Georg von Charasoff's Linear Economic Analysis and Anticipation of von Mises Iteration in Economic Analysis

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I. Introduction

In his main work *Das System des Marxismus. Darstellung und Kritik*, 1910[1], Georg von Charasoff criticized and reconstructed Marx's price theory and, in doing this, anticipated, at an advanced analytical level, most of the results that were going to be achieved later in the course of the 'transformation controversy'. However, his contribution can be considered to go beyond the particular range of Marxian theory. Mori (2011) consistently reformulated his argument by a mathematical model to extract some analytical characteristics of his model. The aim of this presentation consists in examining the contribution of Charasoff's linear economic analysis, in particular his theory of "Urkapital (original capital)", price of production and labour value in the light of the development of matrix theory in the beginning of the 20th century.

The linear algebraic development we would like to refer to as background is in particular the Perron-Frobenius theorem in 1907-12 on the one hand and the so-called Power Method (or von Mises Iteration) devised initially by Richard von Mises and Hilda Pollaczek-Geiringer (1929) on the other. As well known, the Perron-Frobenius theorem (in its generalized version) proposed the existence of the non-negative absolutely largest eigenvalue with a semi-positive associated eigenvector for every non-negative square matrix[2]. 20 years later, without directly using this theorem, and by using iterative procedures unlike this theorem, von Mises, the brother of the economist Ludwig, together with Pollaczek-Geiringer devised and proved practical procedures to calculate eigenvalues and eigenvectors for any (not only non-negative) square matrix under some assumptions. For each square matrix, starting from a suitable vector, one can reach the matrix's eigenvector associated with the dominant eigenvalue by multiplying the initial vector iteratively by the matrix.

von Mises and Pollaczek-Geiringer developed their ideas of iterative procedure based on an iterative procedure which approximately determines eigenvalues and eigenfunctions of boundary-value problems (see Vianello (1898), Stodola (1904), Pohlhausen (1921), Koch (1926)). This so-called Vianello-Stodola method was used by them to compute the "natural frequency (Eigenschwingung)" of an elastic material. This calculation is important in the mechanics because if e.g. the speed of a rotating shaft of steam turbine reaches its natural frequency, the resonance occurs and the material

[1] On recent receptions of Chrasoff's work, see Egidi and Gilibert (1989, 59–74), Kurz (1989, 11–61), Howard and King (1992), Kurz and Salvadori (1995; 1998, 25–56; 2000, 153–79), Egidi (1998, 96–100) , Gehrke (1998) and Stamatis (1999). On Charasoff's bibliography, see Mori (2007).

[2] See Perron (1907) and Frobenius (1908, 1909, 1912).

could break down (so that it is called "critical speed (kritische Drehzahl)"). Just as the case of Stodola (1904) indicates, this research field was closely related to the development of steam turbines which just started to be used in ships and trains from the beginning of the 20th century (we recall that the first turbine ship was constructed in 1894 and the "unsinkable" Titanic sank in 1912). Therefore, von Mises and Pollaczek-Geiringer (1929) can be seen as a natural extension of research on such a historical context. However, their achievement can be acknowledged independently because the area of research is different: linear equations for the former and differential equations for the latter.

Furthermore, von Mises and Pollaczek-Geiringer proposed to use also an iterative procedure to solve inhomogeneous linear equation systems. Before them, there had been a forerunner in this subject, namely Seidel (1874), and they overtook some of his ideas. However, the one of their iterative procedures which matters in respect to Charasoff was developed independently of Seidel (1874).

Georg von Charasoff's analysis uses mainly numerical examples (and this only at most three-dimensionally) and therefore cannot be seen to contain an algebraic general proof. However, it exemplifies *de facto* the existence of the above mentioned Frobenius root and its semi-positive eigenvector. Indeed, it is unknown whether Charasoff knew the papers of Perron or Frobenius, however, the earliness (one or two years after Frobenius) of his publication is as itself already remarkable. Besides, the characteristic feature of this exemplification consists in anticipating those procedures that were going to be discovered 19 years later by von Mises and Pollaczek-Geiringer. The main ideas of both works are quite similar. Furthermore, going beyond von Mises and Pollaczek-Geiringer, Charasoff carried out the iteration in the dual manner, namely in the search of the eigencolumn and eigenrow and also paid more attention to the uniqueness of solution.

This linear algebraic analysis was not carried out by Charasoff in an abstract form, but in an application to the economic context where the matrix was implicitly assumed as an input-coefficient matrix, column and row vectors as activity and price vectors. According to the duality, the vector iteration converging to the eigenvector expressed the iterative regression of an (arbitrary semi-positive) initial good vector to its input vector converging to the ultimate (unique) input vector i.e. "original capital (*Urkapital*)" as the primal problem, and the iterative progression of an (arbitrary positive) price vector to its successively corrected price vector converging to the unique equilibrium price vector i.e. "price of production" as the dual problem.

II. Procedure for solving homogeneous linear equation systems

In 1927, Richard von Mises held a lecture[3] about "Praktische Analysis" and taught a series of

[3] According to his biography, it must have been at University of Berlin.

calculation procedures for solving linear equation systems. In 1929, he published some of these procedures[4] with Hilda Pollaczek-Geiringer, his future wife. The authors treated inhomogeneous equation systems in the first part of their paper, and homogeneous equation systems in the second. We will examine, for convenience, homogeneous systems first.

The authors consider the following homogeneous equation system:

$$x = \lambda \mathfrak{A} x \qquad (1)$$

where $\mathfrak{A} = \{\alpha_{ij}\} \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Note that the non-negativity of constants and variables is not assumed in their paper. The problem is to solve a non-zero vector x and a scalar λ. It is a problem of finding eigenvalues of $\mathfrak{A}$ and their associated eigen (column) vectors. Throughout this part, they made the following assumption:

(MA.1) $\mathfrak{A}$ is symmetric and invertible.

