

**All but one:**

**How pioneers of linear economics**

**overlooked Perron-Frobenius mathematics**

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Paper prepared for the Conference

“The Pioneers of Linear Models of Production”

at the University of Paris Ouest, Nanterre, 17-18 January 2013

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## 1. Introduction

Recently MIT Press published the final two volumes, numbers 6 and 7, of *The Collected Scientific Papers of Paul Anthony Samuelson*. Because he wrote on many and widely different topics, Samuelson has often been considered the last generalist in economics, but it is remarkable how large a proportion of the final two volumes are devoted to linear economics.<sup>1</sup> Besides Samuelson, at least nine other 20<sup>th</sup> century Nobel laureates in economics (Wassily Leontief, Ragnar Frisch, Herbert Simon, Tjalling Koopmans, Kenneth Arrow, Robert Solow, Gérard Debreu, John Hicks, Richard Stone) published interesting work on similar linear models.<sup>2</sup>

Because the inputs in processes of production are either positive or zero, the theory of positive or nonnegative matrices plays a crucial role in many modern treatments of linear economics. The title of my paper refers to “Perron-Frobenius mathematics” in honour of the fundamental papers by Oskar Perron (1907a, 1907b) and Georg Frobenius (1908, 1909, 1912) on positive and on nonnegative matrices.<sup>3</sup>

Today the Perron-Frobenius mathematics of nonnegative matrices enjoys wide applications, the most sensational perhaps being its implicit use by millions of internet surfers, who routinely activate Google’s PageRank algorithm every day (Langville & Meyer, 2006). Such a remarkable worldwide application in an electronic search engine is a far cry from the original theoretical articles, written more than a century ago, by Perron and Frobenius. These two German scholars preferred to concentrate on pure mathematics. Especially Frobenius in Berlin considered applications as inferior subjects that should be relegated to technical schools.

Their preference for pure mathematics might explain why neither Frobenius, who died already in 1917, nor Perron, who remained a very active scholar until half a year before his death in 1975, ever published on mathematical economics, although the Perron-Frobenius mathematics of nonnegative matrices is often the crucial tool to prove the existence of meaningful solutions in linear economic models.<sup>4</sup> In the first half of the 20<sup>th</sup> century, numerous publications in economics (and in many other scientific disciplines) could have benefited from applying Perron-Frobenius results, but failed to do so, even the publications authored by

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<sup>1</sup> By the short expression “linear economics” I mean linear models of production of the input-output type, often called Leontief-Sraffa models.

<sup>2</sup> Some bibliographic examples: Leontief (1928, 1941, 1951), Frisch (1934a, 1934b), Hawkins & Simon (1949), Koopmans (1951), Arrow (1951), Solow (1952), Debreu & Herstein (1953), Dorfman, Samuelson & Solow (1958), Hicks (1960), Stone (1970).

<sup>3</sup> Good introductions for economists are provided by Nikaido (1968, 1970) or Takayama (1985). Nikaido’s 1970 book contains a very long chapter, simply titled “the Frobenius Theorem”, which is a generic name for covering some original results by Frobenius and many related theorems on eigenvalues and eigenvectors, determinants, solutions of related equations or inequalities, etc. The relevant Chapter 4 in Takayama’s book is titled “Frobenius’ theorems, dominant diagonal matrices, and applications”. It contains more historical remarks and bibliographical references than many other books, thanks to Takayama’s wide knowledge of both the Japanese and the Western literature on mathematics and economics. See also Dorfman, Samuelson & Solow (1958), Kurz & Salvadori (1995) and Bidard (2004).

<sup>4</sup> A comprehensive Perron bibliography is provided by Heinhold (1980) and Frank (1982). On the historical origins of Perron-Frobenius mathematics, see Thomas Hawkins (2008, 2013).

mathematicians. Before the Second World War, nearly all pioneers of linear economics overlooked Perron-Frobenius: either they didn't know it, or they didn't realize its relevance. I know of only one amazing exception: the French Jesuit mathematician Maurice Potron (1872-1942) made explicit and intense use of Perron-Frobenius mathematics in the core of his economic model, as early as 1911, one year before the third and final paper by Frobenius (1912) appeared. But Potron's contemporaries didn't recognize his originality, and his economic writings fell into oblivion, until a paper by Émeric Lendjel (2000) started the process of changing Potron's status in economics from zero to hero.

The seminal publications on linear economics in the 1910s and 1920s were written by Charasoff, Potron, Bray and Remak. All four were mathematicians! All except Potron overlooked the usefulness of Perron-Frobenius. It is interesting to study how Perron-Frobenius was overlooked so long in economics, and to provide additional insight into the communication or lack of communication between the different protagonists of economics and mathematics.

My paper is organized as follows. Section 2 starts with Georg Charasoff, because he was the first pioneer who wrote after Perron's 1907 results on positive matrices became available. For many reasons, Charasoff had a higher probability of discovering and applying Perron-Frobenius than the average mathematician. Two of his mathematical interests (irreducibility of equations, and continued fractions) were close to Perron's interests, but it is doubtful whether Charasoff knew Perron's matrix theorems. Maybe Charasoff just assumed that the properties of his simple numerical examples with three commodities also hold in the general case with  $n$  commodities.

Section 3 briefly describes the now-well known case of the French Jesuit mathematician Maurice Potron, whose linear model of production used Perron-Frobenius as early as 1911, in order to prove the existence of positive solutions for the relevant variables. He presented remarkable duality properties that show the connection between his quantity system and his price system, and he anticipated the famous Hawkins-Simon conditions. All his pioneering results were neglected by his contemporaries.

Section 4 contains a few remarks about Hubert Bray's 1922 paper, on Cournot's equations of exchange. Section 5 discusses the mathematically similar system presented by Robert Remak in 1929. I briefly mention his forgotten economic paper of 1918, on the repayment of the national debt, and then I focus attention on the mathematical economics of his 1929 paper on superposed prices. Here Remak neglected the useful results of Frobenius (his own doctoral supervisor in 1911), and moreover it turns out that most of Remak's mathematical arguments concentrate on rather unusual economic systems, in which the most important commodities have zero prices.

Section 6 presents related topics, about the two eponyms of Leontief-Sraffa models. It is now well-known that both Leontief and Sraffa started their research on such topics in the late 1920s, although both were confronted with some delays in their publication processes. Archival evidence suggests that Leontief was well aware of Bray's and Remak's work, and even contacted Remak in Berlin in 1931 (maybe in vain). Ultimately Leontief presented his

input-output algebra to the readers of Anglo-Saxon journals only in 1937, after being rejected by Keynes for the *Economic Journal* in 1933. Even more than Leontief, Sraffa preferred to work in isolation. He didn't discuss his mathematical problems with competent economic colleagues in Cambridge, nor with the specialists of the *Econometric Society*, but preferred help from three non-economists: Ramsey, Watson and especially Besicovitch. The latter surely knew Perron (but not his results on positive matrices), was a student of Markov and had published one paper involving dominant diagonal matrices in 1918. The above description suggests that Besicovitch in his early days "came close" to Perron-Frobenius results, but it is well-known that he didn't know Perron-Frobenius, and invented his own proofs for Sraffa in the 1940s.

Mathematicians like Charasoff, Bray, Remak, Besicovitch, and others didn't use the relevant theorems on nonnegative matrices in economics. This gives additional prestige to the remarkable performance of Maurice Potron, who in 1911 provided the first application of the theorems in any discipline. Economics didn't perform worse than other disciplines, which also overlooked the usefulness of Perron-Frobenius for a long time.

## 2. Charasoff goes to the limit

Georg Charasoff <sup>5</sup> was born in Tbilisi in 1877, in a wealthy family. After his doctoral dissertation on mathematics (Heidelberg in 1902), he was living the free life of a “rentier” and an “independent scholar” in Germany and Switzerland, and he published two books on Marx, *Karl Marx über die menschliche und kapitalistische Wirtschaft* (1909) and *Das System des Marxismus* (1910). Further planned work on economics was never published, due to negative changes in personal and political circumstances.<sup>6</sup> His second wife poisoned herself with cyanide in 1912, and his publisher went out of business. Charasoff himself sexually approached his house-maid in Zurich, refused to pay her wage when she refused him, and then was condemned by a Swiss court on 24 March 1915. Charasoff didn't attend the hearings, where several witnesses offered very negative descriptions of his personality. At that moment, Charasoff was already back in his birth place Tbilisi, trying to save the family wealth in turbulent political times. He never returned to Western Europe, and died in Stalin's Soviet Union in 1931.<sup>7</sup>

For many decades after his death, Charasoff was classified in the lower divisions of heterodox economics. Only a small minority of Marxian studies reviewed or cited his two books, usually concentrating on minor details, without recognizing the striking originality of some analytical tools in his second book *Das System des Marxismus*. In the preface, Charasoff (1910: xii) claims to have presented a “definitive solution” to the main problems of the classical theory of value, thanks to his theory of *Urkapital* (“original capital”), a composite commodity similar to Sraffa's standard composite commodity (Sraffa 1960), but in Charasoff's case it is computed with respect to an augmented input matrix, which includes both the material inputs of the production processes and the wage goods for the workers. In the language of modern mathematical economics, *Urkapital* corresponds to an eigenvector of the augmented input matrix.

Neither the mathematicians nor the economists took notice of Charasoff's innovations. A special example of overlooking Charasoff's originality is provided in a doctoral dissertation on the economics of Bortkiewicz and related topics, written by Werner Klimpt (1900-1978), and defended at the Ruprecht-Karls-Universität Heidelberg in 1931, the same university where Charasoff had received his doctoral degree in 1902, and where Perron had been a professor from 1914 to 1922. Klimpt published his text in Berlin in 1936, and seems to be one of the few Charasoff readers who comments on Charasoff's innovating concept of *Urkapital*.

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<sup>5</sup> He signs his first economic book in 1909 as Dr. Georg von Charasoff, but in his other publications and in his three extant letters to Kautsky he omits the “von”. Some studies refer to him as G. A. Kharazov (G. A. = Georg Artemovich) or Charasov.

<sup>6</sup> Several extracts from his two books were later reprinted in *Die Aktion* or in *Der Gegner*; one extract even in both journals around the same time (see Charasoff 1920).

<sup>7</sup> For many years, biographical details on Charasoff were scarce. Recently more information on Charasoff became available, thanks to Peter Klyukin (2008) and Christian Gehrke (2012). The well-documented biographical paper by Gehrke presents a wealth of new material on Charasoff's activities in Heidelberg, Zurich and Lausanne, and also on his later years in Tbilisi, Baku and Moscow. Gehrke's bibliography contains a long list of publications by and on Charasoff. Mori (2007: 132-133) and Gehrke (2012: 19) mention that Charasoff in 1907 submitted a paper on Marx to Karl Kautsky, editor of *Die Neue Zeit*, but the manuscript was rejected and is not extant.

Despite his mathematical background, Klimpt (1936: 119) rejects Charasoff's device as "highly peculiar", "complicated", "confused" etc.<sup>8</sup> Apparently, Charasoff's innovations were not understood, neither by Klimpt nor by his doctoral referee Emil Julius Gumbel<sup>9</sup> and his co-referee Emil Lederer.

In the 1980s the scientific reputation of Charasoff suddenly started to grow, thanks to the remarkable discoveries of Egidi & Gilibert (1989 [1984]). In the following decades, many other authors studied Charasoff's original theoretical model (see for example, Kurz & Salvadori (1995, 2000), Stamatis (1999) and Mori (2007, 2011)), all awarding to Charasoff the status of a remarkable pioneer.

It is now well-known that Charasoff independently discovered some Sraffa-like tools: his *Grundprodukte* bear a close resemblance to Sraffa's basic commodities, his *Reproduktionsbasis* reminds us of Sraffa's subsystems, his labour values are computed via dated labour series, etc. The most fundamental new concept invented by Charasoff is his notion of *Urkapital*, corresponding to an eigenvector of the augmented input matrix.<sup>10</sup> The way this notion of *Urkapital* was developed in Charasoff's work, is rather unexpected. An attempt to explain Charasoff's approach requires some knowledge of his mathematical background.

### 2.1. Charasoff's mathematical background

Charasoff started studies of medicine in Moscow, but then switched to mathematics in Germany, and presented his doctoral dissertation, *Arithmetische Untersuchungen über Irreduktibilität* in 1902, in the Faculty of Natural Sciences and Mathematics of the Ruprecht-Karls-Universität in Heidelberg. Charasoff cannot be found in the set of more than 300 names mentioned in the autobiography of his doctoral promoter Leo Königsberger (1919). The dissertation looks like a minor exercise in the research program of his promoter. Its results generated no follow up in the mathematical literature, and the dissertation fell into oblivion.

Note that the "irreducibility" in Charasoff's dissertation is different from the problem of reducible and irreducible matrices in Perron-Frobenius mathematics. It simply refers to the irreducibility of equations. Consider for example the equation  $x^2 - 5 = 0$ . It is called reducible if its left hand side (a polynomial of the second degree) can be "reduced" to polynomials of lower degree. If we consider all real numbers, this is possible, because  $x^2 - 5 = 0$  can be written as  $(x - \sqrt{5})(x + \sqrt{5}) = 0$ , but if we consider only rational numbers, the equation is irreducible. Much more sophisticated cases of irreducibility exist, of

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<sup>8</sup> In German: "höchst eigenartig" - "kompliziert" - "unübersichtlich".

