On the Bertrand core and equilibrium of a market

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Abstract

A striking result in economic theory is that price competition between a small number of sellers producing a homogeneous good may result in the perfectly competitive market outcome. We analyze the formation of price-making contracts when there is the possibility of coalitional deviations from the market. We consider a market with a finite number of buyers and sellers and standard market primitives. In this context we introduce a new core notion which we term the *Bertrand core*. A trading price is said to be in the Bertrand core if all sellers quoting this price constitutes an equilibrium *and* no subset of traders, buyers and sellers, can leave the market and improve their outcomes by trading by themselves. Under standard assumptions we show that the Bertrand core is non-empty. Moreover, we are able to obtain a partial equilibrium analogue of the well-known Debreu-Scarf (1963) result by showing that as the set of market traders is replicated then any price other than the competitive equilibrium can be blocked by some subset of traders provided that the market is replicated sufficiently many times.

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1 Introduction

A problematic issue in economic theory is the study of price-making behaviour and the formation of price-making contracts. The original insight by Joseph Bertrand (1883), and the later formalization of his insight, showed that subject to certain technical conditions, such as smoothness of market demand and constant returns to scale costs, price competition between two or more sellers is sufficient to obtain the competitive equilibrium of a market. However, this outcome is well-known to fail under different market conditions such as when sellers have limited capacities or decreasing returns to scale costs.[1] We reconsider the problem of establishing what price a homogeneous good might be traded at in a market where sellers have strictly convex costs and act as strategic price-makers. The difference in this paper is that we introduce the possibility that traders may choose to form coalitions and trade by themselves. To study which prices may result in the market we introduce a new core concept which we term the *Bertrand core*. A trading price is said to be in the Bertrand core if it constitutes a pure strategy Bertrand equilibrium for the grand coalition *and* no subset of buyers and sellers can improve their outcomes trading by themselves.

The Bertrand core is an original combination of the classical ideas of Bertrand and Edgeworth. It is well-known that Edgeworth criticized Bertrand's insight regarding price competition which resulted in the study of markets with capacity constraints and decreasing returns to scale costs. However, Edgeworth's other seminal insight, that of the core of an economy, introduced in Edgeworth (1881), has tended to be studied solely in the context of general equilibrium exchange or cooperative game theory. This paper combines Edgeworth's insight regarding the core with Bertrand price competition. Mas-Colell et al.(1995, p.655) note that there is a close relationship between Bertrand price competition and the market competition in the Edgeworth core.[2] The seminal result of Debreu and Scarf (1963) showed that as an economy is replicated the only allocations which remain in the core are Walrasian allocations.[3] In this paper we find that there are some deep similarities between the Edgeworth core and the Bertrand core. Whereas Walrasian allocations always belong to

[1]For a succinct summary of the Bertrand model see Vives (1999, Ch.5) or Baye and Kovenock (2008).

[2]At a technical level the models display a number of similarities. Walrasian allocations belong to the Edgeworth core and competitive equilibria belong to the set of Bertrand equilibria (subject to the sharing rule). Moreover, generically the Edgeworth core has uncountably many allocations and there are generically uncountably many Bertrand equilibrium prices.

[3]This result still holds even if traders increase arbitrarily provided that all traders do not vanish as a fraction of the limit economy (Hildenbrand and Kirman, 1988, pp.190-9).

the Edgeworth core we show that price-taking equilibria always belong to the Bertrand core. Moreover, we establish a partial equilibrium analogue of the Debreu-Scarf result: as the number of traders in the market is replicated the only price which remains in the Bertrand core is the competitive equilibrium. Remarkably, this result remains valid even when the limit market possesses uncountably many pure strategy Bertrand equilibria. Therefore, we are able to provide a new strategic foundation for price-taking behaviour in large markets.

This work is related to a number of papers which have considered strategic price-making foundations of competitive equilibrium. Dixon (1992) analyzed a model where sellers had symmetric, strictly convex costs and showed that if sellers post prices and can commit to supplying a quantity greater than their competitive supplies, subject to a no-bankruptcy condition, then the price-taking equilibrium can be sustained as a pure strategy Nash equilibrium. A sufficient condition for this was found to be that all but one seller could supply the market demand at the competitive price without incurring a loss. In an influential paper, Dastidar (1995) considered price competition, with a commitment to supply all demand forthcoming, between sellers with strictly convex costs. In a market with symmetric sellers and equal sharing at prices ties it was shown that there are uncountably many pure strategy Bertrand equilibria and the competitive equilibrium belongs to the set (Vives, 1999, p.122). Chowdhury and Sengupta (2004) considered when the refinement of coalition-proofness reduces the equilibrium set in standard Bertrand games. It was established that if sellers have symmetric costs then the game admits a unique coalition-proof Bertrand equilibrium. They showed that if one considers sequences of economies then as the number of sellers in the market becomes large the set of coalition-proof equilibria coincides with the competitive equilibrium of the market provided all sellers are active in the limit. Yano (2006a) analyzed a market model with free entry where sellers had u-shaped average costs. Sellers posted prices and a set of quantities they were willing to sell at the posted prices. It was shown that under certain conditions the competitive outcome is a Nash equilibrium of the game despite only a small number of sellers being active in the market. In a related paper, Yano (2006b) showed that the Bertrand paradox and Edgeworth criticism could be obtained as special cases of the game where sellers post prices and quantities.

We follow the tradition of these papers by analyzing price competition between sellers producing a single perfectly homogeneous good. However, unlike most of the previous literature, we model the demand side of the market in an explicit manner by assuming that there is a finite number of buyers. This framework then permits a rich set of trading possibilities as any subset of buyers and sellers could trade by themselves. We also allow for asymmetries between buyers and sellers so the model imposes few restrictions upon buyers' market demands and sellers' cost functions.

The notion of the Bertrand core introduced here brings new insights to the types of market contracts which price-setting sellers make with buyers. Traditionally, two different approaches have been considered in the literature. First, Bertrand competition assumes that sellers post a price in the market with a commitment to supply all the demand forthcoming from buyers.[4] Second, Bertrand-Edgeworth competition assumes that sellers post prices but do not give any commitment to supply any quantity demanded so that sellers would never produce more than their competitive supply at any given price.[5] The model presented here assumes that the market contracts may be somewhere between these two extremes in that sellers may make contracts with specific buyers to supply all demand forthcoming from these buyers, which may be more than their competitive supply, but that sellers make no commitment to buyers in the market with whom they do not trade. Therefore the market contracts may have elements of both Bertrand competition and Edgeworth competition. A market contact which is in the Bertrand core is immune to a group of traders leaving the market and forming contracts in this way. We also consider which types of contracts remain in the Bertrand core when sellers can communicate with each other, and exhibit limited cooperation, but cannot form binding agreements. To study this possibility we introduce the concept of the coalition-proof Bertrand core which combines the possibility of coalitional improvements with the notion of coalition-proofness analyzed by Chowdhury and Sengupta (2004).