The existence of a solution is obvious, the problem here, however, is not to show the existence but to determine eigenvalues and eigenvectors specifically. Such a problem is trivial today and easy to solve by a computer, but it was very serious and important at that time because „the most voluminous calculation machine does not have enough digits to provide a result of, say, three digits“[5].

To be able to accomplish the task of calculation sufficiently precisely and conveniently, they proposed to use an iterative procedure because it has the advantage that „it shows in each stage an approximation, which can be further improved if necessary. Besides, errors do not spread continuously, but they are in general automatically corrected in the course of calculation“[6]. So, they define the following iteration.

$$z^{(\nu+1)} = \mu^{(\nu)} \mathfrak{A} z^{(\nu)} \quad (\nu = 1, 2, ...) \qquad (2)$$

where $z^{(\nu)} \in \mathbb{R}^n$ and $\mu^{(\nu)} \in \mathbb{R}_{++}$ for $\nu = 1, 2, ...$

[4] von Mises/Pollaczeck-Geiringer (1929).
[5] Ibid, p.62.
[6] Ibid.

In their paper, they showed, using this iteration, how to reach each of *n* eigenvalues and its associated eigenvector. Here, we concentrate on their procedure to calculate especially the dominant eigenvalue (i.e. eigenvalue of minimum modulus) $1/\lambda_1$ and its associated eigenvector[7]. There are three cases to be considered according to the property of the dominant eigenvalue $1/\lambda_1$:

(i) $1/\lambda_1$ is simple and $-1/\lambda_1$ is not an eigenvalue

(ii) $1/\lambda_1$ is multiple and $-1/\lambda_1$ is not an eigenvalue

(iii) $-1/\lambda_1$ is an eigenvalue

Regarding the case (i), they conclude with the following proposition (the first part of „Satz 11").

Proposition 1 (von Mises und Pollaczek-Geiringer):

"For a homogenous linear equation system of the form (1) with parameter λ which can be written shortly as $x = \lambda\mathfrak{A}x$ ($a_{ij} = a_{ji}$), the smallest eigenvalue (dominant eigenvalue[8] – K.M.) and the associated eigensolutions can be found by setting an iteration $z^{(\nu+1)} = \mu^{(\nu)}\mathfrak{A}z^{(\nu)}$ (ν = 1, 2, ...) and starting from an *arbitrary* vector $z^{(1)}$ (italic by K.M.) with suitable coefficients $\mu^{(\nu)}$. If one continues so far that $z^{(\nu+1)}$ is approximately parallel to $z^{(\nu)}$, then the proportion of components of $z^{(\nu)}$ to components of $z^{(\nu+1)}$ provides the value λ_1; the common direction of $z^{(\nu)}$ and $z^{(\nu+1)}$ is an associated eigensolution."[9]

As they paraphrased, $z^{(\nu)}$ converges *except for a factor* to the required eigenvector, and each *quotient* of $\mu^{(\nu)}z_i^{(\nu)}$ to $z_i^{(\nu+1)}$ converges to the required (inverse) eigenvalue λ_1.

Their phrases like "*converge except for a factor* (*bis auf einen Faktor konvergieren*)" and convergence of "*quotient*" of each component of two successive vectors are intuitively understandable but need to be precisely formulated. First, the convergence of $z^{(\nu)}$ *except for a factor* was originally formulated by the authors themselves as follows[10]:

$$z^{(\nu+1)} \to \frac{\mu(\nu)}{\lambda_1} z^{(\nu)} \qquad \text{for } \nu\to\infty$$

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[7] It must be noted that their definition of eigenvalue was different from the usual one of today, i.e. the eigenvalue was defined by λ, *not* $1/\lambda$. Therefore, their eigenvalue is the inverse of today's one so that their "smallest eigenvalue" corresponds to the eigenvalue of maximum modulus in today's definition.

[8] See footnote 7.

[9] Ibid, p.154. Some symbols were adjusted to the usage of this paper. And see their poof of Proposition 1 (Satz 11) on pp. 153-4.

[10] Ibid.

However, this formulation is incorrect because if $z^{(v)}$ converges itself to null vector, this formulation is always valid even if $z^{(v)}$ does not converge *except for a factor.* Alternatively, the following formulation is possible.

$$\lim_{\nu\to\infty}\min\left\{\left\|\frac{z^{(\nu)}}{\left\|z^{(\nu)}\right\|}-\frac{z^{(\nu+1)}}{\left\|z^{(\nu+1)}\right\|}\right\|,\left\|\frac{(-1)^{\nu}z^{(\nu)}}{\left\|z^{(\nu)}\right\|}-\frac{(-1)^{\nu+1}z^{(\nu+1)}}{\left\|z^{(\nu+1)}\right\|}\right\|\right\}=0 \qquad (3)$$

Note that we have to use "min" because the sign of components might be oscillating as v goes if λ_1 is negative.

Second, the convergence of *quotient* of each component of two successive vectors was formulated in the paper as follows[11]:

$$\lambda_1=\lim_{\nu\to\infty}\frac{\mu^{(\nu)}z_1^{(\nu)}}{z_1^{(\nu+1)}}=\cdots=\lim_{\nu\to\infty}\frac{\mu^{(\nu)}z_n^{(\nu)}}{z_n^{(\nu+1)}}$$

This formulation is not precise either because some component i of $z^{(v+1)}$ may vanish after some v, and therefore for this i after this v, the quotient cannot be well-defined anymore. Alternatively, based on the above convergence of $z^{(v)}$ except for a factor according to (3), the following formulation would be possible to express the convergence of the *quotient*.

$$\lambda_1=\lim_{\nu\to\infty}\frac{\mu^{(\nu)}\left\|z^{(\nu)}\right\|}{\left\|z^{(\nu+1)}\right\|} \qquad (4)$$

As regards the case (ii), the authors propose to start the iteration with another initial vector in order to obtain, *in general* (i.e. often but not always), another eigenvector associated with the dominant eigenvalue.