<sup>9</sup> The Gumbel biography by Arthur D. Brenner (2001: 32-34) describes how the pacifist Georg Friedrich Nicolai tried to resume his university lectures in Berlin in 1920, but was shouted down by hundreds of demonstrators. When the Faculty Senate accused Nicolai of desertion and treason, only three persons signed a letter to protest against Nicolai's banning from the university: the publisher Curt Thesing, Emil Gumbel and Otto Buek. The latter was Charasoff's most important helper, the only person that received thanks in the preface of Charasoff's 1909 book. Gehrke (2012) provides interesting details on Charasoff, Buek and Nicolai, but Gumbel is not mentioned. Note that Gumbel is not a protagonist in the present story, but in the 1930s he seems to be the first scholar that could have known the work of both Charasoff and Remak and Potron and Leontief (see infra).

<sup>10</sup> For Charasoff's basics and *Urkapital*, an augmented input matrix has to be used.

course, for example in the differential equations in Charasoff's dissertation. Although the topic itself is not connected to positive or nonnegative matrices, one of the specialists on irreducibility of equations was ...Oskar Perron (1905a, 1907c).<sup>11</sup>

In the late 19<sup>th</sup> and the early 20<sup>th</sup> century, David Hilbert in Göttingen was one of the most important mathematicians of the world, and perhaps the most powerful in the German academic networks of the time. His address at the Second International Congress of Mathematicians in Paris in 1900 contained 23 famous unsolved problems, many of which still fascinate mathematicians today. Numerous scholars sent letters or papers to Hilbert, who acted as one of the editors of the influential *Mathematische Annalen* from 1902 to 1939. It is unknown whether Charasoff ever sent his dissertation results of 1902 to a journal, but in 1904 he had formulated some results on another topic: continued fractions ("Kettenbrüche"). He submitted his results to Hilbert, but obtained a rejection.<sup>12</sup> Charasoff very politely accepted the verdict in his letter of 10 May 1904 to Hilbert.<sup>13</sup> Charasoff mentioned that he had read many papers by Minkowski, but apologized for overlooking a relevant paper in which Minkowski had already worked out a similar theorem on continued fractions. No further details are given by Charasoff, who apologized for using in such an unproductive manner some of Hilbert's very precious time.<sup>14</sup>

This story shows that Charasoff was following the literature on continued fractions at that time, and that Hilbert had suggested Charasoff should get a more complete knowledge of this literature. If Charasoff was stimulated by this advice, then from 1905 on Charasoff must have noticed a series of successful publications on continued fractions by ... Oskar Perron, who would become a well-respected authority on the subject.<sup>15</sup> Perron (1905b, 1905c) wrote two papers on continued fractions for the Academy of Sciences in Munich, and then, more importantly and more visibly, he published his Habilitation thesis on the same subject in the form of the long opening article in the 1907 volume of the leading journal *Mathematische Annalen* (Perron, 1907a). In the course of his study, Perron had to prove some "auxiliary results" on matrices and their characteristic equations. His first treatment of this matrix problem was hidden in the middle of his long paper, but a few months later Perron (1907b) isolated these matrix problems in a separate article "Zur Theorie der Matrizes", again in the *Mathematische Annalen*, and he gave a complete proof of the now classic Perron Theorem on the eigenvalues and eigenvectors of positive matrices. Perron had expressed some dissatisfaction with his own arguments, because he often had to rely on limit methods (in

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<sup>11</sup> Alfred Brauer (1948: 423) writes that Perron "was the first who obtained criteria for the irreducibility of polynomials depending on the comparative size of the coefficients". Studies on irreducibility of equations often mentioned Perron and Königsberger, but never referred to Charasoff.

<sup>12</sup> After Hilbert's rejection, it remained unpublished (and unfound). The name Charasoff is not mentioned in the comprehensive bibliography on continued fractions compiled by Wölffing (1908).

<sup>13</sup> See Nachlass David Hilbert, Cod. Ms. D. Hilbert 59, Abteilung Handschriften und seltene Drucke, Niedersächsische Staats- und Universitätsbibliothek, Göttingen. I thank Dr. Hunger and his staff for providing me copies of some Hilbert correspondence with Charasoff, Remak and Potron.

<sup>14</sup> The very humble, courteous letter to Hilbert is not representative for Charasoff. Gehrke (2012) offers some negative stories on Charasoff's rude, unpleasant character in everyday life in Germany and Switzerland.

<sup>15</sup> In 1905 Charasoff enrolled as an "auditor" for Heinrich Burkhardt's lectures on elliptic functions at the University of Zurich (Gehrke 2012: 17), one of the specialties of Georg Frobenius. Note that Oskar Perron went on writing about continued fractions up until the 1970s, and his textbook *Die Lehre von den Kettenbrüchen* was often reprinted and is still recommended in some reading lists today.

algebra, pure mathematicians prefer purely algebraic tools, and try to avoid limits if possible). In the authoritative historical study on this story, Thomas Hawkins (2008: 679) used the expression “Perron’s Limit Lemma” for the following “inelegant” auxiliary tool that Perron needed in his 1907 proofs:

**Perron’s Limit Lemma** (Perron 1907b: 259-261): *Let the  $n \times n$  matrix  $A$  have all its elements  $a_{ij}$  positive, and denote the elements of its  $v$ -th power  $A^v$  by  $a_{ij}^v$ . Then for  $v \rightarrow \infty$ , we can find a positive row vector  $q = [q_1 \ \dots \ q_n]$ , a positive column vector  $p = [p_1 \ \dots \ p_n]^t$ , and a positive number  $\rho'$  such that*

$$(i) \quad \lim_{v \rightarrow \infty} \frac{a_{1k}^v}{a_{1s}^v} = \lim_{v \rightarrow \infty} \frac{a_{2k}^v}{a_{2s}^v} = \dots = \lim_{v \rightarrow \infty} \frac{a_{nk}^v}{a_{ns}^v} = \frac{q_k}{q_s}$$

$$(ii) \quad \lim_{v \rightarrow \infty} \frac{a_{i1}^v}{a_{r1}^v} = \lim_{v \rightarrow \infty} \frac{a_{i2}^v}{a_{r2}^v} = \dots = \lim_{v \rightarrow \infty} \frac{a_{in}^v}{a_{rn}^v} = \frac{p_i}{p_r}$$

$$(iii) \quad \lim_{v \rightarrow \infty} \frac{a_{ij}^{v+1}}{a_{ij}^v} = \rho' \quad \text{for every } i \text{ and } j$$

$$(iv) \quad qA = \rho'q \text{ and } Ap = \rho'p$$

In more verbal mathematics: in the limit every row of  $A^v$  becomes proportional to the row vector  $q$ , every column of  $A^v$  becomes proportional to the column vector  $p$ , and two consecutive matrices in the sequence become proportional to each other. For Perron, the above limit properties were just auxiliary tools, allowing him to prove that the positive matrix  $A$  has a positive eigenvalue  $\rho'$  that can be associated with a positive row eigenvector  $q$ , and a positive column eigenvector  $p$ . When Georg Frobenius, a pioneer in matrix algebra, read Perron’s second 1907 paper on matrices, he responded to two challenges: to offer proofs of Perron’s main results without using the “ugly” limit lemma, and to extend Perron’s results on eigenvalues and eigenvectors (and minors of certain determinants) from positive matrices to nonnegative ones. Although being in the last years of his life, in three remarkable papers on the subject, Frobenius (1908, 1909, 1912) succeeded *summa cum laude*.<sup>16</sup>

Many ironical aspects exist. Perron and Frobenius wanted to delete the limit lemma, if possible, because in algebra a pure mathematician prefers purely algebraic proofs without limits. But for a mathematical treatment of Charasoff’s economics, Perron’s Limit lemma is extremely useful. It provides the mathematical foundations of Charasoff’s economics. Given his mathematical background, and his interest in continued fractions, Charasoff was in a better position than the average mathematician to know Perron’s work, but it is strange that in the hundreds of pages of his two books, he never explicitly referred to Perron or Frobenius, not even in one footnote. Charasoff limited his algebra to simple numerical examples, and assumed, without general proofs, that the properties of his examples with three commodities also hold for economies with  $n$  commodities.

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<sup>16</sup> For more details on continued fractions and the origins of the papers by Perron and Frobenius, see Hawkins (2008, 2013).



## 2.2. Charasoff's system of production of commodities by means of commodities

The different chapters of Charasoff's 1910 book, *Das System des Marxismus*, start with a quotation from Marx (without further bibliographical details). For example, Charasoff places the following citation below the title of his brilliant chapter 10 (in German):

Coal is required for making gas, but gas lighting is used in producing coal. . . . Coal must replace the wear and tear of the steam-engine used to produce it. But the steam-engine consumes coal. Coal itself enters into the means of production of coal. Thus it replaces itself in kind. Transport by rail enters into the production costs of coal, but coal in turn enters into the production costs of the locomotive. (Charasoff, 1910: 118)

Readers of Marx's *Theories of Surplus Value* (vol. 1, p. 247) will recognize the source of this quotation. By choosing this text of Marx, Charasoff wants to emphasize the circular nature of production. In Charasoff's system commodities are produced by means of commodities, even more than in the core of Sraffa (1960), because Charasoff's lists of inputs usually replace the workers by their wage goods.<sup>17</sup>

In his 1910 book, Charasoff often considers the following numerical example with three production processes. In Sraffian layout the numbers can be presented as follows:

70 units of corn + 30 units of labour time → 100 units of corn

20 units of corn + 20 units of labour time → 100 units of bread

10 units of corn + 50 units of labour time → 100 units of cakes

Workers receive only bread (wage good), one unit of bread per unit of labour time. The cakes are a luxury product eaten only by the capitalists. If we replace "a unit of labour time" by its necessary input "a unit of bread", and express all input magnitudes per unit of output, then the Charasoff example becomes:

0.7 units of corn + 0.3 units of bread + 0 cakes → 1 unit of corn

0.2 units of corn + 0.2 units of bread + 0 cakes → 1 unit of bread

0.1 units of corn + 0.5 units of bread + 0 cakes → 1 unit of cakes

Charasoff provides a rather intuitive analysis, without any explicit use of tools from matrix algebra, although modern approaches prefer matrices in his context. Let  $A$  represent the "augmented" input matrix (containing technically necessary inputs of corn, augmented by the quantities of bread, the wage good). Assume a production period of one year, an unchanging technology, and consider the output of 1 unit of corn. The first row of the matrix  $A$  below

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<sup>17</sup> Elsewhere, in a long footnote, Charasoff (1910: 290-291) cites a similar example from Pareto, also involving coal that is ultimately produced by coal.

describes the input vector needed one year ago: Charasoff calls it “capital of the first order”. The first row of the matrix  $A^2$  describes the input vector needed two years ago, the capital of the second order;  $A^3$  describes the input vector needed three years ago, the capital of the third order, etc. In a similar way, the second rows of  $A, A^2, A^3, A^4$  etc. describe the dated input vectors for bread. The third rows describe the dated input vectors for cakes:

$$A = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.2 & 0 \\ 0.1 & 0.5 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0.55 & 0.27 & 0 \\ 0.18 & 0.10 & 0 \\ 0.17 & 0.13 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0.439 & 0.219 & 0 \\ 0.146 & 0.074 & 0 \\ 0.145 & 0.077 & 0 \end{bmatrix}, A^4 = \begin{bmatrix} 0.3511 & 0.1755 & 0 \\ 0.1170 & 0.0586 & 0 \\ 0.1169 & 0.0589 & 0 \end{bmatrix}$$

The numbers above correspond to Charasoff’s tables (1910: 114). In general, if we trace back an arbitrary output vector, say the row vector  $X$ , then the inputs of one year ago are given by  $X' = XA$ , the inputs of two years ago by  $X'' = XA^2$ , etc. Charasoff calls this a *production sequence* and denotes it as  $X \sim X' \sim X'' \sim \dots$  and he also refers to it as an “equation”.<sup>18</sup>

In a productive economy, the sequence  $A, A^2, A^3, A^4 \dots$  converges to a zero matrix. But if we normalize all matrices  $A^v$  in a suitable manner,<sup>19</sup> then the limit will be a positive multiple of

the following matrix:  $A^\infty = \begin{bmatrix} 6 & 3 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ .

In this limit situation, the rows are all proportional to a basket of Charasoff’s *Urkapital*, which is a basket of 2 units of corn, 1 unit of bread, and 0 cakes (or any positive multiple of this basket).

The intuition of Charasoff (1910: 123) for the existence of *Urkapital* is as follows. Capitals of the first order can be very different from each other. Capitals of the second order are a sort of average of capitals of the first order, and therefore are less different from each other, etc. In the limit all capitals have the same *Urkapital* composition. Charasoff (1910: 114) suggests that in his example the composition of *Urkapital* is already obvious after four rounds. Indeed, each row of  $A^4$  is nearly proportional to the *Urkapital* vector  $q = [2 \ 1 \ 0]$ . To produce  $q$  we need the input vector  $qA = [1.6 \ 0.8 \ 0]$ . The vectors  $q$  and  $qA$  are different quantities of the same composite commodity. Whatever the individual prices of corn, bread and cakes, the basket  $q$  is worth 25% more than  $qA$ , and thus the rate of profit is 25%. In this way, Charasoff (1910: 105-128) shows that the rate of profit is determined by the rate of growth of *Urkapital*. Therefore some authors, for example Kurz & Salvadori (1995: 390) and Mori (2011), give him credit for discovering a simple duality between the price system and the quantity system. In mathematical terms, the equality  $qA = 0.8q$  illustrates that the *Urkapital* vector  $q$  is a row eigenvector of the matrix  $A$ , corresponding to the dominant eigenvalue 0.8.