In the next section we introduce standard mathematical notation used throughout the rest of the paper. In the following section we present the market model, define the Bertrand core and present the results. The final section presents some suggestions for future research.

2 Notation

The following notation is used throughout the rest of the paper.
$\Re^n$ denotes n-dimensional Euclidean space.
$\Re^n_+$ is the non-negative orthant of $\Re^n$.
2^X denotes all the subsets of X.

[4]This assumption is sometimes justified on the basis that there may be large costs involved in turning customers away (Dixon, 1990).

[5]Papers in this tradition include Allen and Hellwig (1986a,b) and Vives (1986).

$|X|$ denotes the cardinality of X.
$\setminus$ denotes set theoretic subtraction.
$\emptyset$ denotes the emptyset.
$\mathbf{a}$ lower-case bold letters denote vectors.
$\mathbb{N}$ denotes the set of natural numbers.
$\mathbb{Q}^+$ denotes the set of positive rational numbers.

3 The trading game

Consider the market for a perfectly homogeneous good. In the market there is a finite set of buyers $B = \{1, ..., b\}$, $b \geq 2$, and a finite set of sellers $S = \{1, ..., s\}$, $s \geq 2$. Each seller in the market has a cost function $C_i : \Re_+ \to \Re_+$ which is C^2, strictly convex and satisfies $C_i(0) = 0$ and $C_i'(0) = 0$. Each buyer in the market has a demand function $D_j : \Re_+ \to \Re_+$ which is C^2 and for each $j \in B$ there exist strictly positive finite real numbers $\bar{p}_j, \bar{q}_j$ such that $D_j(\bar{p}_j) = 0$ and $D_j(0) = \bar{q}_j$. Also, $D_j'(p) < 0$ for all $p \in (0, \bar{p}_j)$. In what follows we shall make frequent use of sellers' competitive supplies. The profit of each seller, as a function of quantity, is $\pi_i(q) = pq - C_i(q)$. The competitive supply of the seller, as a function of price, is $h_i(p) = \arg\max_{q \in \Re_+} \pi_i(q)$. As each seller's cost function is strictly convex the function $\pi_i(q)$ is strictly concave in q and $h_i(p)$ is well-defined and single-valued. Also let $\pi_i^*(p) = ph_i(p) - C_i(h_i(p))$ so $\pi_i^*(p)$ is the value function. We shall want to consider a Bertrand price competition game between possible subsets of buyers and sellers so let $\chi^B = \{M : M \in 2^B \setminus \{\emptyset\}\}$ and let $\chi^S = \{M : M \in 2^S \setminus \{\emptyset\}\}$. The set χ^B is all the non-empty subsets of buyers and χ^S is all the non-empty subsets of sellers. For any set of traders $T' = (B', S') \in \chi^B \times \chi^S$ consider a classical Bertrand price game between these buyers and sellers. Each seller simultaneously and independently chooses a $p_i \in \Re_+$ with a commitment to supply all the demand forthcoming from the buyers B'. If a seller posts the unique minimum price in the market then it serves all the demand forthcoming at that price. If a seller is undercut then it obtains no demand and its profit is zero. If a seller ties with other sellers at the minimum price then a sharing rule describes how the market demand is shared. Throughout we shall assume the market demand is shared according to capacity sharing.[6] Let $\beta_i(p) = h_i(p) / \sum_{j \in A} h_j(p)$ is the share of the market demand which seller i obtains when it ties with $A \setminus \{i\}$ other sellers at minimum price p. Fix a vector of

[6]This sharing rule has been used by, amongst others, Dastidar (1997) and Chowdhury and Sengupta (2004).

prices $\mathbf{p} \in \Re_+^{|S'|}$. Let $D(B',p) = \sum_{j\in B'} D_j(p)$ and $\pi_i(\mathbf{p},T')$ denote the profit of seller i at price vector $\mathbf{p}$ in the market with T' traders. We can summarize this profit as:

$$\pi_i(\mathbf{p},T') = \begin{cases} p_i D(B',p_i) - C_i(D(B',p_i)) & \text{if } p_i < p_k \ \forall k \neq i; \\ p_i \beta_i(p_i) D(B',p_i) - C_i(\beta_i(p_i) D(B',p_i)) & i \text{ ties with } A \setminus \{i\} \text{ at min price;} \\ 0 & \text{if } p_i > p_k \text{ for some } k. \end{cases} \quad (1)$$

We shall let $G = (B,S) \in \chi^B \times \chi^S$ denote the grand coalition of all buyers and sellers. Now in a market with a given set of traders some sellers may be able to improve their outcomes by affecting a coalitional change in their prices.

Definition 1 *Fix a coalition of traders $T' = (B',S') \in \chi^B \times \chi^S$ and a price vector $\boldsymbol{p} \in \Re_+^{|S'|}$. A coalition of sellers $A \subset S'$ has an improvement upon price vector $\boldsymbol{p}$ if there is a vector $\boldsymbol{p}'(A) = \{p'_i\}_{i\in A}$ such that $\pi_i(\boldsymbol{p}'(A), \boldsymbol{p}(S' \setminus A), T') > \pi_i(\boldsymbol{p},T')$ for all $i \in A$.*

A coalition of sellers, $A \subset S'$, has an improvement upon a price vector $\mathbf{p} \in \Re_+^{|S'|}$ if the coalition has another vector of prices which they can post in the market, $\mathbf{p}'(A) = \{p'_i\}_{i\in A}$, which will result in higher profit provided all sellers not in the coalition $S' \setminus A$ continue to post their prices in the price vector $\mathbf{p}$ which we denote by $\mathbf{p}(S' \setminus A)$. Given this concept of an improvement upon a price vector we can now define the concept of a pure strategy Bertrand equilibrium for a market.