Concerning the case (iii), they showed that for $\mu^{(v)} = 1$ for all v, each *quotient* of $z_i^{(\nu-1)}$ to $z_i^{(\nu+1)}$ converges to the square of the required (inverse) eigenvalue λ_1 (i.e. according to our reformulation

[11] Ibid.

$\lambda_1^2 = \lim_{\nu\to\infty} \frac{\left\| z_1^{(\nu-1)} \right\|}{\left\| z_1^{(\nu+1)} \right\|}$), and $z^{'(\nu)}$ and $z^{''(\nu)}$ is converging except for a factor to the eigenvector associated respectively with $1/\lambda_1$ and $-1/\lambda_1$, where

$$z^{'(\nu)} := z^{(\nu)} + \lambda_1 z^{(\nu+1)}$$

$$z^{''(\nu)} := z^{(\nu)} - \lambda_1 z^{(\nu+1)}.$$

From today's point of view, the above analysis seems quite obvious. At that time, however, it must have been an innovative discovery because this procedure has been named after the author "von Mises Iteration" (or alternatively "Power Method"[12]). As it will turn out to be after scrutinizing it, however, Proposition 1 is unfortunately incomplete.

Contrary to Proposition 1 that starting from *any* arbitrary initial vector, the iteration (2) converges except for a factor to the dominant eigenvector (eigenvector associated with the dominant eigenvalue), there must be some vector from which the iteration (2) does not converge except for a factor to the dominant eigenvector. A very simple counterexample is $\mathfrak{A} := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, which satisfies Assumption (MA.1), i.e. $\mathfrak{A}$ is symmetric and invertible. Take a unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as the initial vector. Then, the iteration (2) leads never to the dominant eigenvalue 2 or the associated eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, Proposition 1 is not valid for *every arbitrary* initial vector but for *some suitable* initial vector.

III. Charasoff's theory of "Urkapital" and price of production.

The main ideas of von Mises iteration had been, as stated in Introduction, anticipated by Georg von Charasoff 19 years before in Charasoff (1910). Distinctive features of this anticipation are the following:

- he did not argue in a general algebraic style but in an application to economic context and by using numerical examples
- he used (implicitly) another system of assumptions
- he paid more attention to the uniqueness of solutions

[12] Bodewig (1959), p.231. Proposition 1 is called there just "Theorem of von Mises" (p. 233).

- he paid more attention to the duality of equation systems.

Just as von Mises and Pollazcek-Geiringer, Charasoff set a linear equation system of the form (1) and solved it by employing an iteration of the form (2). As already mentioned in Introduction, however, this calculation procedure was not carried out by Charasoff in an abstract form, but in an application to the economic context where the matrix $\mathfrak{A}$ was implicitly assumed as an augmented input-coefficient matrix (i.e. including not only physical input but also real wage in input-coefficients), the column vector x as quantity vector. The iteration according to (2) starting from an arbitrary initial vector $z^{(1)}$ and setting $\mu^{(v)} = 1$ for all v is called "production series (Produktionsreihe)"[13] of $z^{(1)}$. The iteration "production series" expressed the successive regression of the initial good vector to its input vector so that $z^{(2)}$ is input of $z^{(1)}$, $z^{(3)}$ is input of $z^{(2)}$, and so on. Charasoff shows that the "production series" converges except for a factor to an eigenvector and leads to its associated eigenvalue as limit of quotient of components of two successive vectors just as the analysis in von Mises und Pollaczek-Geiringer (1929) showed[14].

Furthermore, Charasoff went in two respects beyond the explicit scope of von Mises und Pollaczek-Geiringer (1929). First, he paid more attention to the *uniqueness* of limit in the iteration (2) while the analysis of the former was not confined only to the case of uniqueness (see the three cases (ii) in the previous section). He showed namely that starting from any arbitrary good vector the iteration (2) converges except for a factor to a *unique* limit and the quotient of components of two successive vectors converges to a unique limit as well (on the proof, see Appendix 3). The uniqueness means here that the limit is common for any arbitrary semi-positive initial vector (starting from null good vector would be meaningless). Therefore, the unique limit of successive vectors means the ultimate universal input and is called "original capital (Urkapital)"[15]. The unique limit of quotient is interpreted as the rate of growth (plus 1) of "original capital". Since he wanted to see also the composition of components of each vector in the production series, he considered an associated iteration setting $\mu^{(\nu)} = \frac{1}{\left\| \mathfrak{A} z^{(\nu)} \right\|}$ so that all vectors have the same length, i.e. $\left\| z^{(\nu)} \right\| = 1$ holds for all v. The eigenvector attained by this associated iteration is called "*original type* (*Urtypus*)"[16].

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[13] Charasoff (1910), p.120.

[14] As the authors carefully mentioned, if there are several different eigenvalues with the same modulus (like case (iii) in section II), the iteration does not converge even except for a factor.

[15] Charasoff (1910), p.111.

[16] Charasoff (1910), p.124.

Note that the uniqueness of limit is not guaranteed by von Mises und Pollaczek-Geiringer (1929). As already shown in section II, even in the case (i), according to the initial vector, the iteration (2) may converge except for a factor to different eigenvectors. Let us take once again $\mathfrak{A} := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, starting from the initial vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the iteration converges except for a factor to different limits, i.e. to respectively $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The same is true for the case (ii) as an example $\mathfrak{A} := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ shows. In the case (iii), there must be some initial vector which does not converge except for a factor at all. Let us take an example $\mathfrak{A} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and start from $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then, the sequence must oscillate. Therefore, the concept of "Urkapital" could hardly be obtained from von Mises and Pollaczek-Geiringer (1929). As the above examples show, the concept of "Urkapital" would lose its whole meaning (any good vector converges except for a factor to a unique input vector, i.e. "Urkapital" as the ultimate universal input).