The numbers in the sequences for the different goods will also reveal the equilibrium prices. To have an exact view of this, we should go to the limit. But again, in this example, the

<sup>18</sup> On the one hand, the word “Gleichung” (equation) is sometimes used by Charasoff in unexpected places. On the other hand, Charasoff fails to show us the simple system of linear equations that was used a few years earlier by the non-mathematicians Mühlpfordt (in 1893 and 1895) or Dmitriev (in 1898) to determine the labour values of  $n$  commodities.

<sup>19</sup> Divide  $A$  by its dominant root 0.8, and obtain the matrix  $1.25A$ . For  $v \rightarrow \infty$  the sequence  $(1.25A)^v$  converges to a limit matrix that is proportional to the matrix  $A^\infty$  of my example. See also my next footnote.

capitals of the fourth order already suggest the limit outcome. The capital of the fourth order for good 1 (see matrix  $A^4$ , row 1) is approximately three times larger than that of goods 2 and 3 (see matrix  $A^4$ , rows 2 and 3). Intuitively speaking, infinite years ago, the “start” of the long process leading to a unit of good 1 today, required exactly three times more Urkapital than for a unit of goods 2 or 3. Therefore good 1 is three times more expensive than goods 2 or 3, and the column vector of the prices of the three goods will be equal to or proportional to the column vector  $p = [3 \ 1 \ 1]^t$ . In mathematical terms, the equality  $Ap = 0.8p$  illustrates that  $p$  is a column eigenvector of the matrix  $A$ , corresponding to the dominant eigenvalue 0.8.

Charasoff uses his Urkapital and his production sequences for many purposes: determination of prices and the rate of profit (and the law of the rising, not falling rate of profit); distinction between *Grundprodukte* and other products, where only the production conditions of *Grundprodukte* (see corn and bread above) determine the rate of profit; balanced growth with profit rate equal to growth rate; convergence of the transformation of labour values or any other value vector into prices; construction of a sort of Sraffian subsystem and dated labour to determine labour values, etc. (see Mori (2011) for more details).

The latter problem is perhaps symptomatic of Charasoff’s tendency to “go to the limit”, even when simpler methods are available. Since Mühlpfordt (1893, 1895), we know that labour values can be computed by a simple system of linear equations. Consider a productive economy, let  $M$  denote the non-augmented input matrix (thus without the wage goods), and  $\ell$  the column vector of direct labour inputs. It is well-known that the labour values correspond to the vector  $v$  that solves  $v = Mv + \ell$ , which can also be written as  $v = (I - M)^{-1} \ell$  or as a series  $v = \ell + M\ell + M^2\ell + M^3\ell + \dots$ . It is strange that Charasoff never presents a system of the form  $v = Mv + \ell$  (with or without matrix notation); he prefers series and limits.

Note that Charasoff’s Urkapital contains only *Grundprodukte*. In Sraffa’s terminology these are the basic commodities with respect to the augmented input matrix  $A$ . Charasoff knew that these “fundamental” commodities played a decisive role in his system. They can be studied in a smaller system, independently from the other commodities. In the example above, instead of using the whole matrix  $A$ , I could use the submatrix  $A_{11} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}$  of corn and bread only, which is a positive matrix, and then Perron’s Limit Lemma can be applied without worries about zero elements.<sup>20</sup>

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<sup>20</sup> I handled the whole  $3 \times 3$  matrix  $A$ , which is reducible and semipositive, but not positive. In this way, I could stay close to the tables in Charasoff (1910: 114). Perron formulated his Limit Lemma for a positive matrix  $A$ , and then mentioned that it remained valid for a semipositive matrix  $A$  if some power  $A^p$  is positive. In modern terminology, the latter means that  $A$  is semipositive, irreducible and primitive (“not cyclic”). Slightly different convergence theorems in the spirit of Perron’s Limit Lemma can still be constructed for some reducible matrices. See Egidi (1992: 249-250) who does not mention Perron, but provides a related convergence theorem for some well-behaved reducible matrices. The reducible  $3$  by  $3$  matrix  $A$  of the Charasoff example above satisfies Egidi’s requirements: it can be written in the form  $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ , where the submatrix  $A_{11}$  is a semipositive, irreducible, primitive matrix, whose dominant eigenvalue 0.8 is larger than that of the submatrix  $A_{22}$  (zero in this case). Hence, in Charasoff’s well-behaved reducible case, the columns of the limit matrix are proportional to the positive column eigenvector  $p$  associated with the dominant eigenvalue 0.8, and the rows are proportional to the semipositive row eigenvector  $q$  associated with the same dominant eigenvalue 0.8

Nikaido (1968: 110-112) proves a Limit Lemma for semipositive, irreducible, primitive matrices, without referring to the similar findings by Perron. Nikaido's theorem is usefully applied by Egidi & Gilibert (1989), in their mathematical treatment of a Charasoff system with  $n$  commodities. Kenji Mori (2011) also makes interesting use of an analogous procedure. These authors don't mention a connection between their limit theorems and Perron's Limit Lemma. As I already mentioned, Frobenius (1908, 1909, 1912) simplified the proof of Perron's theorem on the existence of a positive eigenvector of a positive matrix, and generalized his research to nonnegative matrices. For enthusiasts of pure algebra like Perron and Frobenius, the Limit Lemma was an "awkward" and "foreign" auxiliary tool. It was discarded in the simpler proofs by Frobenius and later mathematicians. In this way, Perron's Limit Lemma disappeared from the radar of most mathematicians a few years later. But if there is one person who could and should have seen it, it was Charasoff in 1907. For him, such a limit lemma was not an auxiliary tool. On the contrary, it provided the mathematical foundations of his economic argument.

### *2.3. Did Charasoff know Perron-Frobenius?*

Given the scarce archival material on Charasoff's mathematics, it is unclear whether Charasoff did or didn't know Perron's matrix algebra results (or Frobenius'). I can provide arguments for a YES and for a NO.

YES: Charasoff wrote a doctoral dissertation on mathematics, and he was able to read German. Charasoff must have followed Perron's publications in the leading German journal of mathematics in 1907, because both mathematicians were interested in research on irreducible equations, and continued fractions. Charasoff (1910: xiii) mentions that, unlike many scholars in German speaking countries, he doesn't want to intimidate his readers by means of many scholarly references and citations. He gives only numerical examples in his book to avoid problems for the non-mathematical readers.

NO: Why did Charasoff never give at least one reference to Perron in a footnote? Determinant, matrix, eigenvalue, eigenvector, Perron, Frobenius are completely absent from his text. The discovery of his Urkapital was first announced in a short appendix to chapter VIII of his first book (Charasoff 1909: 67-69), a book with preface dated 12 October 1908. In his second book Charasoff (1910: xiv, in his preface dated 24 December 1909) suggested that he found his concept of Urkapital independently from Menger's related economic idea of capitals of higher order, and he claims he started studying Menger four years before finishing his second book.

Note that memories in autobiographical stories are sometimes imperfect, as is obvious in some stories about Leontief in a later section of my paper.

### 3. Potron's tour de force

In the summer of 1902, David Hilbert received five letters from a 30-year old French engineer, Maurice Potron, who expressed his plans to concentrate on pure mathematics in his first letter (13 July 1902):

As a graduate of the École Polytechnique, I have abandoned the career of an engineer [...] to devote myself to the teaching and the study of mathematics.<sup>21</sup>

When Hilbert suggested Potron to come to Göttingen, because helping by correspondence was difficult, Potron agreed in principle, but also signalled a practical problem (letter of 29 August 1902):

I am not completely free in my actions. This year and next year, I have to take courses in theology in England, in Canterbury.

Indeed, Potron was combining his activities in mathematics with the very long training program of a Jesuit (partly in Canterbury, to avoid problems with anticlericalism in France). Ultimately Potron would write a doctoral dissertation in Paris (Potron 1904), on finite group theory, with three eminent French mathematicians in his jury: Émile Picard (his supervisor), Paul Appell and Henri Poincaré. The dissertation and a few other mathematical publications by Potron are occasionally cited in the mathematical literature.<sup>22</sup>

In the periods 1911-1914 and 1935-1942, Potron also published on economics, but only in mathematical or Roman Catholic networks, not in economic journals. These publications failed to make an impression during his lifetime, and were hardly noticed up until more than half a century after his death. The situation changed when Émeric Lendjel (2000) drew the attention of economists to Potron's status as a remarkable precursor of input-output analysis. This led to many other publications on Potron's economics, plus a French and an English edition of his economic texts.<sup>23</sup>

Inspired by Roman Catholic social thought and by his knowledge of mathematics, Potron developed a highly original economic model that used the powerful Perron-Frobenius mathematics of positive and nonnegative matrices, as early as 1911.<sup>24</sup> The core of Potron's economic model is now well-known. I will give a short description of it, with a minimum of variables.<sup>25</sup> To shorten my exposition, I assume that all matrices in this section are

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<sup>21</sup> Potron's five letters to Hilbert are conserved in the Nachlass David Hilbert, Cod. Ms. D. Hilbert 315, Göttingen. (I translated from French into English).

<sup>22</sup> See my Annotated Potron Bibliography (Parys 2010).

<sup>23</sup> See for example Abraham-Frois & Lendjel (2004, 2006), Bidard & Erreygers (2007, 2010), Mori (2008), Bidard, Erreygers & Parys (2009), etc.

<sup>24</sup> The fundamental mathematical text is Potron (1913), but its results were already announced earlier (Potron 1911a, 1911b, 1911c, 1912a, 1912b).

<sup>25</sup> I call vectors and matrices *nonnegative* if all their elements are positive or zero; *positive* if all their elements are positive. My notations for vector inequalities are:  $x \geq y$  means that the vector  $x - y$  is nonnegative,  $x > y$  means that  $x - y$  is positive. The expression  $x \geq y$  is equivalent to  $x = y + b$ ,  $b \geq 0$ , where  $b$  is a nonnegative vector of slack variables. Potron uses many slack variables, but in my simplified representation I will omit them. I often replace the awkward expression "nonnegative and nonzero" by *semipositive*.

semipositive and irreducible (Potron's analysis is more sophisticated and lengthier),<sup>26</sup> and that the dominant root of  $A$  is smaller than one.<sup>27</sup>

### 3.1. The economico-social state

Potron characterizes an *economico-social state* by the following constants (three matrices and one special number):

- the  $n \times k$  matrix  $L$  of labour input coefficients  $\ell_{ih}$
- the  $n \times n$  matrix  $A$  of commodity input coefficients  $a_{ij}$
- the  $k \times n$  matrix  $B$  of annual consumption coefficients  $b_{hj}$
- the crucial number  $N$  (each labourer works at most  $N$  hours per year)

Potron's system of production assumes constant returns to scale, a production period of one year, circulating capital and heterogeneous labour. Production process  $i$  ( $i = 1, 2, \dots, n$ ) produces one unit of commodity  $i$  by means of  $a_{ij}$  units of commodity  $j$  ( $j = 1, 2, \dots, n$ ) and  $\ell_{ih}$  ( $h = 1, 2, \dots, k$ ) hours of labour of type  $h$ . The annual consumption baskets of the labourers depend on the social group they belong to: the annual basket for a worker of type  $h$  consists of  $b_{hj}$  units of commodity  $j$  ( $j = 1, 2, \dots, n$ ).

### 3.2. Satisfactory regimes of production and labour

Let the number of workers in social group  $h$  be represented by  $x_h$  and the gross output of process  $i$  by  $y_i$ . These variables<sup>28</sup> are collected in the  $1 \times k$  vector  $x$  and the  $1 \times n$  vector  $y$ . A so-called *satisfactory regime of production and labour* must satisfy two fundamental principles:

- *the principle of sufficient production*:  $y \geq yA + xB$
- *the right to rest*:  $Nx \geq yL$

On the one hand the gross output  $y$  must be sufficient for industrial and domestic consumption, but on the other hand we must respect the maximum number of  $N$  working hours per year for each type of labour. Are these two fundamental principles compatible? Can both inequalities be satisfied with positive  $y$  and  $x$ ?

Suppose we put  $x = \frac{1}{N}yL$ , then by substitution the principle of sufficient production means  $y \geq y\left(A + \frac{1}{N}LB\right)$ . The latter is an expression of the form  $yQ \leq y$  where  $Q = A + \frac{1}{N}LB$  is a

<sup>26</sup> I borrow most of the notation and the mathematical content from Bidard, Erreygers & Parys (2009).

<sup>27</sup> Same assumption in Potron. Then  $(I - A)^{-1}$  is positive, and the economy can produce a positive net output, which then must be distributed among workers and "non-workers". Note that Potron's matrix  $A$  contains the technical inputs only. Potron collects the consumption coefficients of the workers in another matrix  $B$ .

<sup>28</sup> On the one hand Potron treats  $L, A, B$  and  $N$  as constants; on the other hand he makes the strong assumption that  $x$  is a variable, which means that workers of any group can quickly be converted to any other group (and adopt the consumption basket of their new group).

semipositive irreducible matrix. Then Perron-Frobenius mathematics guarantees that such an inequality has a positive solution  $y$  if the dominant root of the matrix  $Q$  is smaller than or equal to one.<sup>29</sup> In the present context this is expressed by the following fundamental condition:

$$\text{dom} \left( A + \frac{1}{N}LB \right) \leq 1 \quad (1)$$

Potron also considers the following alternative method. The principle of sufficient production,  $y \geq yA + xB$ , leads to  $y \geq xB(I - A)^{-1}$  and then substitution in  $Nx \geq yL$  (right to rest) gives  $Nx \geq xB(I - A)^{-1}L$ . The latter is an expression of the form  $xP \leq Nx$  where the matrix  $P = B(I - A)^{-1}L$  is a semipositive irreducible matrix. Then Perron-Frobenius mathematics guarantees that such an inequality has a positive solution  $x$  if the dominant root  $v$  of the matrix  $P$  is smaller than or equal to  $N$ :

$$v = \text{dom}[B(I - A)^{-1}L] \leq N \quad (2)$$

In Potron's terminology, the dominant root  $v$  is called *the characteristic number of the economico-social state*.