Definition 2 *For any coalition of traders $T' = (B',S') \in \chi^B \times \chi^S$ a price vector $\boldsymbol{p} \in \Re_+^{|S'|}$ is a pure strategy Bertrand equilibrium if no coalition of sellers $A \subset S'$, $|A| = 1$, has an improvement upon price vector $\boldsymbol{p}$.*

A price vector is a pure strategy Bertrand equilibrium of a market if no seller can unilaterally change their price and obtain higher profit. For any set of traders $T' = (B',S') \in \chi^B \times \chi^S$ we shall let $\mathcal{E}(T') \subset \Re_+^{|S'|} \cup \{\emptyset\}$ denote the set of pure strategy Bertrand equilibria of the market formed by the traders. In defining the notion of an improvement upon a price vector we restricted the deviating coalition to the set of sellers. However, when we introduce the Bertrand core we shall want to consider the possibility that a coalition of traders, buyers and sellers, can enact an improvement upon a price vector. We now introduce this concept.

Definition 3 *Fix a coalition of traders $T' = (B',S') \in \chi^B \times \chi^S$ and a price vector $\boldsymbol{p} \in \Re_+^{|S'|}$. A coalition of traders $A = (B'',S'') \subset T'$ has an improvement upon price vector $\boldsymbol{p}$ if there exists a price vector $\boldsymbol{p}' \in \mathcal{E}(A)$ such that $\min \boldsymbol{p}' < \min \boldsymbol{p}$ and $\pi_i(\boldsymbol{p}',A) > \pi_i(\boldsymbol{p},T')$ for all $i \in S''$.*

A coalition of traders, sellers and buyers, has an improvement upon a price vector if the coalition can form a market and there is an equilibrium trading vector which is an improvement for both the buyers and sellers in the coalition. The new trading vector must be an improvement for buyers in that they obtain the good at a lower price which is represented by $\min \mathbf{p}' < \min \mathbf{p}$. The new trading vector must be an improvement for sellers in that they obtain higher profits at the new equilibrium which is represented by $\pi_i(\mathbf{p}', A) > \pi_i(\mathbf{p}, T')$. Note that the difference between Definition 1 and Definition 3 is that in Definition 1 the improvement upon a price vector is enacted by a subset of the sellers whereas in Definition 3 the improvement is enacted by some subset of sellers *and* buyers. The concept of an improvement enacted solely by sellers was considered by Chowdhury and Sengupta (2004) in the context of coalition-proof equilibria. The concern here is about the stronger concept of an improvement enacted by both sides of the market. Later in the section we shall return to the relationship between coalition-proof equilibria and the Bertrand core. We are now ready to introduce the Bertrand core.

Definition 4 *A price vector $\boldsymbol{p} \in \Re^s_+$ is in the Bertrand core if $\boldsymbol{p} \in \mathcal{E}(G)$ and no coalition of traders, $A \subset G$, has an improvement upon $\boldsymbol{p}$.*

A price vector is in the Bertrand core if it constitutes a pure strategy Bertrand equilibrium for the grand coalition, G, and no coalition contained within, or including, the grand coalition, has an improvement upon the price vector. Therefore if a price vector is in the Bertrand core it is a pure strategy Bertrand equilibrium for the grand coalition and no coalition of traders can form a market and trade at a new price vector which is an improvement for both buyers and sellers. We shall let $\mathcal{C}(G) \subset \Re^s_+ \cup \{\emptyset\}$ denote the price vectors in the Bertrand core.

Remark 1 $\mathcal{C}(G) \subset \mathcal{E}(G)$. *The Bertrand core is a subset of the set of pure strategy Bertrand equilibria of the grand coalition as any price in the Bertrand core is a pure strategy equilibrium for the grand coalition. However, a price vector which is a pure strategy Bertrand equilibrium may not be in the Bertrand core. See the example below.*

Remark 2 *It is worth noting that the Bertrand core is neither wholly cooperative nor non-cooperative. It is cooperative in the sense that a coalition of traders, buyers and sellers, may recognize that they can improve their outcomes by forming a market within the grand coalition. However, once the market is formed the sellers act non-cooperatively in offering price-making contracts to the buyers in the market.*

Remark 3 *A price vector which is not in the Bertrand core is a vector which is not robust to traders forming contracts which may be somewhere between the extremes of standard price-making contracts. Price-setting games have traditionally assumed that either sellers commit to supplying all demand from the grand coalition (Bertrand competition) or make no commitment to supply any particular demand (Bertrand-Edgeworth). The notion of an improvement by a coalition in Definition 3 admits sellers to form contracts which are between these two extremes in that sellers may supply more than their competitive supply but less than the demand from the grand coalition.*

Lemma 1 $h_i(0) = 0$ *and* $h_i'(p) > 0$ *for all* $i \in S$*.*

Proof. As $h_i(p) = \arg\max_{q \in \Re_+} \pi_i(q)$ if $p = 0$ then the profit of the seller is $\pi_i(q) = -C_i(q)$. Therefore the profit maximizing output is $q = 0$. To establish the second part of the lemma note that $h_i(p)$ must satisfy the first-order condition for maximization:

$$p - C_i'(h_i(p)) = 0.$$

Differentiating w.r.t. p we obtain:

$$1 - C_i''(h_i(p))h_i'(p) = 0.$$

Rearranging:

$$h_i'(p) = 1/C_i''(h_i(p)).$$

As sellers have strictly convex cost functions $C_i''(\cdot) > 0$ and $h_i'(p) > 0$. ■

Lemma 2 $\pi_i^{*\prime}(p) > 0$ *for all* $p > 0$*.*

Proof. From the definition $\pi_i^*(p) = ph_i(p) - C_i(h_i(p))$ and:

$$\pi_i^{*\prime}(p) = h_i(p) + ph_i'(p) - C_i'(h_i(p))h_i'(p).$$

Factorizing:

$$\pi_i^{*\prime}(p) = h_i(p) + h_i'(p)[p - C_i'(h_i(p))].$$

From the first-order condition $p - C_i'(h_i(p)) = 0$ therefore:

$$\pi_i^{*\prime}(p) = h_i(p).$$

From Lemma 1 we know that $h_i(p) > 0$ for all $p > 0$ which establishes the result. ■

3.1 Price-taking equilibrium and the Bertrand core

Having defined the Bertrand core and related notions of improvements upon price vectors we now show that, under the assumptions made here, the Bertrand core is non-empty. This is established by showing that the competitive equilibrium of the market belongs to the Bertrand core. In a market where all sellers take prices as given a price-taking, or competitive, equilibrium is a price such that the quantities the sellers are willing to supply to the market is exactly equal to the quantity demanded by the buyers.