Second, Charasoff was well aware of the *duality* of linear equation systems. After solving the system (1) as primal problem, he moved on to the dual problem, i.e.

$$p = \lambda p \mathfrak{A}, \qquad (5)$$

and the iteration for solving the problem is

$$w^{(\nu+1)} = \mu^{(\nu)} w^{(\nu)} \mathfrak{A} \quad (\nu = 1, 2, ...) \qquad (6)$$

The iteration according to (6) was interpreted by Charasoff as "capitalistic competition"[17] which tends to level out individual rates of profits. The eigenvector attained by the iteration was interpreted as "price of production (Produktionspreis)"[18], and the (inverse) eigenvalue λ as the rate of profit (plus 1). Thus, the iteration starting from an arbitrary (positive) initial vector $w^{(1)}$ represents an successive progression of $w^{(1)}$ to successively corrected price vectors $w^{(2)}$, $w^{(3)}$, … converging to the unique equilibrium price "price of production". On the proof, see Appendix 3. (Since prices can be

[17] Charasoff (1910), p. 134.
[18] Charasoff (1910), p. 137.

considered to be normalized, $\mu^{(\nu)} = \dfrac{1}{\left\| w^{(\nu)}\mathfrak{A} \right\|}$ is implicitly assumed here so that all price vectors have the length of 1.) Taking particularly the vector of labour values as the initial vector, Charasoff identified this type of iteration as Marxian transformation of value to price of production[19].

IV. Charasoff's theory of "dimensions"

Charasoff drew a consequence from the duality of equation systems (1) and (5). He showed namely the way how one can solve the dual problem at the same time as the primal problem. Indeed, the rate of profit in the dual problem is automatically solved by solving the rate of growth in the primal problem because both rates are identical, i.e. the inverse eigenvalue of $\mathfrak{A}$, λ, minus 1. But he also showed that the price of production as eigenrow of $\mathfrak{A}$ can be attained in the same procedure as the "Urkapital" as eigencolumn.

According to Charasoff's theory of "Urkapital" stated in section III, starting from an arbitrary good vector, the iteration (2) is converging except for a factor to the unique eigenvector. If we take n unit vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_1, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ as initial vectors, their iterations are to converge *except for a factor* to the same eigenvector as their limit. Excepting a factor means ignoring the size (or length) and sign of vectors. What is now at stake, however, is just the size (called "dimension" by Charasoff) of vectors, and it may be different between the iterations at each stage.

Let us consider the ν-th terms of the iteration ("production series") of e_1, e_2, …, e_n and call them $e_1^{(\nu)}, e_2^{(\nu)}, \cdots, e_n^{(\nu)}$ respectively. The vector of their sizes ("dimensions") at this stage is therefore $\eta^{(\nu)} := \left(\left\| e_1^{(\nu)} \right\|, \cdots, \left\| e_n^{(\nu)} \right\| \right)$. Thus, we obtain a sequence of dimension vectors associated with the production series of e_1, e_2, …, e_n. Charasoff showed and exemplified by a numerical example that this sequence $\{\eta^{(\nu)}\}$ converges *except for a factor* as well. According to our reformulation (3):

[19] Charasoff (1910), p. 138. Twenty three years later, Shibata (1933, 49–68) illustrated a similar iteration with a numerical example without referring to Charasoff, and Okishio (1972, 1973, 1974) then provided a formal proof of the convergence.

$$\lim_{v\to\infty}\left\|\frac{\eta^{(v)}}{\left\|\eta^{(v)}\right\|}-\frac{\eta^{(v+1)}}{\left\|\eta^{(v+1)}\right\|}\right\|=0 \quad \text{and} \quad \lim_{v\to\infty}\frac{\eta^{(v)}}{\left\|\eta^{(v)}\right\|}=: p \qquad (7)$$

(note that $\eta^{(v)}$ is non-negative here for all v and therefore we do not need to use "min" in (3)) and that p, i.e. the limit (except for a factor) of the sequence of dimensions $\{\eta^{(v)}\}$, is a eigenrow of matrix $\mathfrak{A}$ and the (normalized) price of production. On the proof, see Appendix 3.

Just as the concept of "Urkapital", the insight into the simultaneous determination of eigencolumn and eigenrow could hardly be obtained from von Mises and Pollaczek-Geiringer (1929). Because the concept of "dimension" would lose its whole meaning if the concept of "Urkapital" could not be established, i.e. if initial vectors may converge to different limits.

V. Charasoff's system of (implicit) assumptions

For Proposition 1 in section II to be valid, the following condition is necessary:

(A.1) there is a real and non-zero dominant eigenvalue.

This condition is necessary and sufficient in order that there exists some vector such that starting from it as the initial vector the von Mises Iteration (2) converges except for a factor to an eigenvalue associated with the dominant eigenvalue. von Mises and Pollaczek-Geiringer made Assumption (MA.1), i.e. symmetry and invertibility of $\mathfrak{A}$, to sufficiently guarantee (A.1) .

As we saw in section III, however, their assumption (MA.1) is not suitable to found the theory of Urkapital and price of production à la Charasoff on it. Assumption (MA.1) does not guarantee in particular the uniqueness of limit of iteration over the whole meaningful domain of initial vector. Contrarily, to guarantee the uniqueness and establish his theory consistently, Charasoff had deliberately elaborated a framework of his argument, particularly by effectively introducing the concept of basic and non-basic products / production ("Grund- und Nebenproduktion"[20]). Note that the concept of basic and non-basic product is defined by Charasoff in terms of *augmented* input coefficients (i.e. including real wage). On his definition of basic and non-basic products, see Appendix 1. Charasoff's framework of argument amounts to the following set of assumptions:

(CA.1) All input coefficients are non-negative.
(CA.2) The real wage vector is non-negative and non-zero (i.e. semi-positive).