In this way, Potron presents two equivalent "dominant root" conditions (1) and (2) for the existence of satisfactory regimes of production and labour. If the economico-social state, i.e. the set of the values of  $L$ ,  $A$ ,  $B$  and  $N$ , is "too bad" for the existence of a satisfactory solution, then we must try to decrease some elements of the matrices  $L$ ,  $A$ , or  $B$  (more efficient production or less consumption) or increase the maximum number of  $N$  working hours per year.

Intuitively, the characteristic number  $v$  describes the "average" number of hours per year that a worker of an "average" social group has to contribute to produce the annual consumption of the workers. If every worker performs  $N$  hours per year, and  $v$  is much smaller than  $N$ , then the surplus labour time can be used for surplus production that sustains many "non-workers", also called "simple consumers" or "rentiers" by Potron. He would have been astonished if he had known that a century after his original analysis, some economists suddenly start to use parts of his "Roman Catholic" model for Marxian theorems on surplus labour and exploitation in economies with heterogeneous labour.<sup>30</sup>

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<sup>29</sup> More general, for a given scalar  $\alpha$  and semipositive irreducible matrix  $M$ , there exists a positive row vector  $z$  that solves the vector inequality  $zM \leq \alpha z$  if and only if  $(\text{dom})M \leq \alpha$ . This result is applied to the matrices  $Q = A + \frac{1}{N}LB$  and  $P = B(I - A)^{-1}L$  in the present section. By means of a vector  $b$  of slack variables the inequality system  $zM \leq \alpha z$ ,  $z \geq 0$  can also be written  $z(\alpha I - M) = b$ ,  $z \geq 0$ ,  $b \geq 0$ . The latter formulation is closer to Potron, but requires more variables.

<sup>30</sup> See for example Mori (2008).

### 3.3. Simply satisfactory regimes of prices and wages

Let  $p_i$  be the price of a unit of commodity  $i$  and  $w_h$  the hourly wage of a worker of type  $h$ . These variables are collected in the  $n \times 1$  vector  $p$  and the  $k \times 1$  vector  $w$ . A so-called *simply satisfactory regime of prices and wages* must satisfy two fundamental principles:

- *the principle of justice in exchange*:  $p \geq Ap + Lw$

- *the right to life*:  $Nw \geq Bp$

On the one hand, justice in exchange requires that the prices of the commodities must be high enough to cover their material input costs plus the wage costs. On the other hand, the right to life means that wages must be high enough to cover the price of the consumption goods. Are these two fundamental principles compatible? Can both vector inequalities be satisfied by positive  $p$  and  $w$ ? Potron applied Perron-Frobenius mathematics again, and arrived at a remarkable case of duality.

### 3.4. Duality

Potron arrived at the same dominant root conditions (1) or (2) as before. In other words, the existence conditions for satisfactory regimes of production and labour are exactly the same as the existence conditions for simply satisfactory regimes of prices and wages. In this way Potron (1911c, 1913) is the first who proved duality results between quantity systems and price systems for the general case of  $n$  commodities.<sup>31</sup>

### 3.5. Anticipating the Hawkins-Simon conditions

Potron also thought about practical applications. He suggested that a *Bureau de Calculs* should collect the relevant coefficients of the matrices  $L$ ,  $A$  and  $B$ . Was it possible to check whether the dominant root of the matrix  $Q = A + \frac{1}{N}LB$  is smaller than one, without computing the root itself via an immense characteristic equation? Inspired by Frobenius' results on matrices and principal minors, Potron (1913: 62-63) developed the following criterion: it is necessary and sufficient that all principal minors of the matrix  $I - Q$  are positive. Later, Potron (1937b, 1937c, 1939) formulated this fundamental criterion in an equivalent and more efficient way: it is necessary and sufficient that the *leading* principal minors of  $I - Q$  are positive.<sup>32</sup>

The Potron criterion anticipated the famous Hawkins-Simon conditions by many years (Hawkins & Simon 1949). Moreover, when David Hawkins and Herbert Simon published their often quoted *Econometrica* article in 1949, they were not familiar with Perron-Frobenius or Potron. After their article, jointly written by correspondence, without seeing each other, Hawkins and Simon concentrated on widely different topics. Hawkins was a philosopher of science, who had contributed to many different disciplines, including economics (Hawkins

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<sup>31</sup> I described only the case of “simply satisfactory regimes of prices and wages”, which refers to the simple case where every worker performs exactly  $N$  hours per year. Potron is more general.

<sup>32</sup> Potron (1939) is listed in the remarkable bibliography of Woodbury (1954), without any further explanation.



1948, 1949). In a letter to Richard Trahair, dated 8 June 1993, Hawkins was very humble about the status of the 1949 Hawkins-Simon conditions: “The theorem [...] was, it turned out later, first published long before by the mathematician Frobenius.”<sup>33</sup> Hawkins described himself as follows: “I was an amateur in the field and have published very little else on economics.” It is obvious that Hawkins didn’t know Potron.

### 3.6. Overlooking Potron

In his articles from 1911-1914 Potron immediately introduced a full-fledged open input-output system and the associated quantity and price equations for the general n-commodity case. Leontief introduced his open price and quantity system in the *Quarterly Journal of Economics* as late as the 1940s (Leontief 1944, 1946a, 1946b). Potron was still unknown. His style of exposition was difficult, and his articles had appeared in non-economic journals that were overlooked by economists.

The 1914-1918 war stopped Potron’s activities on economics. After 1918 he concentrated on his mathematical teachings and research. His scientific network consisted essentially of Jesuits and alumni of the *École Polytechnique*. In the 1930s, the latter group organized regular meetings at the *Centre Polytechnicien d’Études Économiques*. In this context, in 1935 Potron heard of recent work in *Econometrica* by Ragnar Frisch (1934a, 1934b), who used nonnegative matrices in his economic model, without Perron-Frobenius: “Frisch drew on mathematical results which he proved as he went along, without checking up on the literature” (Bjerkholt & Knell 2006: 408n12).

Ironically, Frisch’s former assistant Olav Reiersøl, trying to extend statistical results by Koopmans (1937), was the first to show the power of Frobenius’ results to the econometricians, in his 1941 *Econometrica* paper on statistical confluence analysis. Reiersøl (1941: 8) commented on his discovery of Frobenius matrices: “I vainly attempted to generalize the proof of Koopmans already mentioned until I happened to read a paper by Frobenius...” It is symptomatic that Reiersøl, a graduate in mathematics, hadn’t picked up the relevant matrix algebra in his university courses, but only later: “I hadn’t learned anything about matrices in my studies of mathematics, so I had to read up this subject during my summer holidays in 1935” (Reiersøl 2000: 115). Note that Reiersøl’s statistical confluence analysis is an econometric topic, unrelated to Potron’s system or to Frisch’s model of circulation planning of 1934.

When Potron heard about the Frisch matrix in 1935, he obviously recognized the link between Frisch and his own use of nonnegative matrices in 1911-1914. In this way, Potron again became interested in (highly individual) research on economics, and tried to spread his economic ideas in some new writings (Potron 1935, 1936, 1937a, 1937b, 1937c, 1939, 1942).

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<sup>33</sup> Trahair was working on an encyclopedia of eponymous terms in the social sciences, and had written to Hawkins about the Hawkins-Simon conditions. A copy of Hawkins’ letter is in the David Hawkins Papers, Box 17, Folder 14 (University of Colorado at Boulder Libraries, Archives Dept.). I thank David M. Hays (Archivist) and his assistants for their efficient and friendly help in Boulder.

Although he was hampered by a speech defect, Potron organized a series of six lectures in Paris in 1937, to propagate his economic model. Two of these lectures were attended by François Divisia and René Roy, both alumni of the École Polytechnique, and important figures in the international community of mathematical economists (for example, in the *Econometric Society*). However, neither Divisia nor Roy ever disseminated Potron's message. In his book on the economic contributions of the French engineers, Divisia (1951) never mentions the name Potron.

One year before the 1937 lectures in Paris, Potron had made his only "international live appearance" ever, in the *International Congress of Mathematicians* (Oslo, 13-18 July 1936). About five hundred mathematicians (plus two hundred members of family) attended this congress. Most of the time was devoted to plenary sessions or speeches, but more specialized subjects were treated in different special sections. According to Dunnington (1936: 91) about forty sectional sessions took place, papers were limited to 15 minutes (plus discussion), with the result that 227 papers were presented. Georgiadou (2004: 317) also mentions the 15 minutes rule, and adds that "summaries were to be published in the congress files and the manuscripts themselves had to be submitted before the end of the congress".

In Oslo, Potron presented one paper on mathematical analysis, and one on his economic model. The economic paper (Potron 1936, 1937a)<sup>34</sup> was located in Special Section IV, called *Probability Theory, Mathematical Statistics, Insurance Mathematics, and Econometrics*, organized by Ragnar Frisch and Birger Meidell. Only three papers treated economics: Potron on his linear system, Ragnar Frisch on price index comparisons, and Franz Alt on utility measurement. Arthur Bowley presented two papers on statistics, so did Emil Gumbel and Herman Wold. Émile Borel delivered one paper on games, William Feller on stochastic processes, Harald Cramér on the central limit theorem, Maurice Fréchet on probability, etc. All in all this was an impressive list of first rate scholars. Here Potron had his 15 minutes of potential fame, but his presentation did not convey the originality of his contribution, and he failed to make an impression.

Probably, not only Potron's speech defect and his difficult style of writing were to blame. In addition, his situation was intrinsically difficult. Compare with George Dantzig in 1947, for example. Dantzig (partly inspired by Leontief's input-output model) made the first public announcement of his discoveries on linear programming in the December 1947 meeting of the American Statistical Association & Institute of Mathematical Statistics. This happened in a session with three other speakers on other topics. Each speaker had a 20-minute time constraint, and in this situation Dantzig's paper "was so sketchy that a listener who was not already familiar with the background would not be likely to perceive the importance and the difficulty of the problem whose solution was announced" (Dorfman 1984: 292).

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<sup>34</sup> Potron (1936) is the unpublished manuscript, Potron (1937a) is the summary published in the proceedings. Both texts are difficult to read, and (as usual) Potron provides no references to the mainstream economic literature.

Franz Alt (1910-2011), participant in Karl Menger's famous Mathematical Colloquium in Vienna in the 1930s, was the longest survivor of "Potron's 1936 Oslo session", and wrote in 2007:

I attended the Oslo International Congress of 1936 (as well as several other Congresses of the International Mathematical Union) and I do remember speaking there (and many times subsequently) with Ragnar Frisch and some other participants, but the name of Potron does not come to mind.<sup>35</sup>

In the 20<sup>th</sup> century, Potron remained unknown in the Anglo-Saxon networks. Paul Samuelson was very clear on this:

I have an excellent memory and I know that I and my associates in Cambridge, Massachusetts, and at RAND Corporation in California, did not know anything about the name Potron.

We should not be surprised that different scholars, unknown to each other, should converge toward some common mathematical truths.<sup>36</sup>

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<sup>35</sup> Mail from Franz Alt to the author, 24 May 2007.

<sup>36</sup> Letter from Paul Samuelson to the author, 5 August 2009.

#### 4. Bray on Cournot's equations of exchange

Probably the earliest use of simultaneous equations in economics appeared in the work of Achilles Nicolas Isnard (1781), hidden in the middle of a nondescript lay-out, nearly unnoticeable for readers who quickly browse through the nearly 700 pages of the *Traité des Richesses*. Both the notation and the mathematical content look primitive compared to the more professional and more general approach of modern writers. More sophisticated simultaneous equations appeared in Cournot (1838), in his Chapter Three, on computing the rates of exchange between the currencies of different countries. Here Cournot, probably for the first time in the history of mathematical economics, introduced the professional *double subscripts  $i$  and  $k$* , in his nonnegative coefficients  $m_{ik}$  ( $i$  and  $k$  running from 1 to  $r$ ).<sup>37</sup> Cournot constructed a general system of  $r$  linear equations, but failed to use the general theory of determinants to solve it, and limited his solution to a system with three equations.<sup>38</sup>

Nearly a century later, the general solution to Cournot's system was formulated by Hubert Evelyn Bray (1889-1978) in his article *Rates of Exchange* in the *American Mathematical Monthly* of November-December 1922. A few years earlier, the Rice Institute (today Rice University) in Houston had awarded its first Ph.D. in mathematics to Bray in 1918, on a dissertation *A Green's Theorem in Terms of Lebesgue Integrals*. Bray stayed at Rice throughout his whole career, mostly concentrating on pure mathematics. His 1922 paper on Cournot's problem was inspired by his colleague Griffith Conrad Evans, who had discussed these equations in a course at Rice from 1920 to 1921.<sup>39</sup>

Bray was able to determine the unknown rates of exchange via a homogeneous system of  $n$  linear equations. Equation  $i$  ( $i = 1, \dots, n$ ) described that for currency  $i$  total purchases must equal total sales. From a mathematical point of view, Bray's problem is similar to the 1929 paper of Robert Remak that I will consider in more detail in the next section. Although all his coefficients  $m_{ik}$  were nonnegative, Bray did not use Perron-Frobenius, and developed his own analysis of the special matrix of his system.<sup>40</sup>

Bray was first and foremost a mathematician, and received some international recognition, for example, by being elected a foreign member of the *Société Mathématique de France* in 1930. After his 1922 article on Cournot, he showed occasional interest in economics in 1937, when he participated (without a paper) in the *Third Annual Research Conference on Economics and Statistics*. This was a yearly meeting in Colorado Springs, organized by the *Cowles Commission*, less formal than the meetings of the *Econometric Society*. The list of speakers in 1937 included Bray's former colleague Griffith Conrad Evans (now at Berkeley), Karl Menger, Ragnar Frisch, Jacob Marschak, Harold Davis, Joseph Schumpeter, and others.<sup>41</sup>

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<sup>37</sup> Cournot separated the two subscripts  $i$  and  $k$  by a comma.