Definition 5 *For any coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ a price-taking equilibrium is a $p' \in \Re_+$ such that $\sum_{i \in S'} h_i(p') = D(B', p')$.*

For any coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ we shall let $\mathcal{P}(T') \subset \Re_+ \cup \{\emptyset\}$ denote the price-taking equilibria of the market composed of the T' traders.

Proposition 1 *For any $T' = (B', S') \in \chi^B \times \chi^S$, $\mathcal{P}(T') \neq \emptyset$ and $|\mathcal{P}(T')| = 1$.*

Proof. Define the function $f(p) = D(B', p) - \sum_{i \in S'} h_i(p)$. The function $f(p)$ is the excess demand function. From the first-order condition $h_i(p) = C_i'^{-1}(p)$ and as the cost function $C_i(\cdot)$ is C^2 the first derivative is continuous and the inverse of the first derivative is continuous. Therefore $f(p)$ is a is a continuous function of price. Note that $f(0) = \sum_{j \in B'} \bar{q}_j > 0$ and letting $\bar{p} = \max\{\bar{p}_j : j \in B'\}$ we have $f(\bar{p}) = -\sum_{i \in S'} h_i(\bar{p}) < 0$. As $f(p)$ is continuous, the intermediate value theorem guarantees that there exists a $p' \in (0, \bar{p})$ such that $f(p') = 0$ which implies $D(B', p') = \sum_{i \in S'} h_i(p')$. To see that the price-taking equilibrium is unique note that $f'(p) < 0$. ■

Proposition 2 *For any $T' = (B', S') \in \chi^B \times \chi^S$ if $p' \in \mathcal{P}(T')$ then $(p', ..., p') \in \mathcal{E}(T')$ provided $|S'| \geq 2$.*

Proof. Suppose we have a market with $T' = (B', S') \in \chi^B \times \chi^S$ traders. If each seller quotes price p' to the buyers, with $p' \in \mathcal{P}(T')$, the profit which each seller obtains at this price is:

$$p'\beta_i(p')D(B', p') - C_i(\beta_i(p')D(B', p')).$$

As $\beta_i(p') = h_i(p') / \sum_{j \in S'} h_j(p')$ and $\sum_{j \in S'} h_j(p') = D(B', p')$ the profit of each seller simplifies to:

$$p'h_i(p') - C_i(h_i(p') = \pi_i^*(p').$$

Now consider whether any seller could profitably deviate from quoting this price. If a seller were to quote a price $p'' < p'$ then the maximum profit they could obtain is $\pi_i^*(p'')$. Lemma 2 then implies $\pi_i^*(p'') < \pi_i^*(p')$ and this is not a profitable deviation. If a seller increases their price then as $|S'| \geq 2$ they lose all demand and earn zero profit which is not a profitable deviation. Therefore $(p', ..., p') \in \mathcal{E}(T')$. ■

Proposition 3 *$\mathcal{C}(G) \neq \emptyset$.*

Proof. We shall show that if $p^C \in \mathcal{P}(G)$ then $(p^C, ..., p^C) \in \mathcal{C}(G)$. That is, the price-taking equilibrium for the whole market belongs to the Bertrand core. Suppose a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ deviate from the grand coalition. The profit which a seller $i \in S'$ earned at the price-taking equilibrium was $\pi_i^*(p^C)$. Suppose that $\mathbf{p'} \in \mathcal{E}(T')$ is the equilibrium price at which trade takes place amongst T' traders. Let $p'_j = \min \mathbf{p'}$. If $p'_j < p^C$ then the maximum profit seller j obtains from deviating is $\pi_j^*(p'_j) < \pi_j^*(p^C)$ and deviating is not profitable for seller j. If $p'_j \geq p^C$ then the deviating coalition is not a strict improvement for buyers. Therefore $(p^C, ..., p^C) \in \mathcal{C}(G)$. ■

The results show that the Bertrand core is non-empty as the price-taking equilibrium for the grand coalition belongs to the core. We illustrate these results in an example.

Example 1 *Consider a market with two buyers, $B = \{1, 2\}$, and three sellers, $S = \{1, 2, 3\}$. The market demand of each buyer is given by the piecewise-affine function $D(p) = \max\{0, 5 - \frac{1}{2}p\}$. Each seller's cost function is given by $C(q) = q^2$. Standard calculations[7] reveal that the Bertrand equilibrium set for the grand coalition is $\mathcal{E}(G) = \{\boldsymbol{p} \in \Re_+^3 : p_i = p_j, \forall j \neq i, p_i \in [2\frac{1}{2}, 5\frac{5}{7}]\}$. There are a number of different coalitions which could deviate from the market. One possibility is that a single seller leaves the market and trades with a subset of buyers. However, routine calculation shows that the monopoly price which a seller facing a single buyers would charge is $6\frac{2}{3}$. Therefore this coalition would not benefit buyers. Second, a coalition with two sellers and one buyer, $B' = \{1\}$ and $S' = \{1, 2\}$, could form. Routine calculations show that $\mathcal{E}(B', S') = \{\boldsymbol{p} \in \Re_+^2 : p_i = p_j, \forall j \neq i, p_i \in [2, 4\frac{2}{7}]\}$. Of the possible coalition prices it is straightforward to check that all prices in the interval $[2, 3\frac{13}{19})$ represent profitable deviations from the whole market. The final possible coalition is that of two buyers and two sellers, $B'' = \{1, 2\}$ and $S'' = \{1, 2\}$. The set of equilibria of this market is $\mathcal{E}(B'', S'') = \{\boldsymbol{p} \in \Re_+^2 : p_i = p_j, \forall j \neq i, p_i \in [3\frac{1}{3}, 6]\}$. Of the possible coalition prices the prices in the interval $(4\frac{6}{11}, 5\frac{5}{7}]$ represent profitable deviations from the whole market. Therefore the Bertrand core is $\mathcal{C}(G) = \{\boldsymbol{p} \in \Re_+^3 : p_i = p_j, \forall j \neq i, p_i \in [3\frac{13}{19}, 4\frac{6}{11}]\} \subset \mathcal{E}(G)$. Note that the competitive supply of each seller is $h(p) = \frac{p}{2}$ and the price-taking equilibrium is $p^C = 4$.*

[7]See Vives (1999, pp.120-2) or Dastidar (1995).