[20] Charasoff (1910), p.81.

(CA.3) Labour is directly used in all sectors.
(CA.4) Non-basic products are not used as input in any sector.

(CA.1), (CA.2) and (CA.3) are almost self-evident from the economic point of view, but (CA.4) is somewhat restricting. These assumptions as a whole imply the following properties of augmented input coefficient matrix $\mathfrak{A}$. On the proof, see Appendix 2.

- Matrix $\mathfrak{A}$ has a dominant eigenvalue λ_1 which is unique, positive and simple
- the associated eigenrow *p* is positive
- the associated eigencolumn *x* is semi-positive

These properties of $\mathfrak{A}$ guarantee, unlike Assumption (MA.1), the uniqueness of limit of iteration over the whole meaningful domain of initial vector (semi-positive $z^{(1)}$ and positive $w^{(1)}$), and besides the non-negativity of limit. On the proof, see Appendix 3.

VI. Procedure for solving inhomogeneous linear equation systems

In von Mises and Pollaczek-Geiringer (1929), the authors consider the following inhomogeneous equation system[21]:

$$\mathfrak{A}x - r = 0 \qquad r \neq 0 \qquad (8)$$

where $\mathfrak{A} = (\alpha_{ij}) \in \mathbb{R}^{n \times n}$ is a coefficient matrix, $r \in \mathbb{R}^n$ is a constant, and $x \in \mathbb{R}^n$ is an unknown. Note that the non-negativity of constants and valuables is not assumed also here. The problem is to solve a vector *x*. And throughout this part, the authors made the following assumption[22]:

(MA.2) $\mathfrak{A}$ is *invertible*

The existence of a unique solution *x* of (8) is also obvious, the problem here, however, is not to show the existence but to calculate solutions specifically.

To be able to accomplish the task of calculation sufficiently precisely and conveniently, they

[21] Mises and Pollaczek-Geiringer (1929), p.62.
[22] Ibid.

proposed also here to use the following iteration[23]:

$$x^{(\nu+1)} = (I + C\mathfrak{A})x^{(\nu)} - Cr \quad (\nu = 1, 2, ...) \qquad (9)$$

where $x^{(\nu)} \in \mathbb{R}^n$ for ν = 1, 2, ... and C is a diagonal matrix with positive diagonal elements, i.e.

$$C := \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \in \mathbb{R}^{n \times n}, \ c_i > 0 \text{ for all } i.$$

Then, the authors conclude with the following proposition (which corresponds to "Satz 5"[24]).

Proposition 2 (Mises and Pollaczek-Geiringer)
The iteration (9) converges to the unique solution of (8) if and only if all eigenvalues of the matrix $I + C\mathfrak{A}$ have a modulus less than unity.

In other words, if we can find such a matrix C that the modulus of the dominant eigenvalue of $I + C\mathfrak{A}$ is less than one, then and only than $\lim_{\nu \to \infty} x^{(\nu)} = x$ holds, where $x^{(\nu)}$ and x are defined by (9) and (8) respectively .

As corollaries, von Mises and Pollaczek-Geiringer showed some *sufficient* conditions for the iteration (9) to be successful.

The first sufficient condition consists in $\sum_{i=1}^{n} \frac{|\alpha_{ij}|}{|\alpha_{ii}|} < 1$ for all j[25]. And the second sufficient condition is $\sum_{i,j} \frac{\alpha_{ij}^2}{\alpha_{ii}^2} < 1$[26]. If one of both is valid, then the iteration (9) leads to the solution of (8) by taking $C = I$.

VII. Charasoff's theory of "Reproduktionsbasis" and labour value

[23] Ibid, p.63.
[24] Ibid, p.68.
[25] Ibid, p.64.
[26] Ibid, p.67.

Just as in the case of homogeneous equation systems, Charasoff anticipated the main ideas of the iterative procedure proposed by von Mises and Pollaczek-Geiringer (1929) for solving inhomogeneous equation systems. Also here, his style of argument is characterized by exemplifying the procedure with numerical examples applied to an economic context. The economic problem to which he applied the iterative procedure for solving inhomogeneous equation systems was the problem of calculating labour values of commodities. What was very characteristic of his calculation, is that he intentionally presented two different procedures of calculation, namely a simultaneous method, i.e. by solving the value equation directly on the one hand, and a recursive method, i.e. by counting retroactively a whole series of past expended labour.

For this part of analysis, Charasoff made implicitly the following assumption.

(CA.5) Positive net product in all sectors is possible. In other words, the dominant eigenvalue of the input coefficient matrix *A* is less than unity.

Now, we examine the first procedure. Charasoff considered a 3-sector economy which consists of sectors of means of production (Sector I), means of subsistence (Sector II) and luxuries (Sector III) and has the following production technique[27]. And the real wage rate is 1 unit of means of subsistence.

$$\begin{cases} \text{70 unit of means of production} \oplus \text{30 unit of labour} \rightarrow \text{100 unit of means of production} \\ \text{20 unit of means of production} \oplus \text{20 unit of labour} \rightarrow \text{100 unit of means of subsistence} \\ \text{10 unit of means of production} \oplus \text{50 unit of labour} \rightarrow \text{1 unit of luxuries} \end{cases}$$

Note that the arrow means the production process and its LHS is input and its RHS output. And ⊕ means the union of inputs (the notation is overtaken from Kurz/Salvadori (1995)). Obviously, the calculation of labour values is very easy to be carried out in such a case. Charasoff's procedure of calculation is the following[28]:

$$(100-70)w_1 = 30 \qquad (10)$$
$$20w_1 + 20 = 100w_2 \qquad (11)$$
$$10w_1 + 50 = w_3 \qquad (12)$$

Solving (10)(11)(12) , he conclude $w_1 = 1$, $w_2 = 0.4$ and $w_3 = 60$. Accordingly, he calculated the rate

27 Charasoff (1910), pp.94-5.
28 Ibid.

of surplus value, i.e. (1−0.4)/0.4 = 3/2.