<sup>38</sup> It is very probable that Léon Walras studied the simultaneous equations of Isnard and Cournot at an early stage of his career (see Walras 1905, Jaffé 1969).

<sup>39</sup> See the first footnote of Bray's article (1922) and the preface of Evans' book (1930).

<sup>40</sup> Recently Maroscia (2008) studied Sraffa's Chapter 1 by means of Bray's matrix analysis (thus without Perron-Frobenius).

<sup>41</sup> All reports of these Cowles conferences are available at <http://cowles.econ.yale.edu/archive/reprints/index.htm>

Bray's conference timing was not perfect. Perhaps the 1938 or the 1940 conferences would have been more interesting for him. In 1940 Leontief presented results of his input-output research, announcing his forthcoming 1941 book.<sup>42</sup> So Bray was "too early" to meet Leontief's closed input-output system, whose algebra was mathematically similar to the Cournot-Bray equations of currency exchange. The same bad timing occurred with respect to the 1938 conference, which celebrated the centenary of Cournot's classic, without Bray's participation. In one of the Cournot sessions, René Roy drew special attention to Cournot's mathematical discussion of foreign exchange:

This discussion is the first example of a synthesis which permits clear analysis of the nature and number of variables considered in the economic system, as well as of the relations which exist between them. In other words, Cournot here pointed the way to the founders of the theory of economic equilibrium, who try to show exactly the degrees of freedom of the economic system by comparing the number of variables with the number of relations which connect them. (Roy 1938: 70)

Roy was silent about Bray's work and the existence of positive solutions. In a postwar *Econometrica* paper on rates of exchanges and other problems, Roy (1946) mentioned only the three equations system of Evans' book, and tackled the case of  $n$  equations without referring to Bray's generalization of 1922.

In point of fact, in the first decades after 1922, Evans seemed to be one of the few who referred to Bray's paper.<sup>43</sup> Evans had started his academic career as a mathematician (Ph. D. from Harvard), but paid more intensive and long standing attention to economics than Bray, and became one of the founding members of the Econometric Society. In his well-known textbook *Mathematical Introduction to Economics*, Evans (1930) included a reference to Bray's general treatment of Cournot's rates of exchange, in an exercise at the end of his chapter VII (On Rates of Exchange). But this chapter didn't attract much attention, and Henry Schultz (1931) in his long book review concentrated on other topics. However, note that Leontief was interested in Evans' book; in his letter of 5 January 1932 Leontief thanked Schultz for sending a copy of his review.<sup>44</sup> Even Piero Sraffa took notice of Evans' book. Sraffa's *Cambridge Pocket Diary 1930-1931* has many empty pages, and contains few bibliographical references (less than 10 for the whole year). But on Thursday 15 January 1931, Sraffa's diary makes a reference to "Evans, Math. Introd. to Econ. McGraw-Hill" (see my section 6 for more details on Sraffa and Leontief).

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<sup>42</sup> In the conference report, Leontief announced a forthcoming book titled *Structure of American Industries, 1919-1929* which of course became his 1941 classic *The Structure of American Economy 1919-1929*.

<sup>43</sup> See also the short references to Bray's work in Debreu & Herstein (1953), Woodbury (1954), Morgenstern (1959), Gale (1960) and Dorfman (1973). A long discussion on equations of currency exchange is given by Lorenzen (1969). For many interesting historical remarks on Cournot-Bray and Walras on exchange, see Gilibert (1981).

<sup>44</sup> Wassily Leontief Papers, Harvard University Archives, Accession # 12255, Box 10, Ancient Correspondence 1930-32. I thank the staff of the Harvard archives for their kind help in the Pusey Library.

## 5. Remak fails to normalize

Just like Potron in Paris, Remak in Berlin wrote a doctoral dissertation on finite group theory (Remak, 1911).<sup>45</sup> The choice of this abstract dissertation topic was clearly influenced by his supervisor, who was none other than Frobenius himself. In 1929 Remak used his mathematical abilities to study the existence of positive solutions in his linear system of “superposed prices”, without ever referring to the tailor made theorems of his former master Frobenius (who had died in 1917).

### 5.1. Eccentric and independent thinker

Unlike Frobenius, Remak enjoyed participating in discussions on economics. In Göttingen he even attended some undergraduate lectures in economics. His nonstandard dress and behaviour earned him the nickname “the communist”, and the young students didn’t like Remak’s attendance in the economics lectures. This caused some incidents in the lecture room, and Remak was ejected from the university in Göttingen. Not only the students, but also some professors (Hilbert for example) preferred seminars without the rather extrovert and critical interventions by Remak.<sup>46</sup>

Remak’s academic career proceeded slower than normal for a mathematician of his capacities, just like the slow career of his father and his grand-father.<sup>47</sup> Because of Remak’s eccentric personality, his first two attempts to obtain his Habilitation failed, despite a sufficient number of mathematical publications in German top journals. A typical example of Remak’s despair is shown in a letter of 2 May 1925, from Remak to David Hilbert, begging to end the problems caused by conflicts with Hilbert in Göttingen in 1918-20. Faculty rumours in Berlin gave Remak the impression that Hilbert’s influence had contributed to the first two rejections in Berlin. Remak couldn’t believe that the Göttingen troubles could have such grave consequences in Berlin: “You said me in 1920 that you would put nothing in the way of my career at other universities.” Remak also wrote he didn’t believe Hilbert approved of the “year-long persecutions” (“Verfolgungen”) that Remak suffered in the academic world, and he asked Hilbert a letter attesting the qualities of Remak’s scientific work.<sup>48</sup> Only on 11 January 1929 Remak finally obtained his Habilitation in Berlin (Biermann, 1988: 210). Four years later the Nazi government withdrew his right to give lectures.

After some difficult years in Nazi Germany, Remak was able to move to Amsterdam in 1939. His lack of emotional intelligence worried some mathematical colleagues in Holland. Hans

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<sup>45</sup> The short book by de Séguier & Potron (1938) contains twenty-seven bibliographical references on group theory, including four from Remak (1911, 1913, 1930a, 1930b).

<sup>46</sup> Numerous stories on Remak’s unusual behaviour exist. See for example the memories by Heinrich Behnke (1978) and Werner Fenchel (1980). More bibliographical references can be found in Merzbach (1992) and Hagemann & Punzo (2007). Note that Behnke and Fenchel attended the Oslo conference in 1936, just like Potron.

<sup>47</sup> The Jewish Remak family in Berlin generated four well-known intellectuals. The brother of the mathematician Robert Remak (1888-1942) was Georg Remak (1890-1979), a lawyer, and senior federal prosecutor at the end of his career. Their father Ernst Julius Remak (1849-1911) and especially their grandfather Robert Remak (1815-1865) were famous neurologists, and are still mentioned today in medical journals. See for example the article on the two Remak neurologists, published in *The Lancet* by J.M.S. Pearce (1996).

<sup>48</sup> David Hilbert Nachlass - Cod. Ms. D. Hilbert 326, Göttingen. (I translated from German into English).

Freudenthal, another migrant from Germany to Holland, received a letter from Luitzen Brouwer,<sup>49</sup> dated 2 March 1940, warning him that Remak was in danger to be expelled by the Dutch government, on the grounds of bad behaviour. This happened a few months *before* Holland was occupied by the German troops. Some even claimed that the only way to save Remak's life was to put him in a psychiatric hospital (van Dalen, 2005: 731). Such a difficult person had a less than average chance of surviving after the German occupation of Holland in May 1940. Remak died on transport to Auschwitz on 13 November 1942 (Vogt 1998: 342).

According to Merzbach (1992: 500), Remak had been able to visit Louis Joel Mordell in Manchester in 1939, and was to go to Cambridge in October 1940 for a little while. However, Remak was trapped in Holland when the German troops occupied that country in May 1940. Normally, a mathematician with his set of publications could have obtained a position in an Anglo-Saxon top university, if he had possessed average social skills. Suppose that Remak had been able to settle in Cambridge, and could have entered into contact there with some other Jewish foreign scholars who had emigrated from the dictatorial regimes of their home countries: Besicovitch, Wittgenstein, Sraffa ...

### 5.2. Remak's 1918 paper on the national debt

On 23 May 1918 Remak published what is probably his first economic paper, titled *Vorschläge zur Tilgung der Reichsschuld* (Proposals for Repayment of the National Debt), in the *Europäische Staats- und Wirtschaftszeitung*. This journal was founded in Berlin in 1916, in the middle of the war, and its weekly issues contained many contributions on political and economic problems, mostly written by academic authors like Gustav Stolper, Carl Ballod, Franz Oppenheimer, Karl Diehl, Max Weber, Lujo Brentano, Adolf Löwe, Otto Neurath, etc.

Remak contributed only once to the journal, and included only one bibliographical reference in his article, i.e. a reference to a comprehensive book on wealth taxation from 1917, written by the historian and economist Ignaz Jastrow (1917). Jastrow's proposals for a wealth tax also called the attention of his American colleague Frank W. Taussig. Both Jastrow and Taussig had shown a common interest in the building of business schools at the academic level, like the *Handelshochschule* in Berlin (Jastrow) and the Harvard Business School (Taussig).<sup>50</sup> After the war, Jastrow (1920, 1923) wrote two articles in the *Quarterly Journal of Economics* (Taussig was its editor), in which he compared the proposals of his 1917 book with the actual capital levy tax in Germany.

Remak's 1918 article had no influence on economists in Berlin or elsewhere. He probably published the article because of his wide-ranging political and social interests, including public and private finance. Born in a wealthy family, originally Remak's private capital was sufficient to live the life of a "private scholar", just like Charasoff and Potron, but due to the German hyperinflation in 1923 he was obliged to take a part time job at the *Deutsche Bank*

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<sup>49</sup> The Dutch mathematician Brouwer (1881-1966), the founder of intuitionism, is often cited in economics with reference to his fixed-point theorem. Freudenthal (1905-1990) had already left Germany in 1930, to become Brouwer's assistant in Amsterdam. Unlike many other Jewish inhabitants of Amsterdam, Freudenthal was able to survive in hiding during the German occupation.

<sup>50</sup> See Redlich (1957).

(Vogt, 1998: 337). Perhaps, this led to a further improvement of Remak's practical and theoretical knowledge of finance, and explains the unexpected success of his introductory lectures on insurance mathematics, in the winter semester of 1931-32. These lectures earned Remak more than a thousand Reichsmark of tuition fees, more than ten times more than his earlier lectures on group theory (Vogt, 2008: 414-415).

I draw attention to the 1918 article, because I have never seen it in other Remak studies, but its topic has few connections to Remak's mathematical economic model of 1929. With some goodwill, the first paragraphs of the 1918 paper can be used to illustrate a dated input series or the working of the multiplier. Later on, Remak asserts that

“the state and the empire have the obligation to shorten and to cheapen the way from producers to consumers” (Remak 1918: 389)

Such a precept can also be considered one of the principles of Remak's well-known economic system of 1929, which contains many critical remarks against the high mark ups by merchants, which cause non ideal prices and depress aggregate consumption.

### 5.3. *The Remak price system*

In his short 1933 note, Remak emphasizes the following:

“A price does not emerge from supply and demand, it is rather a number which has to satisfy certain conditions. The price of a commodity must cover the prices of the expenses contained in it including the cost of living, which may be taken to be known, of the people participating in its production. This leads to the superposed price systems.”<sup>51</sup>

This unorthodox statement shows a striking resemblance to the following by Maurice Potron two years later:

“...the prices, the wages, [...] are not imposed by any *physical* necessity. They result from *conventions* which, at least theoretically, are *free*.”<sup>52</sup>

As is well explained by Hagemann & Punzo (2007), one source of inspiration for Remak's 1929 article was provided by the debates on capitalism and socialism, and on the calculation of rational prices in socialism. Remak seems to sustain the optimistic idea that his system of superposed prices provides an “exact” criterion to judge the so-called *extremal* character of an economic system. In modern parlance, Remak's notion of an extremal economy refers to a system with Pareto efficient output, where it is impossible to increase the output of any good, except by decreasing the output of at least one other.

The link between Remak's superposed prices and the extremal output quantities is not clearly visible in his article; there is no mathematics of the quantity system. Remak presents verbal remarks on crises, regional differences, the Taylor system of scientific management, birth

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<sup>51</sup> Remak (1933: 840), citation translated by Kurz & Salvadori (1995: 398).