3.2 A limit result on the Bertrand core

The results in Propositions 1 and 3 show that the price-taking equilibrium of the grand coalition belongs to the Bertrand core. This provides a strategic foundation for price-taking behaviour. However, this strategic foundation is weak in that the Bertrand core will typically contain prices which are different from the price-taking equilibrium, as illustrated in Exanple 1. This raises the question of whether a stronger foundation for price-taking behaviour can be established. In this section we show that as the set of traders in the market becomes large the only price which remains in the Bertrand core is the price-taking equilibrium even when the pure strategy Bertrand equilibria of the limit market remains unchanged. To understand how contracts may be formed in large markets we introduce the notion of a replicated market. Formally, the $r \in \mathbb{N}$ replication of the market with $T' = (B', S') \in \chi^B \times \chi^S$ is the market in which there are r number of each type of buyer and seller. Following the notation used above we shall let $\mathcal{P}_r(T') \subset \Re_+ \cup \{\emptyset\}$ denote the price-taking equilibria of the r-replicated market, $\mathcal{E}_r(T') \subset \Re_+^{r|S'|} \cup \{\emptyset\}$ will denote the set of pure strategy Bertrand equilibria of the r-replicated market, and $\mathcal{C}_r(G) \subset \Re_+^{rs} \cup \{\emptyset\}$ will denote the set of Bertrand core prices of the r-replicated grand coalition.

Proposition 4 *For any $T' = (B', S') \in \chi^B \times \chi^S$, $\mathcal{P}_r(T') = \mathcal{P}(T')$ for all $r \in \mathbb{N}$.*

Proof. Define the excess demand of the replicated market as $f(p, r) = rD(B', p) - \sum_{i \in S'} rh_i(p)$. Factorizing gives $f(p, r) = r(D(B', p) - \sum_{i \in S'} h_i(p))$. As $r \in \mathbb{N}$, $f(p', r) = 0$ if and only if $f(p') = 0$. ■

Proposition 5 *$\mathcal{C}_r(G) \neq \emptyset$ for all $r \in \mathbb{N}$.*

Proof. As $\mathcal{P}_r(G) = \mathcal{P}(G)$ for all $r \in \mathbb{N}$ the same steps used in the proof of Proposition 3 establish that if $p^C \in \mathcal{P}(G)$ the price vector $(p^C, ..., p^C) \in \mathcal{C}_r(G)$ for all $r \in \mathbb{N}$. ■

The result in Proposition 5 shows that the Bertrand core is non-empty for any r-replication of the grand coalition. However, it does not give any insight as to which other prices, if any, remain in the Bertrand core and the set of pure strategy Bertrand equilibria. We now show that as the market becomes large the only trading price which remains in the Bertrand core is the price-taking equilibrium. Let $p^*(r) = \sup\{\max \mathbf{p} : \mathbf{p} \in \mathcal{C}_r(G)\}$ and $p_*(r) = \inf\{\min \mathbf{p} : \mathbf{p} \in \mathcal{C}_r(G)\}$. The price $p^*(r)$ is the supremum of prices quoted by any seller in the r-replicated Bertrand core and $p_*(r)$ is the infimum of prices quoted by any seller in the r-replicated Bertrand core. Also let $A(r) = \{i \in S : p_i = \min \mathbf{p}, \forall \mathbf{p} \in \mathcal{C}_r(G)\}$. The set $A(r)$ is the set of sellers which serve market demand in every price vector in the r-replicated Bertrand core.

Lemma 3 *If $\boldsymbol{p} \in \mathcal{C}_r(G)$ and $p_i = \min \boldsymbol{p}$ then all sellers of the same type as seller i post price p_i.*

Proof. See the Appendix.

Lemma 4 *If $A(r) = S$ for all $r \in \mathbb{N}$ then $p_*(r+1) \geq p_*(r)$ and $p^*(r+1) \leq p^*(r)$. That is, if $A(r) = S$ for all $r \in \mathbb{N}$ the sequences $\{p_*(r)\}_{r\in\mathbb{N}}$ and $\{p^*(r)\}_{r\in\mathbb{N}}$ are monotone.*

Proof. See the Appendix.

We now present the limit result on the Bertrand core.

Proposition 6 *Suppose $A(r) = S$ for all $r \in \mathbb{N}$, then as $r \to \infty$, $p^*(r) \to p^C$ and $p_*(r) \to p^C$ with $p^C \in \mathcal{P}(G)$.*

Proof. The result will be established by showing that if $\lim_{r\to\infty} p_*(r) < p^C$ then there exists $\bar{r} \in \mathbb{N}$ such that the sellers quoting $p_*(r)$ can form a coalition and improve upon quoting $p_*(r)$ for all $r \geq \bar{r}$. Similarly if $\lim_{r\to\infty} p^*(r) > p^C$ then sellers quoting $p^*(r)$ can form a coalition and improve upon quoting $p^*(r)$ for all $r \geq \bar{r}$.

Step 1. Consider the sequence of prices $\{p_*(r)\}_{r\in\mathbb{N}}$. This sequence is bounded above by p^C. From Lemma 4 we know that the sequence is monotone. Therefore the sequence $\{p_*(r)\}_{r\in\mathbb{N}}$ must converge (Rudin, 1976, p.55). Suppose $\lim_{r\to\infty} p_*(r) = p_* < p^C$.

Step 2. As $A(r) = S$, Lemma 3 tells us that all sellers charge the same price in the Bertrand core. Consider the profit which each seller earns when all sellers post price p_*. As $p_* < p^C$ a straightforward check of the excess demand function shows that, for any r-replication, sellers posting p_* serve market demand greater than their competitive supply and earn strictly less than $\pi_i^*(p_*)$. Therefore, by the continuity of the profit function, there exists $\epsilon > 0$ such that $\pi_i^*(p_* - \epsilon)$ is greater than the profit any seller earns from the sequence of prices $p_*(r) \in N_\epsilon(p_*)$.

Step 3. As the sequence $\{p_*(r)\}_{r\in\mathbb{N}}$ is convergent fix an r' such that $p_*(r) \in N_\epsilon(p_*)$ for all $r \geq r'$. Let $\hat{p} = p_* - \epsilon$. If the profit from $\pi_i^*(\hat{p})$ is strictly higher than posting the prices $p_*(r)$, $r \geq r'$ then the continuity of $\pi_i^*(\cdot)$ means there is a $\delta > 0$ such that $\pi_i^*(p)$, $p \in [\hat{p} - \delta, \hat{p}]$, is also strictly greater than the profit from posting prices $p_*(r)$, $r \geq r'$.