It must be noted here that Charasoff calculated the labour values in a *simultaneous* manner and not in a recursive manner, in other words, he calculated them by solving the value equation directly and not by counting retroactively a series of past labour. We can verify this fact by seeing that his procedure (10)(11)(12) is equivalent to the usual value equation[29]:

$$w = wA + l \qquad (13)$$

where $A \in \mathbb{R}^{n \times n}$ is an input coefficient matrix in usual sense (i.e. without real wage), $l \in \mathbb{R}^n$ is a labour input coefficient vector and $w \in \mathbb{R}^n$ is a vector of labour value. In his example, Charasoff set $A = \begin{pmatrix} 0.7 & 0.2 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $l = \begin{pmatrix} 0.3 & 0.2 & 50 \end{pmatrix}$.

However, Charasoff is well aware that this simultaneous method of calculation is very difficult to be carried out in the reality with a large number of sectors "because the value of each means of production contains the value of those means of production which had to be used for producing it. Thus, one has a series of equations in which unkowns appear on both sides of equation. Just as if one moves around in an unsolvable circle"[30]. He then proposed a new concept for a breakthrough by saying: "Only the concept of reproduction basis (Reproduktionsbasis) solves this wrong cycle"[31].

The second procedure for calculating labour values Charasoff presented was indeed a procedure using the concept of "Reprodutionsbasis". The reproduction basis was defined by Charasoff in the following manner.

Let $x^{(1)}$ be an arbitrary good vector. Make a "production series" of $x^{(1)}$ by ignoring real wage in input and denote it $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ The "reproduction basis" of $x^{(1)}$ is defined by $x^{(1)} + x^{(2)} + x^{(3)} + \ldots$

Let us reformulate Charasoff's definition of reproduction basis. Let $A \in \mathbb{R}^{n \times n}$ be an input coefficient matrix in usual sense i.e. an input coefficient matrix including only physical input coefficient and not real wage. For any semi-positive $x^{(1)} \in \mathbb{R}^n_+$, $\tilde{x}$ is called the reproduction basis

29 The value equation had been already published in 1904 by Dmitriev (1974).
30 Charasoff (1910), p. 147.
31 Ibid.

of $x^{(1)}$ if and only if

$$\tilde{x} = \lim_{\nu \to \infty} \sum_{k=0}^{\nu-1} A^k x^{(1)} \qquad (14)$$

holds. Then, Charasoff proposed to calculate the labour value of $x^{(1)}$, $w(x^{(1)})$, as follows:

$$w(x^{(1)}) = l\tilde{x} = \lim_{\nu \to \infty} l \sum_{k=0}^{\nu-1} A^k x^{(1)} \qquad (15)$$

Let w be, as before, the vector of labour values, i.e. the vector of labour values of product units. Then, we have

$$w = \left(w(e_1), \cdots, w(e_n) \right)$$

where e_i is the i-th unit vector. According to Chrasoff's recursive procedure (15), we obtain

$$w = \lim_{\nu \to \infty} l \sum_{k=0}^{\nu-1} A^k \ . \qquad (16)$$

As we have just seen, Charasoff proposed to replace the simultaneous method (13) by the recursive procedure (16) as the only way to avoid the vicious cycle of the simultaneous method. We can show that Charasoff's proposal of the two alternative procedures for calculating labour values anticipated the main ideas of von Mises and Pollaczek-Geiringer's iterative procedure (9).

As we can easily see, if we substitute $I-A$ and l for respectively $\mathfrak{A}$ and r, then the value equation (13) belongs to inhomogeneous equation systems of form (8). On the other hand, according to (9), the iterative procedure proposed by von Mises and Pollaczek-Geiringer for solving the value equation (8) would be, taking $C = I$, the following iteration:

$$w^{(\nu+1)} = w^{(\nu)}(I + (I - A)) - l = w^{(\nu)}A - l$$
$$w^{(\nu+1)} = w^{(1)}A^{\nu} - l \sum_{k=0}^{\nu-1} A^k \qquad \nu = 1,2, \ldots$$

By Assumption (CA.5), we obtain

$$\lim_{\nu\to\infty} A^\nu = 0\,. \qquad (17)$$

And note that recalling $C = I$ and $\mathfrak{A} = I - A,$ Assumption (CA.5) implies that all eigenvalues of $I + C\mathfrak{A}$ have a modulus less than unity, which meets the (necessary and sufficient) condition of Proposition 2 of von Mises and Pollaczek-Geiringer . Therefore, the iteration (9) is guaranteed to succeed, and by (17)

$$w = \lim_{\nu\to\infty} w^{(\nu)} = w^{(1)} \lim_{\nu\to\infty} A^\nu + \lim_{\nu\to\infty} l \sum_{k=0}^{\nu-1} A^k = \lim_{\nu\to\infty} l \sum_{k=0}^{\nu-1} A^k \qquad (18)$$

holds. This iterative procedure of Mises and Pollaczek-Geiringer provides just the same result (18) as Charasoff's recursive procedure (16) for calculation the labour value. Therefore, Charasoff's presentation of two alternative and equivalent procedures for calculating labour values can be interpreted as an anticipation of von Mises and Pollaczek-Geiringer's procedure for solving inhomogeneous equation systems.

VIII. Conclusion

Georg von Charasoff's linear economic analysis (1910) anticipated the main ideas of iterative procedures for solving linear equation systems by von Mises und Pollaczek-Geiringer (1929). Since Charasoff did not argue in a general algebraic style but in an application to economic context and by using numerical examples, his analysis cannot be seen to contain an algebraic general proof. On the other hand, however, his analysis went beyond the explicit scope of von Mises und Pollaczek-Geiringer (1929). He namely elaborated deliberately his framework of argument in order to guarantee the uniqueness of solutions, and in doing so, enabled a profound insight to the duality of equation systems.