<sup>52</sup> Potron (1935: 63), article translated by Bidard & Erreygers (2010: 169).



control and other demographic measures, heterogeneous labour, differences in the cost of living between city and countryside, profit sharing, high profits, etc. But he doesn't offer a formal treatment of these subjects. The mathematical part of his 1929 paper is limited to the price system of a stationary economy with no surplus (no profits), analogous to the first pages of Sraffa (1960).

In Remak's system there are  $n$  sectors,<sup>53</sup> each producing a single output. Sector  $\mu$  produces commodity  $\mu$ . To this end sector  $\mu$  buys  $a_{\mu\lambda}$  units of commodity  $\lambda$  from sector  $\lambda$  ( $\mu, \lambda = 1, 2, \dots, n$ ). Remak's "superposed" prices  $x_1, x_2, \dots, x_n$  require that for each sector  $\mu$  the value of its inputs equals the value of its output:

$$a_{\mu 1}x_1 + a_{\mu 2}x_2 + \dots + a_{\mu n}x_n = x_\mu(a_{1\mu} + a_{2\mu} + \dots + a_{n\mu}) \quad \mu = 1, 2, \dots, n$$

In this way, Remak obtains a system of  $n$  homogeneous equations with  $n$  unknowns  $x_1, x_2, \dots, x_n$ . At this point Remak fails to normalize the units. By a clever choice of units of measurement, he could have put the gross output of each sector equal to one, and then he could have obtained a system in the simple form  $Ap = p$ . This is a routine operation in modern economics; see for example Gale (1960: 261) and Kurz & Salvadori (1995: 398). The system  $Ap = p$  or  $(I - A)p = 0$  can easily be handled by Perron-Frobenius (simple case:  $A$  is a stochastic matrix, with dominant eigenvalue equal to one). Because all columns of  $I - A$  sum to zero, the determinant of  $I - A$  is zero. From the elementary theory of determinants, Remak (1929: 726) knows that such a system has at least one nonzero solution. But for him only nonnegative solutions are acceptable: "on this subject the familiar theorem on determinants is silent" (Remak 1929: 726). He should have added: "But in the final years of his life, my old master, the late Professor Frobenius, provided us with all the material to obtain a quick answer to this problem".

Readers of Remak look in vain for a single reference to Frobenius. Perhaps Remak overlooked Perron-Frobenius, because he failed to normalize the units of measurement for the different commodities. And perhaps he didn't know the details of Frobenius 1912 paper; Remak finished his dissertation in 1911, Frobenius died in 1917, and in his otherwise positive Habilitation report on Remak, Issai Schur wrote: "Despite the wide range of his interests, Herr Remak has only a relatively small knowledge of the literature" (Vogt, 1998: 338).

Remak should have used Frobenius' theory of reducible matrices. A typical strategy for analysing a reducible input matrix  $A$  is to partition it in a *lower triangular block form*, already used by Frobenius (1912: §11), and therefore often called a *Frobenius normal form*:

$$\begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

where all the submatrices on the main diagonal are irreducible.

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<sup>53</sup> At the start of his argument, Remak assumes a system with  $n$  different individuals, each producing a different commodity, but we can also interpret his equations as a system with  $n$  different sectors.

The logic of Remak's problem led him to a similar procedure, but without employing matrices. In his lengthy argument he used the following terminology and results. When sector  $\mu$  supplies commodities (directly or indirectly) to  $\lambda$ , and vice versa, then Remak says that  $\mu$  and  $\lambda$  are equivalent indices, and he partitions the set of  $n$  sectors into  $m$  different subsets, which he calls "groups of equivalent indices". Such a partition corresponds to the  $m$  irreducible submatrices of the Frobenius normal form above. In Remak's terminology, a group  $K$  is higher than a group  $\Lambda$  if there is a delivery of commodities from  $\Lambda$  to  $K$ . Group  $K$  is a highest group if there exists no other group that is higher than  $K$ . There is at least one highest group. Note that several groups can be highest, but only if there is no connection between them.<sup>54</sup> By a patient study of all possible outcomes, Remak arrives at the following results (my notation):

*Fundamental solutions:* Each highest group  $h$  generates a semipositive  $n \times 1$  price vector  $p^h$ , with the following properties:

$$p_j^h > 0 \text{ if commodity } j \text{ belongs to group } h$$

$$p_j^h = 0 \text{ otherwise}$$

Remak call this a *fundamental solution* ("Grundlösung") of his price system. He also proves that a fundamental solution generated by group  $h$  is unique up to a scalar multiple.

*All solutions:* All semipositive price vectors are linear combinations of fundamental solutions. If there are  $k$  highest groups, there are  $k$  linearly independent semipositive price vectors. Positive price vectors exist if and only if all  $m$  groups are highest.

Note that there always is at least one highest group, thus at least one semipositive solution.

In modern parlance, Remak spends nearly all his mathematical efforts on sophisticated reducible matrices, but then he offers a numerical example involving only a very simple positive (and thus irreducible) matrix. His example (Remak 1929: 731-733) studies a system of three individuals, each specializing in the production of one product. Producer 1 produces 27 tons of good 1, with price per ton equal to  $x_1$ . For his production process or for his own consumption, he needs 3 units of his own product 1, plus 5 units of product 2, plus 7 units of product 3. He sells 3 units of his output to himself, plus 11 units to producer 2, plus 13 units to producer 3. The first equation below states that the value of the inputs of producer 1 must be equal to the value of his output. Similar explanations hold for producers 2 and 3:

$$3x_1 + 5x_2 + 7x_3 = x_1(3 + 11 + 13)$$

$$11x_1 + 2x_2 + x_3 = x_2(5 + 2 + 7)$$

$$13x_1 + 7x_2 + 6x_3 = x_3(7 + 1 + 6)$$

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<sup>54</sup>See some numerical examples below. The main diagonal of a Frobenius normal form can contain some  $1 \times 1$  matrices (zero or positive). This corresponds to a group of one industry. If such a "singleton" does not deliver its product to other groups, it must be included in the list of "highest" groups, in order to retain Remak's results on the number of linearly independent semipositive price vectors. Remak never explicitly mentions such groups of one industry.

All the input coefficients are positive, which means that every sector directly delivers to every sector. In this extreme case, the only “group of equivalent indices” is  $\{1, 2, 3\}$ , the set of all indices itself, a trivial case with one highest group, thus one fundamental solution. After slow calculations and controls, Remak finds the superposed price vector  $x = [89 \ 101 \ 233]^t$ , which is unique up to a scalar multiple.

For theoretical purposes, it is better to normalize the units of measurement. The gross output of good 1 is 27; let us call it 27 tons of iron. Then introduce a new unit of measurement for iron: 1 unit of iron = 27 tons, with price  $p_1 = 27x_1$ . In a similar way, I normalize the units for goods 2 and 3, with  $p_2 = 14x_2$  and  $p_3 = 14x_3$ . Such a normalization of gross outputs in such circumstances is a routine one in modern economics, but Remak didn’t think of it in his mathematical proofs. The above normalization would have transformed Remak’s example into the following

$$\frac{3}{27}p_1 + \frac{5}{14}p_2 + \frac{7}{14}p_3 = p_1$$

$$\frac{11}{27}p_1 + \frac{2}{14}p_2 + \frac{1}{14}p_3 = p_2$$

$$\frac{13}{27}p_1 + \frac{7}{14}p_2 + \frac{6}{14}p_3 = p_3$$

This is a simple Perron-Frobenius system of the form  $Ap = p$ , with a stochastic matrix  $A$  (every column sum equals 1). Remak’s numerical example is badly chosen, because it produces a positive matrix  $A$ , whereas his mathematical proofs allow for some zero input coefficients and some zero prices.

#### 5.4. The David Gale example

The 7 by 7 matrix example by Gale (1960: 267-271) can be used to give more insight into the general structure of Remak’s theorems. After renumbering the seven commodities, I obtain the following input matrix in Frobenius normal form:

$$\left[ \begin{array}{cc|ccc|cc} 0.2 & 0.5 & | & 0 & 0 & 0 & | & 0 & 0 \\ 0.5 & 0.3 & | & 0 & 0 & 0 & | & 0 & 0 \\ \hline 0.1 & 0 & | & 0.2 & 0.3 & 0 & | & 0 & 0 \\ 0 & 0 & | & 0 & 0 & 0.4 & | & 0 & 0 \\ 0 & 0 & | & 0.8 & 0.7 & 0.6 & | & 0 & 0 \\ \hline 0.2 & 0 & | & 0 & 0 & 0 & | & 0.5 & 0.9 \\ 0 & 0.2 & | & 0 & 0 & 0 & | & 0.5 & 0.1 \end{array} \right]$$

The system is partitioned into three groups,  $G_1 = \{1, 2\}$ ,  $G_2 = \{3, 4, 5\}$ , and  $G_3 = \{6, 7\}$ . The groups  $G_2$  and  $G_3$  are the two highest groups, generating the two fundamental solutions  $p^2 = [0 \ 0 \ 15 \ 40 \ 100 \ 0 \ 0]^t$  and  $p^3 = [0 \ 0 \ 0 \ 0 \ 0 \ 18 \ 10]^t$ . All semipositive price vectors are linear combinations of the vectors  $p^2$  and  $p^3$ . No positive price vector exists, because not all groups are highest.

Gale presents this in a slightly different setting, in a linear model of international trade between seven countries. His terminology and lay out are different from Frobenius or from the current Sraffian literature, and his book never refers to Frobenius. Neither Gale nor Remak raised an alarm about the following point. In all superposed price vectors the first two components are zero. The first two goods never play a role in Remak's fundamental solutions. In Gale's approach  $G_2$  and  $G_3$  are called "irreducible subsets", while  $G_1$  seems to be treated as less important; it is relegated to a less important part of Gale's matrix; Gale (1960: 269) doesn't use a Frobenius normal form.

### 5.5. *An Extreme Beans system*

Readers of Sraffa (1960) will object to the bizarre hierarchy of the seven commodities in the Remak-Gale approach. In Sraffa's terminology, goods 1 and 2 are not negligible. Quite the contrary. They are the two basic commodities of the economic system. They are directly or indirectly necessary for every production process. The five other goods are Sraffa's nonbasic commodities. Gale's example has the same status as my following *Extreme Beans* example, which is a sort of extreme version of Sraffa's Appendix B on self-reproducing beans:

0.5 units of corn + 0 units of beans  $\rightarrow$  1 unit of corn

0.5 units of corn + 1 unit of beans  $\rightarrow$  1 unit of beans

Note that in the Remak story we consider an augmented input matrix, and thus the above example implies that workers eat only corn. The Remak and Gale solutions present a zero price for corn and a positive price for beans. But will we ever observe such a bizarre system? Not in Sraffa's context of uniform rates of profit. The corn producers (or more general, the basic sectors) form the substance of the economic system, producing the important commodities. A country with such a bad technology for beans will concentrate on corn production only. The corn producers will earn a nice profit rate of 100%, and none of them would invest in beans production. If beans are produced, we live in a bizarre country where corn and labour are worthless (price of corn and thus also the wages are zero).

Remak spent more than 90% of his mathematical efforts on such systems, and the zero prices for various goods.<sup>55</sup> His explanation of the relevance of his results is poor. In his two articles Remak (1929: 734; 1933: 839) seemed to draw parallels between the zero prices in his system and the very low prices that some weak countries with bad currencies receive, when they exchange their products with strong countries that own large stocks of gold and silver.

### 5.6. *Remak's visibility in economics*

Unlike Charasoff and Potron in earlier years, Remak presented his system in a journal that had wide visibility among economists, the prestigious *Jahrbücher für Nationalökonomie und Statistik*. Moreover, four years later he could add a few pages in the same journal, on the practical computability of his price system.

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<sup>55</sup> Potron (1913: §15) devoted one long footnote to "partially reducible matrices" with dominant root equal to one, but used the rest of his mathematical efforts for more relevant systems.

Even though it appeared in an economic journal, Remak's 1929 article was also reviewed in the *Jahrbuch über die Fortschritte der Mathematik*. The review was a neutral summary, written by Erika Pannwitz (1929), who didn't mention the missing link with Perron-Frobenius. At that time, Pannwitz wrote her doctoral dissertation in Berlin (Ph.D. in 1931). Many years later, Waldemar Wittmann consulted her on the origin of Remak's paper. Pannwitz suggested that economists from the network around von Bortkiewicz contacted Remak, about the existence of positive solutions in linear price systems. Remak became interested, and then wrote the 1929 paper (Wittmann, 1967: 401). It is possible that Remak's article provided a challenge for John von Neumann to start the construction of his own more general model of growth (Kurz & Salvadori 1995: 412-414). Remak's results were also discussed by other colleagues of the *Mathematisches Institut* in Berlin, and some ridiculed his economic proposals (Wittmann 1967: 407).<sup>56</sup>

Hans Freudenthal recommended the article to Tjalling Koopmans (1951: 33). Moreover, Gale (1960) included Remak's model in his widely used textbook, and Baumol & Goldfeld (1968) offered a partial English translation of Remak (1929). Remak, unlike Charasoff and Potron, was not in need of a "rediscovery". Some connoisseurs knew his work from the beginning, for example Emil Gumbel, who showed wide interests in mathematical statistics, politics, economics and many other topics, and Wassily Leontief (see next section).

An important letter from Remak to Gumbel was recently published by Annette Vogt (2008). The letter is dated 21 April 1931. About two months earlier, on 19 February 1931, Gumbel had acted as the doctoral promoter in the oral defence of Klimpt's dissertation on Bortkiewicz (including paragraphs on Charasoff: see my section 2). Gumbel must have liked this sort of topics, because he had told Remak about his plans to encourage his doctoral students in Heidelberg to work on the Remak system. Remak's letter of 21 April 1931 thanks Gumbel for his interest in the 1929 paper, and then provides some additional comments on it.