Step 4. Now fix an $i \in S$ and fix a $j \in B$ such that $D_j(p^C) > 0$. Then consider the mapping $g(p) = h_i(p)/D_j(p)$. This is a continuous mapping provided $D_j(p) \neq 0$. As $g(p)$ is continuous, the image of $[\hat{p} - \delta, \hat{p}]$ under $g(\cdot)$ is a compact connected interval which we shall denote by $g([\hat{p} - \delta, \hat{p}])$.

Step 5. By the everywhere denseness of the rationals in the real line there must exist a $z \in int(g([\hat{p}-\delta,\hat{p}]))$ with $z \in \mathbb{Q}^+$. As $z \in \mathbb{Q}^+$ we can write z as $z = x/y$ with $x, y \in \mathbb{N}$ and $y \geq 2$. Now consider the sequence of replicated markets with $\bar{r} \geq \max\{x, y, r'\}$. Let $p(z)$ denote the pre-image of z under $g(\cdot)$.

Step 6. Consider the market formed by x buyers of type j and y sellers of type i. From the mapping we know that $z = g(p(z))$ which means $xD_j(p(z)) = yh_i(p(z))$. That is, $p(z)$ is a price-taking equilibrium for the market x buyers of type j and y sellers of type i. From Proposition 2 we know that all sellers quoting price $p(z)$ is a pure strategy Bertrand equilibrium for the coalition and as $p(z) \in (\hat{p}-\delta,\hat{p})$ the buyers obtain the good at a price lower than any price in the sequence $p_*(r)$, $r \geq \bar{r}$ and each seller obtains strictly higher profit than in the grand coalition. Therefore this coalition improves upon the sequence $p_*(r)$, $r \geq \bar{r}$ and it must be the that $\lim_{r\to\infty} p_*(r) = p^C$.

Step 7. Consider the sequence of prices $\{p^*(r)\}_{r\in\mathbb{N}}$. As the sequence is bounded below by p^C and is monotone the sequence converges. Suppose $\lim_{r\to\infty} p^*(r) = p^* > p^C$. One can repeat Steps 2-6 with $p_* = p^*$ to show that a coalition has an improvement provided the market is replicated sufficiently many times. ■

Example 2 *Consider the market in Example 1. We found that the Bertrand core was* $\mathcal{C}(G) = \{\boldsymbol{p} \in \Re^3_+ : p_i = p_j, \forall j \neq i, p_i \in [3\frac{13}{19}, 4\frac{6}{11}]\}$. *The set of Bertrand equilibria was* $\mathcal{E}(G) = \{\boldsymbol{p} \in \Re^3_+ : p_i = p_j, \forall j \neq i, p_i \in [2\frac{1}{2}, 5\frac{5}{7}]\}$ *and the price-taking equilibrium was* $p^C = 4$. *Now consider what happens as the market is replicated. Routine calculation reveal that* $\mathcal{E}_r(G) = \{\boldsymbol{p} \in \Re^{r3}_+ : p_i = p_j, \forall j \neq i, p_i \in [2\frac{1}{2}, 5\frac{5}{7}]\}$ *for all* $r \in \mathbb{N}$. *The set of pure strategy Bertrand equilibria is not reduced as the market is replicated. However, we know from Proposition 6 that as* $r \to \infty$, $p^*(r) \to 4$ *and* $p_*(r) \to 4$. *Price-taking behaviour prevails in the Bertrand core as the market becomes large.*

3.3 On the relationship between coalition-proof contracts and the Bertrand core

In defining the Bertrand core we assumed that the sellers act non-cooperatively once any market of traders was formed. However, if the sellers could communicate to improve their outcomes and were willing to cooperate, but could not form binding agreements, then the analysis may be quite different. Chowdhury and Sengupta (2004) considered which equilibrium prices survive the Nash equilibrium refinement of coalition-proofness introduced by Bernheim et al. (1987). In this section we review the differences and similarities between the types of price-setting contracts which are coalition-proof and the price-setting contracts in the Bertrand core. As we shall see, the two concepts are independent but can be combined to give a new core notion which we term the *coalition-proof Bertrand core*. Recall the earlier definition of an improvement upon a price vector by a coalition of sellers.

Definition 6 *Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ and a price vector $\boldsymbol{p} \in \Re_+^{|S'|}$. A coalition of sellers $A \subset S'$ has an improvement upon price vector $\boldsymbol{p}$ if there is a vector $\boldsymbol{p}'(A) = \{p_i'\}_{i \in A}$ such that $\pi_i(\boldsymbol{p}'(A), \boldsymbol{p}(S' \setminus A), T') > \pi_i(\boldsymbol{p}, T')$ for all $i \in A$.*

The coalition-proof Bertrand equilibrium is defined inductively. Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$. A coalition of sellers $A \subset S'$, $|A| = 1$, has a *self-enforcing* improvement upon price vector $\mathbf{p}$ if A has an improvement upon price vector $\mathbf{p}$. A coalition of sellers $A \subset S'$, $|A| = 2$, has a self-enforcing improvement upon $\mathbf{p}$ if A has a price vector $\mathbf{p}'(A)$ which is an improvement upon $\mathbf{p}$ and no strict subset of A, which would be a coalition of cardinality one, has an improvement upon $\mathbf{p}'(A)$. We could continue this process to define a self-enforcing improvement for $|A| = 3, 4, ...$ Then any coalition of sellers $A \subset S'$ has a self-enforcing improvement upon $\mathbf{p}$ if A has a price vector $\mathbf{p}'(A)$ which is an improvement upon $\mathbf{p}$ and no strict subset $A' \subset A$, $A' \neq A$, has a self-enforcing improvement upon $\mathbf{p}'(A)$.

Definition 7 *Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$. A price vector $\boldsymbol{p} \in \Re_+^{|S'|}$ is a coalition-proof Bertrand equilibrium of the market with T' traders if no coalition of sellers has a self-enforcing improvement upon $\boldsymbol{p}$.*

We shall let $\mathcal{E}^{CP}(T') \subset \Re_+^{|S'|} \cup \{\emptyset\}$ denote the set of coalition-proof Bertrand equilibria of the market with T' traders.

Proposition 7 *For any coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$, $\mathcal{E}^{CP}(T') \neq \emptyset$.*

Proof. See Chowdhury and Sengupta (2004, Prop. 1). ■

The result in Proposition 7 is interesting because many games fail to possess a coalition-proof Nash equilibrium. We now can consider what the relationship is between the set of coalition-proof Bertrand equilibria and the Bertrand core. The next example shows that the a coalition-proof Bertrand equilibrium may, or may not, belong to the Bertrand core.

