x}_{h2}\right)$, $h = 3,...,n+2$, it is obvious that

$\lim_{n\to\infty}(\hat{x}_{11},\hat{x}_{12})=(\tilde{x}_{11},\tilde{x}_{12})$ for the leader, $\lim_{n\to\infty}(\hat{x}_{21},\hat{x}_{22})=(\tilde{x}_{21},\tilde{x}_{22})$ for the second agent and that $\lim_{n\to\infty}(\hat{x}_{h1},\hat{x}_{h2})=(\tilde{x}_{h1},\tilde{x}_{h2})$ for $h=3,...,n+2$. This completes the proof.

Proposition 1 underlines that the market power of each oligopolist decreases when the number of agents increases unboundedly. Hence, when *n* goes to infinity, the Cournotian behavior tends to the Walrasian one.

We can also notice that these optimal strategies correspond to the competitive plans. Consider now that $\tilde{p}$ is taken as given by each agent *h*, $h=3,...,n+2$. We have to show that this price is associated with the competitive plans for the remaining agents. The optimal plans come from a non-strategic maximization of the utility subject to the budget constraint, i.e. $Arg \max_{\{z_{h2}\}} x_{h1}x_{h2}$ subject to $\tilde{p}x_{h1}+x_{h2}\leq 1/n$ for $h=3,...,n+2$. This leads to $(\tilde{x}_{21},\tilde{x}_{22})=\left(\frac{1}{2n\tilde{p}},\frac{1}{2n}\right)$, $h=3,...,n+2$.

Proposition 2. *There is no Pareto domination between the Stackelberg-Walras and the Stackelberg-Cournot equilibria.*

Proof. From (28), (29) and (30) we have $\hat{U}_1=\frac{2(n-1)}{2n-1}\tilde{U}_1$, $\hat{U}_2=\frac{1}{n(2n-1)}\tilde{U}_2$ and $\hat{U}_h=\frac{4n}{2n-1}\tilde{U}_h$, $h=3,...,n+2$. As $\frac{2(n-1)}{2n-1}<1$, $\frac{1}{n(2n-1)}<1$ and $\frac{2n^2}{(n-1)(2n-1)}>1$, we have $\tilde{U}_1>\hat{U}_1$, $\tilde{U}_2>\hat{U}_2$ and $\tilde{U}_h<\hat{U}_h$, $h=3,...,n+2$. This completes the proof.

Proposition 2 captures that strategic agents of the first sector do better when they face competitive agents than when they struggle with strategic agents. And those agents of the second sector compete better under a Cournotian behavior than under a Walrasian one.

Conclusion

Leaving the Walrasian equilibrium means introducing in some way what is excluded from perfect competition: the strategic interactions. Whereas the CWE and the CE introduce only one kind of strategic behavior, the SWE and the SCE involve two types of this game-theoretic behavior: the active leader's one and the reactive follower's one. The SWE is especially interesting, as it displays three kinds of decision making mode: the competitive one, the monopolistic one and the strategic/parametric one.

References

Amir R., Grilo I. (1999), Stackelberg versus Cournot equilibrium, *Games and Economic Behavior*, 36, pp. 1-21.

Codognato, J., Gabszewicz, J.J. (1991),''Equilibres de Cournot-Walras dans une économie d'échange pure'', *Revue Economique*, 42, pp. 1013-1026.

Cournot A.A. (1838), *Les recherches sur les principes mathématiques de la théorie des richesses*, Dunod (2001).

Friedman, J. (1993), ''Oligopoly theory'', in K.J. Arrow and M.D. Intriligator (Eds), *Handbook of Mathematical Economics*, Elsevier, chapter 11, Volume 2, pp. 491-534.

Gabszewicz, J.J. (2002), *Strategic multilateral exchange, general equilibrium with imperfect competition*, Edward-Elgar, Cheltenham.
Gabszewicz, J.J., Michel, P. (1997), "Oligopoly equilibria in exchange economies", in B.C. Eaton and R.G. Harris (Eds), *Trade, Technology and Economics*. Essays in Honour of R.G. Lipsey, Edward-Elgar, Cheltenham, pp. 217-240.
Gabszewicz, J.J., Vial, J.P. (1972), "Oligopoly 'à la Cournot' in a general equilibrium analysis'', *Journal of Economic Theory*, 4, pp. 381-400.
Shapley, L., Shubik, M. (1977), "Trade using one commodity as a means of payment", *Journal of Political Economy*, 85, pp. 937-967.
Tirole J. (1988), *Theory of industrial organization*, MIT Press.
von Stackelberg H. (1934), *Marktform und Gleichgewicht*, Berlin & Vienna, Springer.