Appendix 1[32]. Basic and non-basic products in Charasoff's system

We first introduce the following symbols for the following appendices:

- input coefficient of good *i* for Sector *j*: $a_{ij} \in \mathbb{R}_+$
- input coefficients matrix: $A := (a_{ij}) \in \mathbb{R}_+^{n\times n}$
- labour input coefficient for Sector *j*: $l_j \in \mathbb{R}_+$
- vector of labour input coefficients: $l := (l_1, \cdots, l_n) \in \mathbb{R}_+^n$
- vector of real wage (wage basket) per labour unit: $d \in \mathbb{R}_+^n$

[32] In order to make the argument in this paper easier to understand, we attached a summary of important parts of Mori (2011) as appendices.

- augmented input coefficients matrix: $B=(b_{ij}):=A+dl\in\mathbb{R}_{+}^{n\times n}$

Note that we use inequality signs for vectors and matrices in this paper so that $X>Y$, $X\gneq Y$ and $X\geq Y$ denote that $X-Y$ is positive, semi-positive and non-negative, respectively.

Now, consider then the matrix $\sum_{t=1}^{n}B^{t}$. A good i is a basic product ('Grundprodukt') if and only if the i-th row of this matrix is positive. Goods that are not basic products are non-basic products.

An equivalent definition can be given in the following manner. If B is indecomposable, all n goods are basic products. If B is decomposable, it can be transformed into the following form by suitable simultaneous substitutions of rows and columns.

$$B=\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n_0} \\ 0 & B_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & B_{n_0n_0} \end{pmatrix}$$

where $B_{11},\cdots,B_{n_0n_0}$ ($n_0\leq n$) denote either a square null-matrix or a non-negative indecomposable square matrix. If B_{11} is a null matrix, there is no basic product. Otherwise, choose an index i such that b_{ii} is an element of B_{11}, and that for each j = 2,...,n_0, $\begin{pmatrix} B_{1j} \\ \vdots \\ B_{j-1j} \end{pmatrix}\gneq 0$ holds if B_{jj} is indecomposable and $\iota\begin{pmatrix} B_{1j} \\ \vdots \\ B_{j-1j} \end{pmatrix}>0$ ($\iota:=(1,...,1)$) holds if B_{jj} is a null-matrix. Then, the good i is a basic product.

Appendix 2. Implications of Charasoff's assumptions

It is useful to note the following: First, since, according to Assumptions (CA.2) and (CA.3), labour is directly used in all sectors and the real wage vector is semi-positive, B has at least one positive row, say i. According to the definition, good i is a basic product, and therefore, the set of basic products is not empty.

Second, from (CA.2), (CA.3) and (CA.4), it follows that B have the following forms if a non-basic product exists:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}, \text{ and } B_{11} \gneq 0,\ B_{12} \gneq 0 \qquad \text{(a1)}$$

If no non-basic products exist, we have $B = B_{11}$.

Now, let $\{1,...,k\}$, $1 \le k \le n$ be the set of indices of basic products. Then, because of the form of B in (a1), the (k + 1)-th to n-th rows are null. The characteristic equation of B is $t^{n-k}\left|tI - B_{11}\right| = 0$ (where I is the unit matrix). All eigenvalues of B except n-k noughts are therefore the same as those of B_{11}, and both B_{11} and B have the same multiplicity for each identical eigenvalue except for zero. Because B_{11} is indecomposable according to the definition of the basic product, the Frobenius root of B_{11} is the same as that of B and a simple positive root of the characteristic equation of B. It is denoted as λ_0. Now, choose an index of good i_0 such that the i_0-th component of d is positive (which is possible according to (CA.2)). Then, by (CA.3), the wage good i_0 is used in all sectors, particularly in Sector i_0. Therefore, at least one diagonal element of B_{11} is positive, which implies that the matrix B_{11} is primitive, i.e. the modulus of all eigenvalues of B_{11} and B except λ_0 is less than λ_0.

Next, we examine the eigenvectors of B associated with λ_0. Because B_{11} is indecomposable, we can choose a positive eigencolumn u of B_{11} and a positive eigenrow $\tilde{u}$ associated with the eigenvalue λ_0. Then, $v := \begin{pmatrix} u \\ 0 \end{pmatrix}$ and $\tilde{v} := (\tilde{u}, \tilde{u}B_{12}/\lambda_0)$ are eigencolumn and eigenrow of B associated with λ_0, respectively, where $u > 0$, $\tilde{u} > 0$, $\lambda_0 > 0$ and $\tilde{v} > 0$. Because the eigenspace of B associated with λ_0 is one-dimensional, all eigencolumns and eigenrows associated with λ_0 are respectively equal to v and $\tilde{v}$ except for a scalar.

Appendix 3. Uniqueness of limits in Charsoff's iterations

Define a matrix $\overline{B}$ as $\overline{B} := B/\lambda_0$, and consider the limit of $\overline{B}^t$ for $t \to \infty$. As discussed above, the Frobenius root of $\overline{B}$ is unity and simple, and v and $\tilde{v}$ are its eigencolumn and eigenrow, respectively. It is also clear that the modulus of all eigenvalues of $\overline{B}$ except for unity is smaller than unity. Therefore, for $t \to \infty$, $\overline{B}^t$ converges to a limit that is a semi-positive matrix. Letting $\overline{B}^*$ be the limit, we have:

$$\overline{B}^* = \lim_{t\to\infty} \overline{B}^t = \lim_{t\to\infty} \overline{B}^{t+1} = \lim_{t\to\infty} \overline{B}\overline{B}^t = \overline{B}\overline{B}^* = \overline{B}^*\overline{B}$$