Example 3 *Consider a market with a grand coalition of two sellers and two buyers $S = \{1, 2\}$ and $B = \{1, 2\}$. Each seller has a cost function given by $C(q) = q^2$. Each buyer has a market demand given by $D(p) = 1 - \frac{1}{2}p$. Routine calculations show that the unique coalition-proof Bertrand equilibrium is for each seller to quote price $p_i = \frac{6}{5}$ (see Chowdhury* *and Sengupta (2004, Example 2)). Moreover, if one considers each of the possible coalitions of traders which could form it is straightforward to show that no coalition can improve upon each seller quoting this price. Therefore in this example the intersection of the set of coalition-proof Bertrand equilibria and the Bertrand core is nonempty $\mathcal{C}(G) \cap \mathcal{E}^{CP}(G) \neq \emptyset$. However, consider the $r = 2$ replication of this market. Routine calculations reveal that the unique coalition-proof Bertrand equilibrium price is unchanged $\mathcal{E}_2^{CP}(G) = \{\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}\}$. In this coalition-proof equilibrium each seller earns profit of $\frac{8}{25}$. Now consider a coalition of traders $T' = (B', S')$ composed of three buyers and two sellers $B' = \{1, 2, 3\}$ and $S' = \{1, 2\}$. Suppose each seller quotes the price $\frac{47}{40}$ to the buyers. A routine check shows that $(\frac{47}{40}, \frac{47}{40}) \in \mathcal{E}(T')$ and the profit which each seller earns is strictly higher than $\frac{8}{25}$. Moreover, $\frac{47}{40} < \frac{6}{5}$ so the coalition is a strict improvement for the buyers. Therefore this coalition improves upon the coalition-proof Bertrand equilibrium and $\mathcal{C}_2(G) \cap \mathcal{E}_2^{CP}(G) = \emptyset$.*

Remark 4 *The results in Example 3 illustrate that it is not easy to compare the set of coalition-proof Bertrand equilibria and the Bertrand core. This is because the Bertrand core is in a sense stronger and weaker than the requirement of coalition-proofness. It is stronger in that it permits a wider range of coalitions to form and improve upon price vectors. The Bertrand core permits buyers* and *sellers to form a coalition whereas the coalition-proof Bertrand equilibrium only permits sellers to form a coalition. However, the Bertrand core is weaker in that it permits a coalition to trade at a price vector which may not be a coalition-proof Bertrand equilibrium.*

Remark 5 *In a market with symmetric sellers Chowdhury and Sengupta (2004) showed that the coalition-proof Bertrand equilibrium is unique and in the equilibrium all sellers quote the same price. This price is different from the competitive equilibrium. Therefore we can use the result in Proposition 6 to establish that the coalition-proof Bertrand equilibrium will not belong to the Bertrand core if a market with symmetric sellers is replicated sufficiently many times. This is what happens in Example 3.*

The refinement of coalition-proofness is interesting because it applies to situations where traders are able to communicate but cannot write binding contracts. In defining the Bertrand core it was implicitly assumed that once a coalition forms and an equilibrium trading vector is agreed upon then this is how trade takes place. However, coalition-proofness raises the possibility that if the contracts are not binding then some subset of sellers in a coalition market may be able to change their trading prices at the expense of other traders in the coalition. To rule out these possibilities we now introduce the concept of a coalition-proof improvement upon a price vector.

Definition 8 *Fix a coalition of traders* $T' = (B', S') \in \chi^B \times \chi^S$ *and a price vector* $\boldsymbol{p} \in \Re_+^{|S'|}$. *A coalition of traders* $A = (B'', S'') \subset T'$ *has a coalition-proof improvement upon price vector* $\boldsymbol{p}$ *if there exists a price vector* $\boldsymbol{p}' \in \mathcal{E}^{CP}(A)$ *such that* $\min \boldsymbol{p}' < \min \boldsymbol{p}$ *and* $\pi_i(\boldsymbol{p}', A) > \pi_i(\boldsymbol{p}, T')$ *for all* $i \in S''$.

We can now define a coalition-proof analogue of the Bertrand core in which buyers and sellers can communicate but the sellers can not make binding contracts.

Definition 9 *A price vector* $\boldsymbol{p} \in \Re_+^s$ *is in the coalition-proof Bertrand core if* $\boldsymbol{p} \in \mathcal{E}^{CP}(G)$ *and no coalition of traders,* $A \subset G$, *has a coalition-proof improvement upon* $\boldsymbol{p}$.

We shall let $\mathcal{C}^{CP}(G) \subset \Re_+^s \cup \{\emptyset\}$ denote the price vectors in the coalition-proof Bertrand core. The next example shows that the coalition-proof Bertrand core may be empty.

Example 4 *Consider a market with five sellers,* $S = \{1, ..., 5\}$, *and ten buyers,* $B = \{1, ..., 10\}$. *Each seller has a cost function given by* $C(q) = q^2$ *and the demand of each buyers is given by* $D(p) = 1 - \frac{1}{10}p$. *The routine calculations reveal that the unique coalition-proof Bertrand equilibrium for the grand coalition each for each seller to quote price* $p_i = 5\frac{5}{11}$ *to the buyers. Therefore* $\mathcal{E}^{CP}(G) = \{5\frac{5}{11}, ..., 5\frac{5}{11}\}$. *At this coalition-proof equilibrium each seller earns a profit of* $4\frac{16}{121}$. *Now suppose a coalition of two sellers,* $S' = \{1, 2\}$, *and six buyers,* $B' = \{1, ..., 6\}$, *were to form. The unique coalition-proof Bertrand equilibrium for the coalition market is for each seller to quote price* $p_i = 4\frac{14}{19}$. *Therefore* $\mathcal{E}^{CP}(B', S') = \{4\frac{14}{19}, 4\frac{14}{19}\}$. *This coalition market is a strict improvement for buyers as* $4\frac{14}{19} < 5\frac{5}{11}$. *Moreover, the profit which each seller earns in the coalition market is* $4\frac{356}{361} > 4\frac{16}{121}$. *As the coalition market is a coalition-proof improvement upon the grand coalition in the example the coalition-proof Bertrand core is empty* $\mathcal{C}^{CP}(G) = \emptyset$.