Just like his article, Remak's letter emphasizes the problem of distortions in the actual prices due to exaggerated mark ups and profits.<sup>57</sup> It would take many statisticians and mathematicians to construct and solve the necessary equations and compute Remak's superposed prices. Remak (1929: 735) hoped that improved electrical circuits would help to solve the equations somewhere in the future. In his letter to Gumbel, he thought about the day when this can be done for the first time: "it will be a big event for us, somewhat like the breakthrough in the middle of the Simplon tunnel" (Remak 1931). Two years later, Remak still seemed optimistic; earth scientists can perform even more extensive geodetic computations (Remak 1933: 841). In economics a lot of data collection is necessary, but

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<sup>56</sup> My paper doesn't discuss Bortkiewicz. His important publications on Marx were already finished before the Perron-Frobenius story started. Von Neumann (1937) also falls outside the scope of my paper. Note that the existence theorems in his very general dynamic model with joint production and choice of technique needed stronger tools than Frobenius' theorems of 1912.

<sup>57</sup> Remak's expressions like "die Kaufmännische Zuschläge" literally mean "the merchant's mark ups", but he also thought of the high rates of profit. Markups or profits depress the purchasing power of the workers, and lead to a sort of underconsumption and depression. I doubt whether Remak saw the following problem: suppose equal value composition of capital exists in all industries, then *relative* prices would not be distorted by high or low rates of profit (of course, the income distribution would be different).

Remak hoped that this problem could be treated by other scholars who were closer than he to practical statistics (Remak 1933: 842).<sup>58</sup>

Both Gumbel and Remak shared similar political outlooks; both were expelled from the German academic networks in the 1930s. Gumbel first fled to France, and later to the U.S., where he became a well-respected statistician, especially after the publication of his classic *Statistics of Extremes* (Gumbel, 1958). Remak's fate was different.

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<sup>58</sup> Remak and Potron both wanted statistical help for the practical computations of the input-output data. Potron (1914: 168) was optimistically comparing this task with the computation of ten volumes of logarithmic tables. In the 1930s Potron was more pessimistic, because of the massive amount of numbers required (Bidard, Erreygers & Parys 2009: 143-148).

## 6. The two eponyms of this age of Leontief and Sraffa

A few additional comments are in order about the two eponyms of “Leontief-Sraffa models” or “this age of Leontief and Sraffa”.<sup>59</sup> Both Leontief and Sraffa started their pioneering work on linear economics in the late 1920s, and both preferred to do so in isolation, but sometimes they tried to benefit from discussions with competent mathematicians.<sup>60</sup>

### 6.1. *Leontief and Remak*

In his article on Leontief’s Nobel Prize, Robert Dorfman (1973: 431) wrote:

I still remember Leontief’s gleeful excitement when he came across the work of Remak, who proposed a theoretical input-output formulation of an economy seven years before Leontief’s earliest paper on the subject. A mathematician, H. E. Bray, had written in similar vein seven years before that.

Dorfman apparently had no access to Leontief’s 1928 dissertation, published one year before Remak’s 1929 paper. More corrections to Dorfman’s story are necessary in the light of archival evidence. Leontief’s letters to Adolf Löwe (3 June 1931) and to Joseph Schumpeter (7 June 1931) provided some news about the economists Leontief had visited while being back in Germany. Leontief also saw Emil Gumbel, and mentions that discussions with Gumbel were very useful to clarify some mathematical questions. Moreover, Gumbel referred Leontief to Remak’s work, and a few days later (11 June 1931) Leontief wrote to Remak himself:

In a recent conversation Prof. Gumbel signaled me that you turned your mind, from the mathematical point of view, to the same problems of mathematical political economy that I have arrived at from the economic point of view: determinant analysis of the economic systems of equations, etc. I would enjoy it very much if I had the, alas so rare, opportunity to discuss all these questions and to see you occasionally. I stay in Berlin till the end of August and would be very grateful for your answer.<sup>61</sup>

It is unclear what happened next. Anyway, the Dorfman story of Leontief’s “gleeful excitement” about Remak reminds us of the stories of Leontief’s “amusement” about his exact year of birth. The latter story appears in many recent papers. For example, Baumol & ten Raa (2009: 511) write: “His date of birth was presumed to be 5 August 1906, but after the collapse of the Soviet Union Leontief apparently first found out, and reported with much amusement, that it was precisely one year earlier, in 1905.”

In his very early days, Leontief indeed thought that he was born in St. Petersburg in 1906, but the Harvard archives contain ample evidence that Leontief knew the exact details many decades before the fall of the Soviet Union. In a letter to the Local Board No. 68 in Berkeley,

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<sup>59</sup> This expression was coined by Samuelson (1971: 400).

<sup>60</sup> In the interview by DeBresson (2004: 146 / 5 April 1997), Leontief said: “I think very separately from my friends and colleagues”. His wife added: “When Wassily’s mind is at work, he puts a wall around himself”. The classic books by Leontief (1941, 1951) and Sraffa (1960) contain hardly any modern bibliographical reference.

<sup>61</sup> Wassily Leontief Papers, Harvard University Archives, Accession 12255, Box 10, Folder: Ancient Correspondence 1930-1932. (I translated from German into English).

dated 14 June 1943, Leontief writes that his parents, who were students in Munich in 1905, had just informed him about his exact year of birth. Leontief enclosed a correct birth certificate in the letter (the Harvard file contains a certificate dated October 1932).<sup>62</sup>

## 6.2. Keynes rejects Leontief

Schumpeter at Harvard thought very highly of Leontief. In a letter of 3 December 1932 to Keynes, editor of the *Economic Journal*, Schumpeter announced that Leontief “will furnish a very interesting piece of work during the Christmas recess. It would make an article of about fifteen pages. Would you care to have it for the Journal?” After four months of waiting, Keynes must have sent Leontief a card to ask him about the article, because Leontief (in a letter of 12 April 1933) thanks Keynes for sending such a card, and for remembering his article, and of course then Leontief apologizes for the delay. So does Schumpeter who gives more details in a letter to Keynes (19 April 1933): “I want to apologize for Leontief. He has indeed produced what I consider a most original and interesting piece of work but then instead of finishing the article he got married in spite of my disapproval of this step. This explains the delay.”

Wesley Mitchell (letter to Leontief, 27 April 1933) had heard from Simon Kuznets that Leontief expected “to have an article on the Theory of International Trade in the next number of the *Economic Journal*”. In his reply to Mitchell (28 April 1933), Leontief claimed the relevant paper was not “his small note on international trade”, which he called “a rather unimportant by-product of my lectures”.<sup>63</sup> Leontief wanted to impress the readers of the *Economic Journal* (then the top journal for economics) with a much more fundamental item. He wrote Mitchell that the relevant item was “a condensed article for the *Economic Journal* dealing with the theoretical scheme on which the statistical part of the analysis of my work for the National Bureau is based” (in other words: the theory of input-output analysis).

Given Leontief’s research reports in that period, given the nature of his 1928 doctoral dissertation in Berlin, given his letter to Remak of 11 June 1931 (see Section 6.1 above), this article must have been Leontief’s first attempt to present the equations of his theoretical input-output model to the Anglo-Saxon readers. The paper was finally submitted to Keynes, accompanied by a letter of 2 June 1933: “I am sending you under separate cover the manuscript of my article *Economic Changes and General Equilibrium*”.

Instead of appearing in the next issue of the *Economic Journal*, Leontief’s paper was rejected within a few weeks. In his letter to Keynes, dated 25 July 1933, Leontief was very polite: “I have received my manuscript and I thank you for the letter, which accompanied it. I am sorry the paper did not meet your approval, still feel that you owe me no apology for this”.<sup>64</sup>

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<sup>62</sup> Wassily Leontief Papers, Harvard University Archives, HUG 4517.6, Box 1, Folder: Misc., Subfolder: Various Important Documents.

<sup>63</sup> The small note would appear in the next issue of the *Quarterly Journal of Economics* (Leontief, 1933).

<sup>64</sup> The two quoted Schumpeter letters (and many others) were published in chronological order in Schumpeter (2000). The letters from Leontief to Keynes are available in the Harvard University Archives, HUG 4517.5, Box 1, Folder: General Correspondence 1932-1941. I couldn’t find Leontief’s manuscript or Keynes’s letter of rejection.



After its rejection by Keynes, the title *Economic Changes and General Equilibrium* wasn't used by Leontief elsewhere. It is very likely that the content of his rejected and lost manuscript of 1933 was used in his 1937 article in the *Review of Economic Statistics*, which is the first paper that presents the algebraic equations of Leontief's closed input-output system.<sup>65</sup> Contrary to Remak, Leontief didn't discuss the positivity of the solutions of his equations, but he presented many comparative static exercises, showing the effect of changes in some input coefficients. Maybe Keynes rejected the rather abstract paper because it was too far outside the Marshallian mainstream, or because it contained too many equations, presented in a rather long-winded manner, or too many partial derivatives and determinants?

### 6.3. Sraffa and the mathematicians in Cambridge.

In 1928, a few years before rejecting Leontief's paper, Keynes was able to read a draft of the opening propositions of *Production of Commodities by Means of Commodities* by Piero Sraffa (1960). Although this book required more than three decades of preparation, the preface expressed thanks to only three scholars, namely Frank Ramsey, Alister Watson and especially Abram Besicovitch, for their mathematical help. Note that all three are "non-economists". Thanks to extensive archival research by Kurz and Salvadori, we now have a better picture of Sraffa's contacts with these three "mathematical friends" in Cambridge (U.K.). It is now clear that Ramsey, Watson and Besicovitch overlooked the powerful Perron-Frobenius theorem:

Sraffa's papers would seem to imply that none of his mathematical friends referred him to this theorem (Kurz & Salvadori 2001: 264).

As early as 1928, Sraffa presented to Frank Ramsey a system of simultaneous equations involving the prices of three commodities and a uniform rate of profit (rate of interest). Ramsey then wrote down the characteristic equation of the underlying 3 by 3 input matrix, but didn't use matrix algebraic terminology, and ignored Perron-Frobenius mathematics (Kurz & Salvadori 2001: 262-264).

Kurz and Salvadori also show how, on various occasions in the 1940s and 1950s, Alister Watson was of great help to Sraffa to tackle the problems of basics and nonbasics in the model of single product industries.<sup>66</sup> In modern parlance, the problem was the study of a reducible input matrix, its economically meaningful eigenvalues, its nonnegative eigenvectors, the connections between the group of basic commodities and the different groups of interconnected nonbasics, in other words, the connections between the different parts of the Frobenius normal form of the input matrix. However, Watson never referred Sraffa to the relevant mathematical literature and discussed it in more intuitive terms. Finally, in a letter dated 9 June 1961, one year after Sraffa's 1960 book was published, Watson wrote to Sraffa:

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<sup>65</sup> The famous paper of one year earlier (Leontief 1936) contained no equations, only empirical results.

<sup>66</sup> Kurz & Salvadori (2001) discuss Sraffa's meetings with Watson from 1945 onward. When looking at Sraffa's diaries in Cambridge, I noticed that there are also numerous meetings with Watson in 1938 and 1939, but I have no further information on their relevance for Sraffa's book.

I have recently come across a paper giving a brief statement and bibliography of the theorems of the type you prove and use that have been dealt with mathematically (cited by Kurz & Salvadori, 2001: 278).

According to Kurz & Salvadori (2001: 284), Sraffa wrote “yes, send bibliography” on the top of Watson’s letter, but no further traces were found in the Sraffa papers. The Watson story reveals how Sraffa’s research was done in isolation from some of his economic colleagues in Cambridge. Sraffa could easily have obtained the Frobenius’ reference from his colleague Richard Goodwin, who was appointed in Cambridge (U.K.) in 1950, after Harvard denied him tenure. Goodwin (1950) used Frobenius’ results in his paper on the oscillation of the matrix multiplier. According to Goodwin (1953: 83)<sup>67</sup> it was Per Göran Ohlin, a Harvard student of him, who had drawn his attention to Frobenius. In this context Velupillai (1982: 78-79) provides the English translation of part of a Swedish letter from Göran Ohlin to Björn Thalberg, dated 5 October 1976: the letter showed that Goodwin had challenged his Harvard students to solve a certain matrix problem, and promised an A to anyone who solved it. Ohlin succeeded thanks to Frobenius.

Even though Ramsey and Watson were very helpful, the preface of Sraffa’s book left no doubt about the crucial role of Abram Besicovitch:

My greatest debt is to Professor A.S. Besicovitch for invaluable mathematical help over many years (Sraffa 1960: vi).

From the many detailed examples by Kurz & Salvadori (2004, 2008), we learn that Sraffa’s preface didn’t exaggerate Besicovitch’s altruistic contribution. On 21 September 1944 Besicovitch even provided Sraffa with a self-made proof of the existence of a standard commodity.<sup>68</sup>

Jonathan Smith (2011) described the relationship between Sraffa and Trinity College in Cambridge, and stressed the importance of Besicovitch’s membership of the same College:

Whether Besicovitch was the best man in Cambridge to attack Sraffa’s problems is unclear, though he was certainly able to produce the equations Sraffa wanted. It seems to me that the important factor was that Sraffa had come to know him through Trinity, clearly got on with him; and he was at hand, which made it easier for both parties (Smith 2011: 107).