We can see that each column and row of $\overline{B}^*$ is either a eigenvector of $\overline{B}$ associated with the Frobenius root or the null vector. Therefore, there is a row vector $q \in \mathbb{R}^n$ such that

$$\overline{B}^* = vq \qquad \text{(a2)}$$

Because of $\tilde{v}\overline{B} = \tilde{v}$, we also have:

$$\tilde{v} = \tilde{v}\overline{B} = \tilde{v}\overline{B}^2 = \tilde{v}\overline{B}^3 = \cdots = \tilde{v}\overline{B}^* \qquad \text{(a3)}$$

From (a2) and (a3), we obtain:

$$\tilde{v} = \tilde{v}\overline{B}^* = \tilde{v}vq \qquad \text{(a4)}$$

$\tilde{v}v > 0$ holds because $v \gneq 0$ and $\tilde{v} > 0$. Then, from (a2) and (a4), we obtain:

$$\overline{B}^* = vq = \frac{1}{\tilde{v}v} v\tilde{v} = \frac{1}{\tilde{v}v}(\tilde{v}_1 v, \cdots, \tilde{v}_n v) \qquad \text{(a5)}$$

where $\tilde{v}_i$ is the *i*-th component of $\tilde{v}$ and $\tilde{v}_i > 0$ for all *i*. We can see that $\overline{B}^*$ is a matrix of rank one and each column is a (semi-positive) eigencolumn, and each row is either a (positive) eigenrow or null vector.

Now, take an arbitrary semi-positive vector $x \in \mathbb{R}^n_+$ and define a sequence $\{z^{(t)}\}$ by $z^{(t)} := B^{t-1}x$.

Then, because of $\tilde{v} > 0$, $x \gneq 0$ and the definition of $\overline{B}$ and $\overline{B}^*$, we have

$$\lim_{t\to\infty} \frac{z^{(t)}}{\left\|z^{(t)}\right\|} = \lim_{t\to\infty} \frac{B^{t-1}x}{\left\|B^{t-1}x\right\|} = \lim_{t\to\infty} \frac{\lambda_0^{t-1}\overline{B}^{t-1}x}{\left\|\lambda_0^{t-1}\overline{B}^{t-1}x\right\|} = \lim_{t\to\infty} \frac{\overline{B}^{t-1}x}{\left\|\overline{B}^{t-1}x\right\|} = \frac{\overline{B}^*x}{\left\|\overline{B}^*x\right\|} = \frac{\frac{1}{\tilde{v}v} v\tilde{v}x}{\left\|\frac{1}{\tilde{v}v} v\tilde{v}x\right\|} = \frac{v}{\|v\|}.$$

Therefore, for any $x \gneq 0$, the sequence $\{z^{(t)}\}$ ("production series" of *x*) converges except for a factor to a unique eigencolumn $\frac{v}{\|v\|}$.

Similarly, Take an arbitrary positive vector $p \in \mathbb{R}^n_{++}$ and define a sequence $\{w^{(t)}\}$ by $w^{(t)} := pB^{t-1}$.

Then, because of $p > 0$, $v \gneq 0$ and the definition of $\overline{B}$ and $\overline{B}^*$, we have

$$\lim_{t\to\infty} \frac{w^{(t)}}{\left\| w^{(t)} \right\|} = \frac{\frac{1}{\tilde{v}v} pv\tilde{v}}{\left\| \frac{1}{\tilde{v}v} pv\tilde{v} \right\|} = \frac{\tilde{v}}{\left\| \tilde{v} \right\|}.$$

Therefore, for any $p>0$, the sequence $\{w^{(t)}\}$ ("capitalistic competition" starting from p) converges except for a factor to a unique eigenrow $\frac{\tilde{v}}{\left\| \tilde{v} \right\|}$.

Appendix 4. Convergence of the sequence of "dimensions" to a eigenrow

Now, let us take unit vectors $e_1, e_2, \ldots, e_n$ as initial vectors of "production series" $\{z^{(t)}\}$ in Appendix 3. Then, we can define the sequence of dimensions associated with "production series" of n unit vectors, $\{\eta^{(t)}\}$, as follows:

$$\eta^{(t)} := \left(\left\| B^{t-1} e_1 \right\|, \cdots, \left\| B^{t-1} e_n \right\| \right).$$

Then, because of $\tilde{v} > 0$, $e_i \gneq 0$ and the definition of $\overline{B}$ and $\overline{B}^*$, we have

$$\lim_{t\to\infty} \frac{\eta^{(t)}}{\left\| \eta^{(t)} \right\|} = \lim_{t\to\infty} \frac{\left(\left\| B^{t-1} e_1 \right\|, \cdots, \left\| B^{t-1} e_n \right\| \right)}{\left\| \left(\left\| B^{t-1} e_1 \right\|, \cdots, \left\| B^{t-1} e_n \right\| \right) \right\|} = \lim_{t\to\infty} \frac{\left(\left\| \overline{B}^{t-1} e_1 \right\|, \cdots, \left\| \overline{B}^{t-1} e_n \right\| \right)}{\left\| \left(\left\| \overline{B}^{t-1} e_1 \right\|, \cdots, \left\| \overline{B}^{t-1} e_n \right\| \right) \right\|}$$

$$= \frac{\left(\left\| \overline{B}^{*} e_1 \right\|, \cdots, \left\| \overline{B}^{*} e_n \right\| \right)}{\left\| \left(\left\| \overline{B}^{*} e_1 \right\|, \cdots, \left\| \overline{B}^{*} e_n \right\| \right) \right\|} = \frac{\frac{1}{\tilde{v}v} \left\| v \right\| \tilde{v}}{\frac{1}{\tilde{v}v} \left\| v \right\| \left\| \tilde{v} \right\|} = \frac{\tilde{v}}{\left\| \tilde{v} \right\|}$$

Thus, we can verify that the sequence of dimensions $\{\eta^{(t)}\}$ converges except for a factor to a eigenrow.

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