The market in Example 4 illustrates that when sellers may communicate but cannot form binding contracts it is difficult to provide insights as to which price-making contracts may be formed in the market as the coalition-proof Bertrand core is empty. Unfortunately it is not easy to identify conditions which guarantee the non-emptyness of the coalition-proof Bertrand core but it seems to be the case that the core is more likely to be non-empty when the demand side of the market contains few buyers compared with the supply side of the market as this limits the types of coalition markets which could form. If one reconsiders Example 4 with just two buyers, each of whom has a demand of $D(p) = 5 - \frac{1}{2}p$, then the coalition-proof Bertrand core is non-empty.

4 Conclusion

The concept of the Bertrand core is an original combination of the early insights of Bertrand and Edgeworth regarding the formation of contracts and exchange. The exchange game presented here provides an elegant and tractable model for studying the formation of price-making contracts. The results show that even if we permit coalitions of traders to form within the grand coalition the Bertrand core is non-empty and as market become large we should expect price-making contracts to be close to the competitive equilibrium. However, if traders can communicate, form coalitions, but cannot commit to binding contracts, then the coalition-proof Bertrand core may be empty.

Given that the Bertrand core is a new concept for analyzing market exchange there are several possibilities for future research. First, Chowdhury and Sengupta (2004) presented a limit result regarding the set of coalition-proof Bertrand equilibria which showed that under certain conditions the limit equilibrium set is the competitive equilibrium as the number of sellers becomes large. It may be possible to use this result to show that the coalition-proof Bertrand core is non-empty in markets with large numbers of sellers. This would then provide a foundation for price-taking behaviour even when binding contracts cannot be formed. Aumann (1964) showed that in an economy with an atomless measure space of traders the Edgeworth core is equal to the set of Walrasian allocations. A similar result is likely to hold regarding the Bertrand core in markets with demand generated by an atomless measure space of buyers.

Second, the exchange game which we studied assumed that the set of traders was known with certainty and there was complete information regarding traders' types. This is clearly a restrictive assumption. Janssen and Rasmussen (2002) analyzed a price game where the set of traders was inactive with some exogenous probability. This possibility of inactivity could be studied in the context of the Bertrand core. Given a probability that some traders are inactive coalitions may prefer not to form because sellers could end up supplying the whole of the demand from the coalition and contracts may be quite different from those considered here. Alternatively, the exchange game could be extended to admit the possibility that traders have incomplete information regarding each others' cost types.

Third, we assumed that there were no non-convexities in the market. The analysis could be extended to cover the cases where each of the sellers incurs a sunk/avoidable fixed cost upon trading. This then produces non-convexities in the cost function. Saporiti and Coloma (2010) considered a market with this characteristic and showed that the existence of a competitive equilibrium guarantees the non-emptyness of the set of Bertrand equilibria. However, a Bertrand equilibrium may exist but a market may fail to possess a competitive equilibrium.

Finally, we remarked that the Bertrand core is neither wholly cooperative nor non-cooperative. It should be possible to analyze refinements of the core which admit greater cooperation between traders within any coalition.[8] In the coalition-proof Bertrand core there is limited cooperation between traders, but the inability to form binding contracts means that the solution must be a Nash equilibrium of the game. However, if traders could write binding contracts and were cooperative it would of great interest to study the core and bargaining set of the game and compare it with the Bertrand core and the coalition-proof core.

5 Appendix

Proof of Lemma 3. If a seller of type i posts the minimum price in the market they must earn non-negative profit. If A denotes the set of sellers tied at the minimum price and we let $x = h_i(p_i)/\sum_{j\in A} h_j(p_i)$ then:

$$xD(B,p_i)p_i - C_i(xD(B,p_i)) \geq 0. \tag{2}$$

If a seller of type i posted a price above this price then the share of the demand they could obtain by joining the minimum price tie, which we shall denote by y is:

$$y = \frac{h_i(p_i)}{\sum_{j\in A} h_j(p_i) + h_i(p_i)}.$$

The demand shares are such that:

$$0 < y < x.$$

Therefore there exists a $\gamma \in (0,1)$ such that:

$$\gamma x = y. \tag{3}$$

By the convexity of the cost function:

$$\gamma C_i(xD(B,p_i)) + (1-\gamma)C_i(0) > C_i(yD(B,p_i)).$$

[8] Kaneko (1977) studied a price-setting game, similar to the exchange game presented here, and characterized both the core and the bargaining set.

As $C_i(0) = 0$ this simplifies to:

$$\gamma C_i(xD(B, p_i) > C_i(yD(B, p_i)). \tag{4}$$

Combining eq.(2) and $\gamma > 0$:

$$\gamma(xD(B, p_i))p_i - C_i(xD(B, p_i))) \geq 0.$$

$$\gamma xD(B, p_i)p_i - \gamma C_i(xD(B, p_i)) \geq 0. \tag{5}$$

By eq.(3) and eq.(5):

$$yD(B, p_i)p_i - \gamma C_i(xD(B, p_i)) \geq 0. \tag{6}$$

Then eq.(4) and eq.(6) yield:

$$yD(B, p_i)p_i - C_i(yD(B, p_i)) > 0.$$

Which means the seller of type i posting a price above the minimum price has a profitable deviation by joining the minimum price tie. ■

Proof of Lemma 4. Suppose a contradiction that $p_*(r) > p_*(r+1)$. As $A(r) = S$, Lemma 3 tells us that in any Bertrand equilibrium all sellers post the same prices. Then there exists a p' such that $p_*(r) > p' \geq p_*(r+1)$ and $\mathbf{p}' \in \mathcal{C}_{r+1}(G)$. As $p_*(r) > p'$ all sellers quoting p' does not belong to the r-replicated Bertrand core. Therefore there is a subset of the r-replicated traders which have an improvement upon $\mathbf{p}'$. The profit which sellers would obtain from posting p' in the r-replicated market is:

$$p' r D(B, p')\beta_i(p') - C_i(rD(B, p')\beta_i(p')).$$

This simplifies to:

$$\frac{p' D(B, p') S_i(p')}{\sum_{j \in S} S_j(p')} - C_i(\frac{p' D(B, p') S_i(p')}{\sum_{j \in S} S_j(p')}).$$

Note that is profit does not depend on r. Therefore sellers would obtain the same profit from posting price p' in the $r+1$-replicated market. However, the same subset of traders which had an improvement upon $\mathbf{p}'$ would also have an improvement upon $\mathbf{p}'$ in the $r+1$-replicated market. This contradicts $\mathbf{p}' \in \mathcal{C}_{r+1}(G)$. The same proof can be used to establish that $p^*(r) \geq p^*(r+1)$. ■

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