It is obvious that Besicovitch did not know Perron’s articles on matrix algebra, although he surely knew some other work by Perron. Indeed, Perron (1928) himself wrote a paper titled *Über einen Satz von Besicovitsch*,<sup>69</sup> which simplified a proof that had been given by Besicovitch (1928) in his own paper *On Kakeya’s problem and a similar one*. In 1958 the Kakeya-Besicovitch problem was the subject of a film commissioned by the Mathematical

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<sup>67</sup> Goodwin presented this paper at the first input-output conference in Driebergen (Holland) in September 1950.

<sup>68</sup> Reproduced and discussed in Salvadori (2011). See also Lippi (2008) and Salvadori (2008). De Vivo (2003: 15-16) mentions that Besicovitch already in 1942 presented to Sraffa an example where his original system was transformed into a system with  $n$  “compounded” commodities, but involving some negative elements.

<sup>69</sup> In this title I copy “Besicovitsch” as spelled by Perron.

Association of America, and with an exposition by Besicovitch.<sup>70</sup> We should not underestimate the opportunity cost of Besicovitch’s help to Sraffa; he was an important mathematician in his own fields of specialization.<sup>71</sup>

#### 6.4 Markov and diagonal dominance: Besicovitch came close to Frobenius

Current textbooks on nonnegative matrices often provide a unified treatment of three topics under one roof: the theory of Markov chains, the Perron-Frobenius theory of nonnegative matrices, and the theorems on diagonal dominance.

For a simple example of the latter, consider the following matrix  $C$ :

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$$c_{ii} > 0; \quad c_{ij} \leq 0 \quad \text{if } i \neq j$$

$$c_{i1} + c_{i2} + \cdots + c_{in} > 0$$

In every row  $i$ , the diagonal element  $c_{ii}$  “dominates” the rest of the row.<sup>72</sup> Such diagonally dominant matrices appeared in some early papers on electrical equilibria (Lévy 1881), and later also in many articles on pure algebra. The best known example is Minkowski (1900). A rather unknown paper in Russian, containing results on such matrices and implicitly also on their leading principal minors, was written by nobody less than Besicovitch (1918) himself.<sup>73</sup>

I presented an exact copy of Besicovitch’s notation and assumptions about the matrix  $C$  above. Input-output theory often studies a related situation. Suppose that we consider a nonnegative input matrix  $A$ , with suitably normalized input coefficients, satisfying the so-called Brauer-Solow row sum criterion: *all row sums of  $A$  are smaller than one*.<sup>74</sup> Then  $C = I - A$  is diagonally dominant. It is well-known (Takayama 1985, Chapter 4) that the Brauer-Solow criterion is sufficient for the Hawkins-Simon criterion: *all principal minors of  $C = I - A$  are positive*. Besicovitch’s 1918 paper considers only difference equations and pure matrix algebra, no economics. But a modern reader can easily recognize the implicit result that is contained in the second part of Besicovitch’s paper: “Brauer-Solow implies Hawkins-Simon”.

Why did Besicovitch, a former student of Markov and author of a paper on diagonally dominant matrices, overlook Perron-Frobenius? When answering this question, we must see

<sup>70</sup> Besicovitch (1963) gives a good approximation of the script of “his” film.

<sup>71</sup> Another revealing example is *Classics on Fractals* (Edgar 1993), a book containing the historic seminal papers on fractal geometry, a topic that had become “popular” thanks to Benoit Mandelbrot. It turns out that 4 out of 19 papers in this compilation are authored or co-authored by Besicovitch. See Domb (1989) on Hausdorff and Besicovitch as precursors of Mandelbrot. Domb knew Besicovitch personally. See also Burkill (1971) or Taylor (1975) for more biographical details and a list of Besicovitch’s publications.

<sup>72</sup> The matrix  $C$  has row dominance. Its transpose has column dominance.

<sup>73</sup> This Besicovitch article is mentioned in the bibliographical references provided by Ostrowski (1937: 73) and Taussky (1949: 672).

<sup>74</sup> Takayama (1985: 364).

Besicovitch's paper in the right context of 1918. Today many textbooks on Markov probability theory heavily rely on Perron-Frobenius mathematics, but Markov himself developed his pioneering results independently.<sup>75</sup> Today dominant diagonals often appear in the same chapter as Perron-Frobenius, but in the early decades of the 20<sup>th</sup> century Perron-Frobenius and diagonal dominance were not well-known textbook topics. Quite the contrary: Taussky (1949) shows how many journal articles "invented" the same theorems on diagonal dominance again and again, often unnoticed by journal editors or referees.

To explain this ignorance, Schneider (1977) distinguishes between "inward matrix theory" and "outward matrix theory". Inward theory looks downward and inward to the minors of a matrix, its diagonals, its eigenvalues, eigenvectors etc. Outward theory of matrices looks "outward and upward to those great societies of groups and algebras of which they are members" (Schneider 1977: 205). In the first half of the 20<sup>th</sup> century, abstract algebra (outward theory) started to flourish and became a prestigious subject. Inward matrix theory looked old-fashioned and less prestigious, and in the first decades after Perron-Frobenius few significant new mathematical results were published. In this context, it becomes less bizarre that even the mathematicians in linear economics overlooked the old Perron-Frobenius results.

#### 6.5. Sraffa and the Econometric Society: *Che barba*

Sraffa didn't consult his economic colleagues in Cambridge, and neither did he use the mathematical knowledge of the specialists of the *Econometric Society*. In point of fact, his relation to this Society was rather special.

In a letter dated 18 February 1938, Ragnar Frisch, the editor of *Econometrica* himself, asked Sraffa whether he had correctly received the journal for the last years, and whether he wished to remain a member of the *Econometric Society*. Such a personal letter suggests that more than routine administration was involved. In a handwritten unsigned draft of 22 February 1938, Sraffa replied that Keynes informed him that Sraffa's name was on a black-list circulated by Alfred Cowles (sponsor of the *Econometric Society*). Therefore Sraffa announced he wanted to resign membership of the *Econometric Society*.<sup>76</sup>

According to Bjerkholt (1998: 55) four candidates had passed the nomination for the prestigious title of *Fellow of the Econometric Society* in 1938 (Lange, Leontief, Stamp and Yntema), but Sraffa was disapproved of by Cowles. It is not clear what happened next. If Sraffa effectively resigned, then it is strange that in the following decades, until the early 1960s, the name Sraffa remained on the lists of members that were regularly published in *Econometrica*.

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<sup>75</sup> Markov's doctoral students include Besicovitch and Romanovsky. The latter published influential texts on Markov's probability theory, he overlooked Perron-Frobenius until 1933, but afterwards included long paragraphs on Frobenius' results in some of his later work; see Hawkins (2008: 702-704).

<sup>76</sup> The letter by Frisch and the handwritten draft of a reply by Sraffa are conserved in the Sraffa Papers, File C 101. My reference numbers follow the catalogue prepared by Jonathan Smith, Archivist and Modern Manuscript Cataloguer, Trinity College, Cambridge. I thank him for his very friendly and competent help in the Wren Library.

Note that in 1931 Sraffa had attended the very first *European Meeting of the Econometric Society* at the University of Lausanne. Only 14 different speakers appeared on stage in this elite gathering, including Piero Sraffa, who gave the only international scientific conference presentation of his entire career, on the evening of Wednesday 22 September 1931, with a paper *Un économiste mathématicien du 18<sup>me</sup> siècle: le general Lloyd*, a paper on the (slightly) mathematical monetary theory of Henry Lloyd (1771). From the report by Hans Staehle (1933: 79), it seems that Sraffa's paper drew no questions from the audience.

Many weeks of Sraffa's *Cambridge Pocket Diary 1930-1931* are blank, but for that special day of 22 September 1931, I found *Che barba*, which in plain English means: "What a drag" or "What a bore". No further details on Sraffa's comment are available, but he never repeated the conference experiment again. Most participants thought the meeting was a great success, but maybe Sraffa found some other papers rather boring, or had expected some helpful comments on his paper.

Maybe he was surprised by the last minute planning of the program; Divisia couldn't meet his logistic commitments, due to family reasons, and at short notice, Hans Staehle and Ragnar Frisch had to step in, and worked nearly the whole night, because the evening before the start of the Lausanne conference, there was still no program (Bjerkholt 1998: 45).

The most important reason for Sraffa's *Che barba* was probably his strong dislike of giving lectures. Moreover, he preferred to do his own research in isolation and secrecy, and preferred to discuss his mathematical problems with a few mathematical colleagues in Cambridge, not with the many mathematicians or mathematical economists that joined the *Econometric Society*.

## 7. Epilogue

After the Second World War, Perron-Frobenius mathematics finally infiltrated mainstream linear economics. Anglo-Saxon economists could learn it from Goodwin (1950), Japanese economists from Takuma Yasui (1948) or Michio Morishima (1950). According to Takayama (1985: 366), citing Morishima (1955: 1), the rediscoveries were independent. Maybe Yasui's work should have been given more attention (Weintraub, 1987).

However, a debate on priority claims for the explicit use of Perron-Frobenius mathematics in economics is now outmoded. Maurice Potron applied it as early as 1911, in the core arguments of many of his writings. Potron's discovery of the connection between Perron-Frobenius and linear models of the input-output type should be held in high esteem, especially in the light of a comparison with the other mathematicians (Charasoff, Bray, Remak) who overlooked this connection in their seminal writings on linear economics.

It is tempting to claim that the other pioneers of the 1910s and 1920s (Charasoff, Bray, Remak) or Sraffa's mathematical helpers (Ramsey, Watson, Besicovitch) missed some "open goal opportunities", to use available tools of mathematics that, from today's viewpoint, seem simple to apply. But were Charasoff, Bray, Remak, Ramsey, Besicovitch, Watson, etc. all suffering from scientific blindness or absent-mindedness? And what about other disciplines (statistics, demography, etc.) where Perron-Frobenius entered in the same slow way? Or, for example, what about the slow infiltration of linear inequalities and convex sets in game theory?

Tinne Hoff Kjeldsen (2001) describes the analogous situation in the history of game theory, taking into account von Neumann's 1953 remarks on the relation between his 1928 minimax theorem and the theory of linear inequalities and convex sets. From today's elementary textbooks on game theory, this connection seems simple and straightforward. But for more than a decade von Neumann himself didn't see the connection, until Oskar Morgenstern in 1941 showed him a 1938 paper by Jean Ville (for more details, see Kjeldsen, 2001: 58, 65). The lesson from this and related stories is clear:

It is common and tempting fallacy to view the later steps in a mathematical evolution as much more obvious and cogent after the fact than they were beforehand (von Neumann 1953: 125).

A similar remark can be made with respect to the use of Perron-Frobenius mathematics. We should see a scholar's work in its context. Results or connections that seem obvious today were not trivial for the original pioneers.

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## **All but one: How pioneers of linear economics overlooked Perron-Frobenius mathematics**

*Wilfried Parys*

### Summary

In the period 1907-1912 Oskar Perron and Georg Frobenius developed the fundamental results of the theory of nonnegative matrices. Today Perron-Frobenius mathematics enjoys wide applications in many disciplines. It is often a crucial tool in linear economic models of the Leontief-Sraffa type.

In the 1910s and 1920s, several pioneering publications in linear economics could have benefited from applying Perron-Frobenius results, but failed to do so, even the economic publications authored by the mathematicians Charasoff, Bray and Remak. Either they didn't know Perron-Frobenius, or they didn't realize its usefulness. The only exception is the French Jesuit mathematician Maurice Potron, who used Perron-Frobenius mathematics in the core of his economic model, in many of his writings, as early as 1911. He constructed a sort of disaggregated open input-output system, formulated duality theorems between his quantity system and his price system, and anticipated the Hawkins-Simon conditions. Potron's economic or mathematical contemporaries didn't recognize his originality.

A general treatment of Charasoff's economic system needs Perron-Frobenius mathematics. Although some of Charasoff's mathematical interests (irreducibility, continued fractions) were close to Perron's interests, the theory of nonnegative matrices is never explicitly used in Charasoff's work. It is doubtful whether Charasoff knew the relevant matrix theorems. Probably he just assumed that the properties of his numerical examples with three commodities also hold in the general case with  $n$  commodities.

Frobenius had been Remak's doctoral supervisor in 1911. After a forgotten non-mathematical paper in 1918, on the repayment of the national debt, Remak presented his mathematical system of superposed prices in 1929, twelve years after Frobenius' death. With suitable units of measurement, Remak's system can easily be handled by Perron-Frobenius tools. However, Remak failed to normalize his units, and provided lengthy proofs of his own. Moreover, he misdirected most of his mathematical efforts to price systems in which the most important commodities have zero prices. A few years earlier, in 1922, Bray also had overlooked Perron-Frobenius in a mathematically similar model that studied Cournot's equations of currency exchange.

I provide evidence that Wassily Leontief knew Bray's and Remak's results already in the early 1930s, before he submitted a theoretical paper on his input-output model to Keynes for the *Economic Journal* in 1933. Keynes rejected Leontief's paper. A few years later Leontief published his Nobel prize winning empirical and theoretical results in American journals.

Just like Leontief, Sraffa started related research in the late 1920s. He didn't discuss his mathematical problems with competent economic colleagues in Cambridge, nor with the specialists of the *Econometric Society*, but preferred help from three non-economists: Ramsey, Watson and especially Besicovitch. I suggest that Besicovitch in his early days "came close" to Perron-Frobenius results, but it is well-known that he didn't know Perron-Frobenius, and invented his own proofs for Sraffa in the 1940s.

We should see a scholar's work in its context. Results or connections that seem obvious today were not trivial for the original pioneers.